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The covariance structure of the bivariate weighted Poisson distribution and application to the Aleurodicus data

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Abstract. We meet in the literature the bivariate Poisson distribution put in evidence by Berkhout and Plug. From this distribution, Elion et al. put in evidence the bivariate weighted Poisson distribution like crossing of two univariate weighted Poisson distributions. The structure of the covariance of this bivariate weighted Poisson distribution has been put not again in evidence in the literature. Thus, in this paper, we remedy this hiatus. The overdispersion, underdispersion and the equidispersion will be valued with the help of the Fisher dispersion index for multivariate count distributions introduces by Kokonendji et al. An illustrative example based on the Aleurodicus data is presented.

Keywords. Bivariate dispersion index; Moment generating function; Conditional law.
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Résumé Nous rencontrons dans la littérature la distribution de Poisson bivariée mise en évidence par Berkhout et Plug. De cette distribution, Elion et al. ont mis en évidence la distribution de Poisson pondérée bivariée comme le croisement de deux distributions de Poisson pondérées univariées. La structure de la covariance de cette distribution de Poisson pondérée bivariée n'a pas encore été mise en évidence dans la littérature. Ainsi, dans ce papier, nous remédions à cette lacune. La surdispersion, la sous-dispersion et l'équidispersion seront évaluées à l'aide de l'indice de dispersion de Fisher, pour les distributions de comptage multivariées, introduit par Kokonendji et al. Un exemple illustratif basé sur les données Aleurodicus est présenté.

1. Introduction

The concept of weighted distributions originated with Fisher (1934) who demonstrated a need for adjustement in the way models are specified depending on how the data are obtained. It was Rao who first unified this concept in a series of papers Patil & Rao (1978) and Rao (1985). The weighted distributions have been used during the past forty years as useful tools in the selection of appropriate models for the observed data, especially when samples are drawn without a proper frame.

Let Y be a discrete univariate random variable with *probability mass function* (*pmf*) $\mathbb{P}(Y = y)$, $y \in \mathbb{R}$. A weighted version of Y with univariate non-negative weight function $w(y)$ with $0 < \mathbb{E}[w(Y)] < \infty$, is denoted by Y^ω. The *pmf* of Y^ω is given by (see, e.g., Kokonendji & Perez-Casany (2012))

$$\mathbb{P}[Y^\omega = y] = \frac{w(y)}{\mathbb{E}[w(Y)]}\mathbb{P}[Y = y]. \tag{1}$$

Afterward, let us suppose that Y follows the Poisson distribution of parameter δ. Expression (1) is equivalent to

$$\mathbb{P}[Y^\omega = y] = \frac{w(y)}{\mathbb{E}_\delta[w(Y)]}\frac{\delta^y}{y!}e^{-\delta}, \qquad y \in \mathbb{N}.$$

It follows, in this case, that the mathematical expectation and the variance of Y^ω are respectively given by (Kokonendji, Mizère & Balakrishnan (2008)):

$$\mathbb{E}_\delta[Y^\omega] = \delta\left(1 + \frac{d}{d\delta}\ln(\mathbb{E}_\delta[\omega(Y)])\right), \tag{2}$$

and

$$\mathrm{Var}(Y^\omega) = \mathbb{E}_\delta[Y^\omega] + \delta^2\frac{d^2}{d\delta^2}\ln(\mathbb{E}_\delta[\omega(Y)]).$$

The *probability generating function* (pgf) and the *moment generating function* (mgf) that we denote by $g_{Y^\omega}(z)$ and $M_{Y^\omega}(z)$ respectively are obtained by the formulas

$$g_{Y^\omega}(z) = \frac{e^{\delta(z-1)}\mathbb{E}_{z\delta}[w(Y)]}{\mathbb{E}_\delta[w(Y)]} \quad \text{and} \quad M_{Y^\omega}(z) = \frac{e^{\delta(e^z-1)}\mathbb{E}_{e^z\delta}[w(Y)]}{\mathbb{E}_\delta[w(Y)]}, \ -1 < z \leq 1.$$

In fact,

$$\begin{aligned}
g_{Y^\omega}(z) &= \mathbb{E}_\delta\left[z^{Y^\omega}\right] = \sum_{k\geqslant 0} z^k \frac{\omega(k)}{\mathbb{E}_\delta[\omega(Y)]}\frac{\delta^k}{k!}e^{-\delta} = \frac{1}{\mathbb{E}_\delta[\omega(Y)]}\sum_{k\geqslant 0}\omega(k)\frac{(z\delta)^k}{k!}e^{-\delta} \\
&= \frac{1}{\mathbb{E}_\delta[\omega(Y)]}\sum_{k\geqslant 0}\omega(k)\frac{(z\delta)^k}{k!}e^{-z\delta}\frac{e^{-\delta}}{e^{-z\delta}} = \frac{e^{\delta(z-1)}}{\mathbb{E}_\delta[\omega(Y)]}\sum_{k\geqslant 0}\omega(k)\frac{(z\delta)^k}{k!}e^{-z\delta} \\
&= e^{\delta(z-1)}\frac{\mathbb{E}_{z\delta}[\omega(Y)]}{\mathbb{E}_\delta[\omega(Y)]},
\end{aligned}$$

and

$$M_{Y^\omega}(z) = \mathbb{E}_\delta\left[e^{Y^\omega}\right] = \mathbb{E}_\delta\left[(e^z)^{Y^\omega}\right] = g_{Y^\omega}(e^z) = e^{\delta(e^z-1)}\frac{\mathbb{E}_{e^z\delta}[\omega(Y)]}{\mathbb{E}_\delta[\omega(Y)]}.$$

We note, in the literature, that the bivariate Poisson distribution is a popular distribution for modelling bivariate count data as illustrated in Famoye (2010). This distribution has been introduced by Campbell (1934) who considered the limit of the distribution of contingency table with two distributions. Pratically at the same time, Guldberg (1934) obtains the independent binomial distribution is due a few years later in 1944 to Aitken (1994). It is however necessary to await Holgate (1964) to obtain a bivariate variables starting from three univariate variables of Poisson independent. The model of Holgate puts in evidence a strictely positive correlation, what is not always realistic. To remedy this problem, Berkhout & Plug (2004) proposed a bivariate Poisson distribution accepting the correlation as well negative, equal to zero, that positive. Elion *et al.* (2016), while leaving from the ideas of Berkhout & Plug (2004), proposed the bivariate weighted Poisson distribution (*bwPd*).

The main purpose of this paper is to continue the work of Elion *et al.* (2016) while calculating the covariance of a couple of a random variables whose conjoint density is the bivariate weighted Poisson distribution.

The remainder of this paper is organized as follows. In Section 2, we first recall the *bwPd* and introduces the structure of the associated covariance. Afterward, we study the over-, equi- and underdispersion of the bwPd from some examples while using a multivariate dispersion index of Kokonendji & Puig (2018). In Section 3, we present the estimation of parameters. Next, an illustrative example based on the *Aleurodicus* data, which is available in Mizère (2006) & Mizère (2007), is presented in Section 4. Finally we conclude this paper in Section 5.

2. Structure of the covariance in the bivariate weighted Poisson distribution

In this section, we first recall the definition of the bivariate weighted Poisson distribution highlighted by Elion *et al.* (2016). We then study the structure of the covariance which is the main object of this paper.

2.1. The bivariate weighted Poisson distribution

Let us consider two random variables $Y_1^{\omega_1}$ and $Y_2^{\omega_2}$ which follow univariate weighted Poisson distributions of mass functions:

$$\mathbb{P}\left[Y_i^{\omega_i}=y_i\right]=p_{\omega_i}\left(y_i;\delta_i\right)=\frac{\omega_i\left(y_i\right)}{\mathbb{E}_{\delta_i}\left[\omega_i\left(Y_i\right)\right]}\frac{\delta_i^{y_i}}{y_i!}e^{-\delta_i},\quad y_i\in\mathbb{N};\delta_i\in\mathbb{R}_+^*;i=1,2. \tag{3}$$

Here, the second parameter δ_2 depends on the values y_1 taken by $Y_1^{\omega_1}$. It is therefore a function of y_1.

The ordered pair $(Y_1^{\omega_1}, Y_2^{\omega_2})$ follows the bivariate weighted Poisson distribution if its mass function is equal to :

$$\begin{aligned} p_{\omega_1,\omega_2}\left(y_1,y_2;\delta_1,\delta_2\right) &= \mathbb{P}\left(Y_1^{\omega_1}=y_1,Y_2^{\omega_2}=y_2\right) \\ &= \frac{\omega_1\left(y_1\right)}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\frac{\omega_2\left(y_2\right)}{\mathbb{E}_{\delta_2}\left[\omega_2\left(Y_2\right)\right]}\frac{\delta_1^{y_1}}{y_1!}\frac{\delta_2^{y_2}}{y_2!}e^{-\delta_1-\delta_2}, \end{aligned} \tag{4}$$

where $\quad (y_1,y_2)\in\mathbb{N}^2, (\delta_1,\delta_2)\in{\mathbb{R}_+^*}^2,$

under the following conditions:

$$\ln\delta_1 = x'\beta_1 \tag{5}$$

$$\ln\delta_2 = x'\beta_2+\eta y_1, \tag{6}$$

where $x'=(x_1,x_2,...,x_p)$ is the vector of the predictor variables or factors. It is clear that δ_2 depends y_1 in Formula (4). Without an appropriate notation Formula (4) can be misleading to a ordered pair with independent margins. $Y_1^{\omega_1}$is the response variable of the model (5) and $Y_2^{\omega_2}$ that of the model (6). Thus, in the model (6), $Y_1^{\omega_1}$ is considered as a predictor variable in order to highlight the dependence between the variable $Y_1^{\omega_1}$ and $Y_2^{\omega_2}$. The models (5) and (6) permit to highlight the effect of the factors on the variable (6) $Y_1^{\omega_1}$ and $Y_2^{\omega_2}$, but also to detect the interaction between the factors.

The expression (6) leads the probability conditional $p_{\omega_2}(y_2,\delta_2) = \mathbb{P}\left[Y_2^{\omega_2}=y_2/Y_1^{\omega_1}=y_1\right]$, of conditional mean (Cf. Expression (2)):

$$\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}/Y_1^{\omega_1}=y_1\right]=\mu_2=\delta_2\left(1+a_2\right), \text{ with } a_2=\frac{d}{d\delta_2}\ln\mathbb{E}_{\delta_2}\left[\omega_2\left(Y_2\right)\right]. \tag{7}$$

This lead to

$p_{\omega_1,\omega_2}\left(y_1,y_2;\mu_1,\mu_2\right)=p_{\omega_1}\left(y_1,\mu_1\right)\mathbb{P}\left[Y_2^{\omega_2}=y_2/Y_1^{\omega_1}=y_1\right].$

The probability $p_{\omega_1}\left(y_1,\mu_1\right)=\mathbb{P}\left(Y_1^{\omega_1}=y_1\right)$ is the marginal law with mean

$$\mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right]=\mu_1=\delta_1\left(1+a_1\right),$$

where $a_1=\frac{d}{d\delta_1}\ln\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]$.

When $\eta=0$, the variables $Y_1^{\omega_1}$ and $Y_2^{\omega_2}$ are independent.

We notice that the generalized linear models of response variables $Y_1^{\omega_1}$ ($Y_2^{\omega_2}$) having for link functions $\ln\mu_1$ and $\ln\delta_1$ ($\ln\mu_2$ and $\ln\delta_2$) produce the same estimators of the coefficients β_1 (β_2 and η). Indeed, $\ln\mu_1=\ln\delta_1+\ln\left(1+a_1\right)=x'\beta_1+c_1$, with $c_1\in\mathbb{R}$ and $\ln\mu_2=\ln\delta_2+\ln\left(1+a_2\right)=x'\beta_2+\eta y_1+c_2$, with $c_2\in\mathbb{R}$.

2.2. Structure of the covariance

We present, through the proposition below, the structure of the covariance in the bivariate weighted Poisson distribution.

Proposition 1. *We have the following formulas*

$$a)\&\ \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]=\ e^{x'\beta_2+c_2+\delta_1\left(e^{\eta}-1\right)}\frac{\mathbb{E}_{e^{\eta}\delta_1}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}.$$

$$b)\&Var\left(Y_2^{\omega_2}\right)=\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]+\left[\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\right]^2\left(e^{\delta_1\left(e^{\eta}-1\right)}\frac{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]\mathbb{E}_{\delta_1e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\left(\mathbb{E}_{\delta_1e^{\eta}}\left[\omega_1\left(Y_1\right)\right]\right)^2}-1\right).$$

$$c)\qquad Cov\left(Y_1^{\omega_1},Y_2^{\omega_2}\right)=\mathbb{E}_{\delta_2}\left[Y\right.$$

Proof of Proposition 1.

(a) For the Expression (8), we have:

$$\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\ =\ \mathbb{E}_{\delta_1}\left[\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}/Y_1^{\omega_1}\right)\right]=\mathbb{E}_{\delta_1}\left(e^{x'\beta_2+c_2+\eta Y_1^{\omega_1}}\right)=e^{x'\beta_2+c_2}\mathbb{E}_{\delta_1}\left(e^{\eta Y_1^{\omega_1}}\right).$$

Thus, the Expression (3) becomes:

$$\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] = e^{x'\beta_2+c_2} M_{Y_1^{\omega_1}}(\eta) = e^{x'\beta_2+c_2} e^{\delta_1(e^{\eta}-1)} \frac{\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}.$$

(b) For the Expression (9), we have :

$$\begin{aligned}
\mathbb{E}_{\delta_2}\left[\left(Y_2^{\omega_2}\right)^2\right] &= \mathbb{E}_{\delta_1}\left(\mathbb{E}_{\delta_2}\left[\left(Y_2^{\omega_2}\right)^2 / \left(Y_1^{\omega_1}\right)\right]\right) = \mathbb{E}_{\delta_1}\left(\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}/Y_1^{\omega_1}\right]\right) + \mathbb{E}_{\delta_1}\left(\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}/Y_1^{\omega_1}\right]\right)^2 \\
&= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] + \mathbb{E}_{\delta_1}\left[\left(e^{x'\beta_2+c_2+\eta Y_1^{\omega_1}}\right)^2\right]. \qquad (11)
\end{aligned}$$

Let us notice that

$$\begin{aligned}
\mathbb{E}_{\delta_1}\left[\left(e^{x'\beta_2+c_2+\eta Y_1^{\omega_1}}\right)^2\right] &= \mathbb{E}_{\delta_1}\left[\left(e^{2x'\beta_2+2c_2+2\eta Y_1^{\omega_1}}\right)\right] = e^{2(x'\beta_2+c_2)} \mathbb{E}_{\delta_1}\left[e^{2\eta Y_1^{\omega_1}}\right] \\
&= e^{2(x'\beta_2+c_2)} M_{Y_1^{\omega_1}}(2\eta) \\
&= e^{2(x'\beta_2+c_2)} e^{\delta_1\left(e^{2\eta}-1\right)} \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}.
\end{aligned}$$

Since we have

$$e^{2\eta}-1 = \left(\left(e^{\eta}\right)^2-1\right) = \left(e^{\eta}-1\right)\left(e^{\eta}+1\right) = \left(e^{\eta}-1\right)\left[\left(e^{\eta}-1\right)+2\right] = \left(e^{\eta}-1\right)^2 + 2\left(e^{\eta}-1\right),$$

we obtain

$$\begin{aligned}
\mathbb{E}_{\delta_1}\left[\left(e^{x'\beta_2+c_2+\eta Y_1^{\omega_1}}\right)^2\right] &= e^{2(x'\beta_2+c_2)} e^{\delta_1(e^{\eta}-1)^2+2\delta_1(e^{\eta}-1)} \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]} \\
&= e^{2(x'\beta_2+c_2)} e^{\delta_1(e^{\eta}-1)^2} e^{2\delta_1(e^{\eta}-1)} \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]} \\
&= e^{\delta_1(e^{\eta}-1)^2}\left[e^{x'\beta_2+c_2+\delta_1(e^{\eta}-1)}\right]^2 \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]} \\
&= e^{\delta_1(e^{\eta}-1)^2}\left[e^{x'\beta_2+c_2+\delta_1(e^{\eta}-1)} \frac{\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\right]^2 \times \\
&\times \frac{\left(\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]\right)^2}{\left(\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right]\right)^2} \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]} \\
&= e^{\delta_1(e^{\eta}-1)^2}\left(\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\right)^2 \mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right] \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\left(\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right]\right)^2}.
\end{aligned}$$

The Expression (11) writes itself then as follows.

$$\mathbb{E}_{\delta_2}\left[\left(Y_2^{\omega_2}\right)^2\right] = \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] + e^{\delta_1(e^{\eta}-1)^2}\left(\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\right)^2 \mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right] \frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}^2_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right]}.$$

So the formula

$$\text{Var}\left(Y_2^{\omega_2}\right) = \mathbb{E}_{\delta_2}\left[\left(Y_2^{\omega_2}\right)^2\right] - \mathbb{E}_{\delta_2}^2\left[Y_2^{\omega_2}\right],$$

gives

$$\text{Var}\left(Y_2^{\omega_2}\right) = \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] + \left(\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\right)^2\left(e^{\delta_1(e^\eta-1)^2}\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]\frac{\mathbb{E}_{\delta_1 e^{2\eta}}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}^2_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]} - 1\right).$$

(c) For the Expression (10), we have

$$\begin{aligned}
\mathbb{E}_{\delta_1\delta_2}\left[Y_1^{\omega_1}Y_2^{\omega_2}\right] &= \mathbb{E}_{\delta_1}\left[\mathbb{E}_{\delta_2}\left[Y_1^{\omega_1}Y_2^{\omega_2}/Y_1^{\omega_1}\right]\right] = \mathbb{E}_{\delta_1}\left(Y_1^{\omega_1}\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}/Y_1^{\omega_1}\right]\right) = \mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}e^{x'\beta_2+c_2+\eta Y_1^{\omega_1}}\right] \\
&= e^{x'\beta_2+c_2}\mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}e^{\eta Y_1^{\omega_1}}\right] = e^{x'\beta_2+c_2}\mathbb{E}_{\delta_1}\left[\frac{d}{d\eta}\left(e^{\eta Y_1^{\omega_1}}\right)\right] \\
&= e^{x'\beta_2+c_2}\frac{d}{d\eta}\left(\mathbb{E}_{\delta_1}\left[e^{\eta Y_1^{\omega_1}}\right]\right) = e^{x'\beta_2+c_2}\frac{d}{d\eta}\left[M_{Y_1^{\omega_1}}\left(\eta\right)\right] \\
&= e^{x'\beta_2+c_2}\frac{d}{d\eta}\left\{e^{\delta_1(e^\eta-1)}\frac{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\right\}. \qquad (12)
\end{aligned}$$

However

$$\begin{aligned}
\frac{d}{d\eta}\left\{e^{\delta_1(e^\eta-1)}\frac{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\right\} &= \delta_1 e^\eta e^{\delta_1(e^\eta-1)}\frac{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]} + \frac{e^{\delta_1(e^\eta-1)}}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\frac{d}{d\eta}\left(\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]\right) \\
&= e^{\delta_1(e^\eta-1)}\frac{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\left(\delta_1 e^\eta + \frac{\frac{d}{d\eta}\left(\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]\right)}{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}\right).
\end{aligned}$$

Therefore Expression (12) becomes

$$\begin{aligned}
\mathbb{E}_{\delta_1\delta_2}\left[Y_1^{\omega_1}Y_2^{\omega_2}\right] &= e^{x'\beta_2+c_2+\delta_1(e^\eta-1)}\frac{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}{\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right]}\left(\delta_1 e^\eta + \frac{\frac{d}{d\eta}\left(\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]\right)}{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}\right) \\
&= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\delta_1 e^\eta + \frac{\frac{d}{d\eta}\left(\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]\right)}{\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]}\right) \\
&= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\delta_1 e^\eta + \frac{d}{d\eta}\ln\left(\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]\right)\right).
\end{aligned}$$

Finally we have

$$\begin{aligned}
\text{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) &= \mathbb{E}_{\delta_1\delta_2}\left[Y_1^{\omega_1}Y_2^{\omega_2}\right] - \mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right]\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] \\
&= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\delta_1 e^\eta + \frac{d}{d\eta}\ln\left(\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right]\right) - \mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right]\right).
\end{aligned}$$

The proof is complete. ■

Example 1

(1) When $Y_1^{\omega_1}$ follows the univariate Poisson distribution of parameter δ_1, we have: $\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right] = 1$ and $\mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right] = \delta_1$.

So the Expression (10) is also

$$\mathrm{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) = \left(e^\eta - 1\right)\delta_1 \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right].$$

We recover the result of Berkhout & Plug (2004).

If

- $\eta = 0$, $Cov\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) = 0$.
- $\eta > 0$, $Cov\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) > 0$.
- $\eta < 0$, $Cov\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) < 0$.

(2) When $Y_1^{\omega_1}$ follows the binomial distribution $\mathcal{B}\left(n, p\right)$, let us pose $\delta_1 = \frac{p}{1-p}$ $\left(0 < p < 1\right)$, we have:

$$\begin{aligned}
\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right] &= \left(\delta_1 + 1\right)^n e^{-\delta_1} \\
\mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right] &= \left(\delta_1 e^\eta + 1\right)^n e^{-\delta_1 e^\eta} \\
\ln \mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right] &= n \ln\left(\delta_1 e^\eta + 1\right) - \delta_1 e^\eta \\
\frac{d}{d\eta} \ln \mathbb{E}_{\delta_1 e^\eta}\left[\omega_1\left(Y_1\right)\right] &= \frac{n\delta_1 e^\eta}{\delta_1 e^\eta + 1} - \delta_1 e^\eta \\
\mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right] &= np = n\frac{\delta_1}{\delta_1 + 1}.
\end{aligned}$$

Expression (10) is:

$$\begin{aligned}
\mathrm{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) &= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\delta_1 e^\eta + \frac{n\delta_1 e^\eta}{\delta_1 e^\eta + 1} - \delta_1 e^\eta - \frac{n\delta_1}{\delta_1 + 1}\right) \\
&= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\frac{n\delta_1 e^\eta}{\delta_1 e^\eta + 1} - \frac{n\delta_1}{\delta_1 + 1}\right) \\
&= n\delta_1 \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\frac{e^\eta}{\delta_1 e^\eta + 1} - \frac{1}{\delta_1 + 1}\right) \\
&= n\delta_1 \frac{e^\eta\left(\delta_1 + 1\right) - \delta_1 e^\eta - 1}{\left(\delta_1 e^\eta + 1\right)\left(\delta_1 + 1\right)} \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] \\
&= \frac{n\delta_1\left(e^\eta - 1\right)}{\left(\delta_1 e^\eta + 1\right)\left(\delta_1 + 1\right)} \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right].
\end{aligned}$$

Therefore

$$\text{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) = \frac{n\delta_1\left(e^{\eta}-1\right)}{\left(\delta_1 e^{\eta}+1\right)\left(\delta_1+1\right)}\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]$$

(3) When $Y_1^{\omega_1}$ follows the negative binomial distribution of mass function

$$p\left(y;\delta_1,\phi\right) = \delta_1^y\left(1-\delta_1\right)^{\phi}\frac{\Gamma\left(\phi+y\right)}{y!\Gamma\left(\phi\right)}, \quad y = 0,1,2,...; \ \phi \in \mathbb{R}_+^*, \ 0 < \delta_1 < 1.$$

We have:

$$\begin{aligned}
\mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right] &= \phi\frac{\delta_1}{1-\delta_1} \\
\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right] &= e^{-\delta_1}\left(1-\delta_1\right)^{-\phi} \\
\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right] &= e^{-\delta_1 e^{\eta}}\left(1-\delta_1 e^{\eta}\right)^{-\phi}.
\end{aligned}$$

We get:

$$\begin{aligned}
\ln\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right] &= -\delta_1 e^{\eta} - \phi\ln\left(1-\delta_1 e^{\eta}\right) \\
\frac{d}{d\eta}\ln\mathbb{E}_{\delta_1 e^{\eta}}\left[\omega_1\left(Y_1\right)\right] &= -\delta_1 e^{\eta} + \frac{\phi\delta_1 e^{\eta}}{1-\delta_1 e^{\eta}}.
\end{aligned}$$

$$\begin{aligned}
\text{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) &= \phi\delta_1\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\frac{e^{\eta}}{1-\delta_1 e^{\eta}} - \frac{1}{1-\delta_1}\right) \\
&= \phi\delta_1\left(\frac{e^{\eta}}{1-\delta_1 e^{\eta}} - \frac{1}{1-\delta_1}\right)\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] \\
&= \frac{\phi\delta_1}{\left(1-\delta_1\right)\left(1-\delta_1 e^{\eta}\right)}\left(e^{\eta}-1\right)\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right].
\end{aligned}$$

(4) When $Y_1^{\omega_1}$ follows the translated Poisson distribution of mass function:

$$P\left[Y^{\omega} = y\right] = \frac{\delta_1^{y-\phi}}{(y-\phi)!}e^{-\delta_1}, y = \phi, \phi+1, ...; \delta_1 \in \mathbb{R}_+^*, \phi \in \mathbb{N}^*.$$

We have:

$$\begin{aligned}
\mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right] &= \phi + \delta_1 \\
\mathbb{E}_{\delta_1}\left[\omega_1\left(Y_1\right)\right] &= \delta_1^{\phi} \\
\mathbb{E}_{e^{\eta}\delta_1}\left[\omega_1\left(Y_1\right)\right] &= \left(e^{\eta}\delta_1\right)^{\phi} = \delta_1^{\phi}e^{\eta\phi} \\
\ln\mathbb{E}_{e^{\eta}\delta_1}\left[\omega_1\left(Y_1\right)\right] &= \phi_1\ln\delta_1 + \eta\phi_1 \\
\frac{d}{d\eta}ln\mathbb{E}_{e^{\eta}\delta_1}\left[\omega_1\left(Y_1\right)\right] &= \phi_1.
\end{aligned}$$

So we get the result:

$$\text{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) = \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left[\left(\delta_1 e^{\eta}+\phi\right) - \phi - \delta_1\right] = \delta_1\left(e^{\eta}-1\right)\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right],$$

this is the result obtained by Elion *et al.* (2016).

2.3. The bivariate dispersion indexes

The main purpose of this subsection is to study the over-/equi-/underdispersion in the *bwPd* through some examples.

Let us consider the couple of random variables $Y^{\omega_1,\omega_2} = (Y_1^{\omega_1}, Y_2^{\omega_2})$ whose conjoined law is the bwPd. The multiple marginal dispersion index of Y^{ω_1,ω_2} writes itself as follows (Kokonendji & Puig (2018)):

$$MDI\left(Y^{\omega_1,\omega_2}\right) = \frac{\left[\mathbb{E}_{\delta_1}\left(Y_1^{\omega_1}\right)\right]^2 DI\left(Y_1^{\omega_1}\right) + \left[\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\right]^2 DI\left(Y_2^{\omega_1}\right)}{\left[\mathbb{E}_{\delta_1}\left(Y_1^{\omega_1}\right)\right]^2 + \left[\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\right]^2} \quad (13)$$

The generalized dispersion index of Y^{ω_1,ω_2} writes itself as follows (Kokonendji & Puig (2018)):

$$GDI\left(Y^{\omega_1,\omega_2}\right) = MDI\left(Y^{\omega_1,\omega_2}\right) + \frac{2\rho\mathbb{E}_{\delta_1}\left(Y_1^{\omega_1}\right)\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\sqrt{DI\left(Y_1^{\omega_1}\right)DI\left(Y_2^{\omega_2}\right)}}{\left[\mathbb{E}_{\delta_1}\left(Y_1^{\omega_1}\right)\right]^2 + \left[\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\right]^2}, \quad (14)$$

with $\rho = \text{Cor}Y^{\omega_1,\omega_2}$, $DI\left(Y_1^{\omega_1}\right) = \frac{\text{Var}\left(Y_1^{\omega_1}\right)}{\mathbb{E}_{\delta_1}\left(Y_1^{\omega_1}\right)}$ and $DI\left(Y_2^{\omega_2}\right) = \frac{\text{Var}\left(Y_2^{\omega_2}\right)}{\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)}$.
If $GDI\left(Y^{\omega_1,\omega_2}\right) < 1$, then one says that the distribution joined of the couple Y^{ω_1,ω_2} is underdispersed. Besides, if $GDI\left(Y^{\omega_1,\omega_2}\right) = 1$, then one says that the distribution joined of the couple Y^{ω_1,ω_2} is equidispersed.
Finally, the couple Y^{ω_1,ω_2} is overdispersed if

$$GDI\left(Y^{\omega_1,\omega_2}\right) > 1.$$

Now we are going to examine two examples while taking into account the two previous indexes.

Example 2

(1) When the variable $Y_1^{\omega_1}$ follows the Poisson distribution of parameter δ_1, we have:

$$DI\left(Y_1^{\omega_1}\right) = 1,\ DI\left(Y_2^{\omega_2}\right) = 1 + \mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\left(e^{\delta_1\left(e^{\eta}-1\right)} - 1\right)$$

and

$$\text{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) = \left(e^{\eta} - 1\right)\delta_1\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right).$$

Let us make the choice of the value of η.

i. $\eta = 0$, we have $\rho = 0$ and $DI\left(Y_2^{\omega_2}\right) = 1$. In this case, $MDI\left(Y^{\omega_1,\omega_2}\right) = GDI\left(Y^{\omega_1,\omega_2}\right) = 1$. The vector $\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right)$ is equidispersed following the two indexes.
ii. $\eta > 0$, we have $DI\left(Y_2^{\omega_2}\right) > 1$ and $\rho > 0$. In this case, $MDI\left(Y^{\omega_1,\omega_2}\right) > 1$, $GDI\left(Y^{\omega_1,\omega_2}\right) > 1$. For the indexes MDI and GDI, the vector $\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right)$ is overdispersed; when we know that $Y_1^{\omega_1}$ is equidispersed and $Y_2^{\omega_2}$ overdispersed.
iii. $\eta < 0$, we have $DI\left(Y_2^{\omega_2}\right) < 1$ and $\rho < 0$. In this case, $MDI\left(Y^{\omega_1,\omega_2}\right) < 1$ and $GDI\left(Y^{\omega_1,\omega_2}\right) < 1$. The vector $\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right)$ is underdispersed for the two indexes, when we know that $Y_1^{\omega_1}$ is equidispersed and $Y_2^{\omega_2}$ underdispersed.

(2) When the variable $Y_1^{\omega_1}$ follows the translated Poisson distribution of parameter δ_1, we have:

$$DI\left(Y_1^{\omega_1}\right) = \frac{\delta_1}{\phi + \delta_1} < 1, \quad DI\left(Y_2^{\omega_2}\right) = 1 + \mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\left(e^{\delta_1(e^{\eta}-1)} - 1\right)$$

and

$$\text{Cov}\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) = \left(e^{\eta} - 1\right)\delta_1 \mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right).$$

Let us make the choice of value of η

i. $\eta = 0$, we have $\rho = 0$ and $DI\left(Y_2^{\omega_2}\right) = 1$. In this case, $MDI\left(Y^{\omega_1,\omega_2}\right) = GDI\left(Y^{\omega_1,\omega_2}\right) < 1$. The vector $\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right)$ is underdispersed following the two indexes MDI and GDI.
ii. $\eta > 0$, we have $\rho > 0$ and $DI\left(Y_2^{\omega_2}\right) > 1$. This case is not easy to treat because it is difficult to compare the numbers MDI, GDI and 1.
iii. $\eta < 0$, we have $\rho < 0$ and $DI\left(Y_2^{\omega_2}\right) < 1$. In this case, $MDI\left(Y^{\omega_1,\omega_2}\right) < 1$ and $GDI\left(Y^{\omega_1,\omega_2}\right) < 1$. The vector $\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right)$ is underdispersed following the two indexes MDI and GDI.

3. Estimations of the parameters β_1, β_2 and η

Proposition 2. *The derivates of the log-likelihood function are equal to: $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\eta}$ of β_1, β_2 and η is solutions of the equations:*

i. $\frac{\partial}{\partial \beta_1} \ln p_{\omega_1,\omega_2} = x'\left(y_1 - \mu_1\right)$ *with* $\frac{\partial^2}{\partial \beta_1 \partial \beta_1'} \ln p_{\omega_1,\omega_2} = - \parallel x \parallel^2 \textit{Var}\left(Y_1^{\omega_1}\right)$
ii. $\frac{\partial}{\partial \beta_2} \ln p_{\omega_1,\omega_2} = x'\left(y_2 - \mu_2\right)$ *with* $\frac{\partial^2}{\partial \beta_2 \partial \beta_2'} \ln p_{\omega_1,\omega_2} = - \parallel x \parallel^2 \textit{Var}(Y_2^{\omega_2} \mid Y_1^{\omega_1} = y_1)$
iii. $\frac{\partial}{\partial \eta} \ln p_{\omega_1,\omega_2} = y_1\left(y_2 - \mu_2\right)$ *with* $\frac{\partial^2}{\partial \eta \partial \eta} \ln p_{\omega_1,\omega_2} = -y_1^2 \mu_2$.
iv. $\frac{\partial^2}{\partial \eta \partial \beta_1} \ln p_{\omega_1,\omega_2} = \frac{\partial^2}{d\beta_1 d\beta_2} \ln p_{\omega_1,\omega_2} = 0$
v. $\frac{\partial^2}{\partial \eta \partial \beta_2} \ln p_{\omega_1,\omega_2} = -x' y_1 \textit{Var}(Y_2^{\omega_2} \mid Y_1^{\omega_1} = y_1)$.

$\parallel x \parallel$ *designate the euclidean norm of* x.

Corollary 1. *The estimators of the maximum likelihood $\hat{\beta}_1$ $\left(\hat{\beta}_2 \text{ and } \hat{\eta}\right)$ of β_1 $(\beta_2 \text{ and } \eta)$ are the coefficients of the generalized linear models $\ln \mu_1 = x'\beta_1$ $(\ln \mu_2 = x'\beta_2 + \eta y_1)$ of response variable $Y_1^{\omega_1}$ $(Y_2^{\omega_2})$.*

Proof of Proposition 2. For the vector (y_1, y_2), the $\log$-likelihood of p_{ω_1,ω_2} is equal to:

$$\ln p_{\omega_1,\omega_2} = y_1 \ln \delta_1 + y_2 \ln \delta_2 - \delta_1 - \delta_2 - \ln \mathbb{E}_{\delta_1} [\omega_1 (Y_1)] - \ln \mathbb{E}_{\delta_2} [\omega_2 (Y_2)] - \ln y_1! - \ln y_2!$$

.

(i) We have:

$$\begin{aligned}\frac{\partial \ln p_{\omega_1,\omega_2}}{\partial \beta_1} &= y_1 \left(x' - x' e^{x'\beta_1} - x' e^{x'\beta_1} \frac{d}{d\delta_1} \ln \mathbb{E}_{\delta_1} [\omega_1 (Y_1)] \right) = y_1 x' - x' e^{x'\beta_1} - x' e^{x'\beta_1} a_1 \\ &= x' \left(y_1 - e^{x'\beta_1} - e^{x'\beta_1} a_1 \right) = x' \left(y_1 - e^{x'\beta_1} (1 + a_1) \right).\end{aligned}$$

However $\mu_1 = \delta_1 (1 + a_1)$, therefore $\frac{\partial \ln p_{\omega_1,\omega_2}}{\partial \beta_1} = x' (y_1 - \mu_1)$.

Otherwise, we have

$$\begin{aligned}\frac{\partial^2}{\partial \beta_1 \partial \beta_1'} \ln p_{\omega_1,\omega_2} &= -x' \frac{\partial}{\partial \beta_1'} \left[e^{\beta_1' x} (1 + a_1) \right] = -x' \left[x e^{\beta_1' x} (1 + a_1) + e^{\beta_1' x} \frac{\partial}{\partial \beta_1'} (1 + a_1) \right] \\ &= -x' \left[x \mu_1 + e^{\beta_1' x} \frac{\partial a_1}{\partial \beta_1'} \right] = -x' \left[x \mu_1 + e^{x \beta_1'} \frac{\partial \delta_1}{\partial \beta_1'} \times \frac{\partial a_1}{\partial \delta_1} \right] \\ &= -x' \left[x \mu_1 + x e^{2\beta_1' x} \frac{\partial^2}{\partial \delta_1^2} \ln \mathbb{E}_{\delta_1} [\omega (Y_1)] \right] = -x' x \left(\mu_1 + \delta^2 \frac{\partial^2}{\partial \delta_1^2} \ln \mathbb{E}_{\delta_1} [\omega_1 (Y_1)] \right).\end{aligned}$$

However Var $(Y_1^{\omega_1}) = \mathbb{E}_{\delta_1} (Y_1^{\omega_1}) + \delta_1^2 \frac{d^2}{d\delta_1^2} \ln \mathbb{E}_{\delta_1} [\omega (Y_1)] = \mu_1 + \delta_1^2 \frac{d^2}{d\delta_1^2} \ln \mathbb{E}_{\delta_1} [\omega (Y_1)]$.
Therefore

$$\frac{\partial^2}{\partial \beta_1 \partial \beta_1'} \ln p_{\omega_1,\omega_2} = -x' x \,\text{Var}\, (Y_1^{\omega_1}).$$

So

$$\frac{\partial^2}{\partial \beta_1 \partial \beta_1'} \ln p_{\omega_1,\omega_2} = -\|x\|^2 \text{Var}\, (Y_1^{\omega_1}).$$

(ii) We have

$$\begin{aligned}\frac{d}{d\beta_2} \ln p_{\omega_1,\omega_2} &= y_2 x' - x' e^{x'\beta_2 + \eta y_1} - x' e^{x'\beta_2 + \eta y_1} a_2 = x' \left(y_2 - e^{x'\beta_2 + \eta y_1} - x e^{x'\beta_2 + \eta y_1} a_2 \right) \\ &= x' \left(y_2 - e^{x'\beta_2 + \eta y_1} (1 + a_2) \right) = x' (y_2 - \mu_2).\end{aligned}$$

Besides,

$$\begin{aligned}
\frac{\partial^2}{\partial\beta_2\partial\beta_2'}\ln p_{\omega_1,\omega_2} &= -x'\frac{\partial}{\partial\beta_2'}\left[e^{\beta_2' x+\eta y_1}\left(1+a_2\right)\right] = -x'\left[xe^{\beta_2' x+\eta y_1}\left(1+a_2\right)+e^{\beta_2' x+\eta y_1}\frac{\partial}{\partial\beta_2'}\left(1+a_2\right)\right]\\
&= -x'\left[x\mu_2+e^{\beta_2' x+\eta y_1}\frac{\partial a_2}{\partial\beta_2'}\right] = -x'\left[x\mu_2+e^{\beta_2' x+\eta y_1}\frac{\partial\delta_2}{\partial\beta_2'}\times\frac{\partial a_2}{\partial\delta_2}\right]\\
&= -x'\left[x\mu_2+xe^{2\beta_2' x+2\eta y_1}\frac{\partial^2}{\partial\delta_2^2}\ln\mathbb{E}_{\delta_2}\left[\omega_2\left(Y_2\right)\right]\right] = -x'x\left(\mu_2+\delta_2^2\frac{\partial^2}{\partial\delta_2^2}\ln\mathbb{E}_{\delta_2}\left[\omega_2\left(Y_2\right)\right]\right).
\end{aligned}$$

Let us notice that $\mu_2+\delta_2^2\frac{\partial^2}{\partial\delta_2^2}\ln\mathbb{E}_{\delta_2}\left[\omega_2\left(Y_2\right)\right] = \mathrm{Var}\left(Y_2^{\omega_2}|Y_1^{\omega_1}=y_1\right)$. Thus get

$$\frac{\partial^2}{\partial\beta_2\partial\beta_2'}\ln p_{\omega_1,\omega_2} = -x'x\,\mathrm{Var}\left(Y_2^{\omega_2}|Y_1^{\omega_1}=y_1\right).$$

We therefore have

$$\frac{\partial^2}{\partial\beta_2\partial\beta_2'}\ln p_{\omega_1,\omega_2} = -\|x\|^2\,\mathrm{Var}\left(Y_2^{\omega_2}|Y_1^{\omega_1}=y_1\right).$$

(iii) We have

$$\begin{aligned}
\frac{d}{d\eta}\ln p_{\omega_1,\omega_2} &= y_1y_2-y_1e^{x'\beta_2+\eta y_1}-y_1e^{x'\beta_2+\eta y_1}a_2 = y_1\left(y_2-e^{x'\beta_2+\eta y_1}-e^{x'\beta_2+\eta y_1}a_2\right)\\
&= y_1\left(y_2-e^{x'\beta_2+\eta y_1}\left[1+a_2\right]\right) = y_1\left(y_2-\mu_2\right).
\end{aligned}$$

Hence

$$\frac{d^2}{d\eta d\eta}\ln p_{\omega_1,\omega_2} = -y_1^2\mu_2.$$

The points (iv) and (v) are demonstrated in the same way as the previous points. □

4. Application to the Aleurodicus data

The *aleurode* (or *Aleurodicus Russel dispersus*, of the order of the *Homopteras*, of the family of the *aleyrodidae*) is a devastating parasite of the plants that it infests (suction of the sap, decrease of the photosynthesis activity, drying up of the leaves...).

We are going to study the adjustment of the data *Aleurodicus* collected in Republic of Congo (see, e.g., Mizère (2006) & Mizère (2007)) to the probabilistic models studied to the section (2.2). This whitefly becomes adult when some wings push him on the back. The experimental raisings of parasites have been achieved on several plants hosts among which (*dacryodes edulis*), the citrus fruit (*citrus paradisi*) and the hura (*hura crepita*). The statistical units (*a statistical unit is one bug*) are described by the variables:

- DDPR : lasted of development preimaginal measured (of the egg to the adult) in number of days.
- LONG : longevity of the adult bug measured in number of days.
- NJPO : number of punter days.

In this work, the variable DDPR corresponds in the variable $Y_1^{\omega_1}$, the LONG variable corresponds in the variable $Y_2^{\omega_2}$ and the variable NJPO to the factor x. We have the following tables (see Mizère (2006) & Mizère (2007)).

Table 1. Distribution of DDPR

DDPR	20	21	22	23	24	25	26	27
Eff. observed	2	10	17	25	8	11	5	4

Table 2. Elementary statistics

Variable	Mean	Variance
DDPR	23.2195	2.8648

Table 3. Distribution of LONG

LONG	1	2	3	4	5	6	9
Eff. observed	29	16	22	8	2	4	1

The observations of the variable DDPR are very big and belong to the set $\{k \in \mathbb{N} : 20 \leqslant k \leqslant 27\}$. The sample mean is higher than the sample variance (Table 2). We have the presumption that these data are underdispersed. They will be adjusted that by left censored models. Thus, we propose the translated Poisson distribution as model of the variable DDPR (See Mizère (2007)), of mass function:

Table 4. Elementary statistics

Variable	Mean	Variance
LONG	2.4634	2.4246

Table 5. Distribution of NJPO

NJPO	0	1	3
Eff. observed	48	33	1

$P\left[Y^{\omega}=y\right]=\frac{\delta_1^{y-\phi}}{(y-\phi)!}e^{-\delta_1}, y=\phi,\phi+1,...; \delta_1\in\mathbb{R}_+^*, \phi\in\mathbb{N}^*$. The estimators are contained in the Table 6.

Table 6. Estimate of ϕ et δ_1

$\hat{\phi}$	20
$\hat{\delta_1}$	3.2195

Table 7. Test of adequacy of χ^2 of the variable DDPR

DDPR	Efficient observed	Efficient theoretical
20	2	3.2779
21	10	10.5533
22	17	16.9882
23	25	18.2313
24	8	14.6739
25	11	9.4486
26	5	5.0700
27	4	3.7569
	N	82
	χ^2	6.0490
	p-value	0.1955
	df	4

The relative data to the adjustment of the variable DDPR onto the probabilistic model are contained in the Table 7. It is evident from this table, that the level $\alpha = 5\%$ of significance, p-value is higher than α. Therefore the null hypothesis according to which the data follow the translated Poisson distribution cannot be rejected.

The observations of the LONG variable belong to the set $\{k \in \mathbb{N} : 1 \leqslant k \leqslant 9\}$. The sample mean is higher than sample variance (Table 4). We have the presumption that these data are underdispersed. Longevity can be only positive, it is for it these data can be adjusted that by laws underdispersed of support equal to the integer set $\mathbb{N} - \{1\}$. We propose that LONG follows the Poisson distribution truncated in zero (Mizère (2006) & Mizère (2007)) of parameter canonical δ_2 whose estimate is contained in the Table 4.

captionTest of adequacy of χ^2 of the variable LONG

LONG	Efficient observed	Efficient theoretical
1	29	22.6786
2	16	24.7973
3	22	18.0760
4	8	9.8823
5	2	4.3222
6	4	1.5753
7	0	0.4215
8	0	0.1345
9	1	0.0416
	$\hat{\delta}_2$	2.1868
	N	82
	χ^2	6.1231
	p-value	0.1058
	df	3

It is evident from this Table 4, that at level $\alpha = 5\%$ of significance, p-value is higher than α. Therefore the null hypothesis according to which the LONG variable follows the Poisson distribution truncated in zero cannot be rejected.
Let us consider the following hypothesis of test H_0 against H_1 :

(H_0) : $\beta_j = 0$ (The x_j factors does not have an effect on the response variable $Y_1^{\omega_1}$ (or $Y_2^{\omega_2}$)).
(H_1) : $\beta_j \neq 0$ (The x_j factors has an effect on the response variable $Y_1^{\omega_1}$ (or $Y_2^{\omega_2}$)).

Thus, we propose the following generalized linear models (See McCullagh & Nelder (1989)).

(1) $\ln \mu_1 = \beta_1 NJPO + c_1$ (of response variable DDPR).
(2) $\ln \mu_2 = \beta_2 NJPO + \eta DDPR + c_2$ (of response variable LONG).

Table 8. Coefficient of model (1)

Variable	$\hat{\beta}_j$	$S_{\hat{\beta}}$	$t_j = \frac{\hat{\beta}_{1j}}{S_{\hat{\beta}}}$	$P(>\lvert t_j\rvert)$
NJPO	-0.0348	0.1102	-0.3160	0.7520
Intercept	1.1843	0.0776	15.2720	$<$ 2e-16
		AIC = 317.8700		

It is evident from Table 8, that at the level $\alpha = 5\%$ of significance, p-value equal to 0.7520 is higher than α . Therefore the coefficient β_1 of estimate $\hat{\beta}_1 = -0.0348$ is significantly null: the variable NJPO doesn't have an effect on the variable DDPR answer. On the other hand, to the same level of significance, the constant c_1 of estimate $\hat{c}_1 = 1.1843$ are not null significantly.

Table 9. Coefficient of model (2)

Variable	$\hat{\beta}_j$	$S_{\hat{\beta}}$	$t_j = \frac{\hat{\beta}_{1j}}{S_{\hat{\beta}}}$	$P(>\lvert t_j\rvert)$
NJPO	0.2459	0.1605	1.5320	0.1250
DDPR	0.3652	0.0683	5.3430	0.0000
Intercept	-8.1689	1.5988	-5.1100	0.0000

It is evident from the Table 9, that at the level $\alpha = 5\%$ of significance, p-value equal to 0.0000 is smaller than α ; therefore the coefficient η of estimate $\hat{\eta} = 0.3652$ is not null significantly; what confirms the dependence between the LONG variables (that is $Y_2^{\omega_2}$) and DDPR (that is $Y_1^{\omega_1}$). It is also evident from this table that to the same level of significance, the coefficient β_2 of estimate $\hat{\beta}_2 = 0.2459$ is null significantly: the variable NJPO doesn't have an effect on the response variable LONG . Still to the same level of significance, the constant c_2 of estimate $\hat{c}_2 = -8.1689$ is not null significantly.

5. Conclusion

We established the structure of the covariance of a couple of random variables whose conjoined law is the *bwPd*. Some illustrative examples have been taken, about this structure of covariance, and a recent result (see Elion *et al.* (2016)) has been recovered.
Then, from the covariance that we established, we could study the over-/equi-/underdispersion, through some examples, in the bwPd while using the generalized dispersion index of Kokonendji & Puig (2018).

In the Aleurodicus data that we considered, the number of punter days does not have an effect on the preimaginal development stage duration (egg to adult) and the longevity of the bug. On the other hand the preimaginal development stage duration has an effect on the longevity. What is in agreement with the found results by Mizère (2006) & Mizère (2007).

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References

Aitken, A.C. (1994). *Statistical Mathematics*. Ed. Oliver and Boyd.

Berkhout, P. & Plug, E. (2004). A bivariate Poisson count data model using conditional probabilities, Statist. Neerlandica 58(3), 349-364.

Campbell, J.T. (1934). The Poisson correlation function. *Proc. Edinburgh Math. Soc.* 4, 18-26,

Elion, L.-L, Koukouatikissa Diafouka M., Bidounga, R., Louzayadio, C.G., Mizère, D., Makany, R. and Kissita, G. (2016). The bivariate weighted Poisson distribution: Statistical study of the arterial hypertension data according to the weight and blood sugar level, Far East Journal of Theorical Statistics. Vol 52, pp. 365-393.

Famoye, F. (2010). A new bivariate generalized Poisson distribution. *Statistica Neerlandica* Vol. 64, Vol. 1, pp. 112-124.

Fisher, R.A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *The Annals of Eugenics*. Vol. 6, pp 13-25.

Guldberg, A. (1934). On discontinuous frequency functions of 2 variables. *Skand. Aktuar.*, Vol. 17, pp. 89-117.

Holgate, P. (1964). Estimation for the bivariate Poisson distribution. *Biometrika*. Vol. 51, 241-245.

Kokonendji, C.C., Mizère, D. & Balakrishnan, N. (2008). Connections of the Poisson weight function to overdispersion and underdispersion J. Statist. Plann. Inférence 138, pp. 1287-1296.

Kokonendji, C.C. & Perez-Casany, M. (2012). A Note on Weighted Count Distributions. *Journal of Statistical Theory and Applications*. Vol. 11, 337-352.

Kokonendji, C.C. & Puig, P. (2018) Fisher dispersion index for multivariate count distributions: A review and a new proposal, J. Multivariate Anal. 165, 180-193.

McCullagh, P. & Nelder, T.A. (1989). *Generalized Linear Models*. 2nd ed. Chapman and Hall.

Mizère, D. (2006). *Contributions à la Modélisation et à l'Analyse Statistique des Données de Dénombrement*. Thèse pour obtenir le grade de Docteur d'Université de Pau et des Pays de l'Adour, soutenue le 26.01.2006, Pau.

Mizère, D. (2007). Modélisation Statistique des données *Aleurodicus* sur agrume par les lois de Poisson pondérées. *Afrika Statistika*. Vol. 2 (1), pp- 59-72.

Patil, G.P. & Rao, C.R. (1978). Weighted distributions and size-biased sampling with applications to wild-life populations and human families. Biometrics.

Rao, C.R. (1985). Weighted distributions arising out of methods of ascertainment: what population does a sample represent? In: Atkinson, A.C., Fienberg, S.E. (Eds.) : *A Celebration of Statistics: The ISI Centenary Volume*. Springer, New York, 543-569.