THE ASYMPTOTIC THEORY OF THE POVERTY INTENSITY
IN VIEW OF EXTREME VALUE THEORY FOR TWO SIMPLE
CASES.

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Abstract. Let $Y_1, Y_2, \ldots$ be independent observations of the income variable
of some given population, with underlying distribution $G$. Given a poverty
line $Z$, then for each $n \geq 1$, $q = q_n$ is the numbers of poor in some given the
population. The general form of poverty measures used by economists to
monitor the welfare evolution of this population is

$$P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^{q} c(n, q, j) d \left( \frac{Z - Y_{j,n}}{Z} \right).$$

This class includes the most popular poverty measures like the Sen, Shorrocks
and Greer-Foster-Thorbecke statistics. We give a complete asymptotic normal-
ity theory in the frame of extreme value theory. In this paper, the two simple
cases of the Pareto and exponential distributions are studied with the intensity
poverty. Simulations and applications to the senegalese data are driven.

Résumé

Soit $Y_1, Y_2, \ldots$ des observations indépendantes de la variable revenu d’une
population donnée. Étant donné un seuil de pauvreté $Z$, alors $q_n$ est pour
tout $n \geq 1$ le nombre de pauvres dans la population. La forme générale des
indicateurs de pauvreté est

$$P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^{q} c(n, q, j) d \left( \frac{Z - Y_{j,n}}{Z} \right).$$

Cette classe de mesures contient les indicateurs classiques tels que ceux de Sen,
de Shorrocks et de Greer-Foster-Thorbecke. Dans cet article, nous préparons
la théorie asymptotique complète de ces indicateurs dans le cadre des valeurs
extrêmes. Nous nous concentrerons ici sur l’intensité de la pauvreté pour deux
cas simples : La loi de Pareto et la loi exponentielle. Des simulations et des
applications sur les données du Sénégal sont présentées.

1. INTRODUCTION.

Poverty is measured in Economics by statistics of the general form

$$P_N = \frac{1}{a(Q)b(N)} \sum_{j=1}^{Q} c(N, Q, j) d \left( \frac{Z - Y_{1,N}}{Z} \right)$$

where $a$, $b$, $c$, $d$ are given functions, $Q$ is the number of poor in the studied
population $P$ of size $N$, $Z$ the poverty line and $Y_{1,N} \leq Y_{2,N} \leq \ldots \leq Y_{N,N}$ are

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the ordered incomes of the individuals of \( P \). The poverty line \( Z \) is defined by economic specialists or governmental authorities so that any individual or household with income (say yearly) less than \( Z \) is considered as poor. The economic welfare of the population \( P \) is monitored both by the measures (1.1) and other measures of the inequality of the income distribution. A very large class of poverty measures can be found in the literature. One may divide them into two classes. The first includes the non weighted ones, for which \( c(N, Q, j) \equiv 1 \). The most popular of them is the Foster-Greer-Thorbecke class which is, for \( \alpha \geq 0 \),

\[
FGT(\alpha) = \frac{1}{N} \sum_{j=1}^{Q} \left( \frac{Z - Y_{j,N}}{Z} \right)^{\alpha}.
\]

For \( \alpha = 0 \), (1.2) reduces to \( Q/N \), the headcount of the poor while for \( \alpha = 1 \) and \( \alpha = 2 \), it is respectively interpreted as the severity of poverty and the depth in poverty. The second class consists of the weighted measures. We mention here two of its famous members, the Sen(1976) measure

\[
P_{SE,N} = \frac{2}{N(Q+1)} \sum_{j=1}^{Q} (Q-j+1) \left( \frac{Z - Y_{j,N}}{Z} \right)
\]

and the Shorroks(1995) one

\[
P_{SH,N} = \frac{1}{N^2} \sum_{j=1}^{Q} (2N-2j+1) \left( \frac{Z - Y_{j,N}}{Z} \right).
\]

These discrete statistics have been widely prospected in the frame of poverty reduction. The interested readers are referred to [9], [10], [14], [18].

We now want to settle an asymptotic theory for the statistics (1.1). We suppose that we have independent and identically observations of the income \( Y_1, Y_2, \ldots \) with underlying distribution \( G \) with lower endpoint \( y_0 = \inf \{ x, G(x) > 0 \} \geq 0 \). Given a poverty line \( Z \), we have the (random) number of poor \( q_n = q \) in the sample of size \( n : Y_1, Y_2, \ldots, Y_n \). The sample measure of poverty becomes

\[
P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^{q} c(n,q,j) \ d \left( \frac{Z - Y_{j,n}}{Z} \right).
\]

Clearly, this depends on the \( q \) lower extreme values and suggests to use the extreme value theory to handle such statistics. If \( q/n \to 0 \), the study will only need the lower tail of \( G \). But for many poor countries \( q/n \) is greater than 50% so that the reasonable hypothesis must be \( q/n \to \xi = G(Z) \in ]0,1[ \). Thus, at most the lower tail and the center of \( G \) will be used. This is in conformity with the focalization hypothesis of poverty measure which says that any poverty statistics is strictly function of the poor income. In général, the law of such statistics are usually guided by the extremal domain of attraction of the sequence of minima \( Y_{1,n}, n \geq 1 \). However, observe that much of the theory of extreme value is set up by using the extremal domain of attraction of the maxima. We make a brief recall of our needs in extreme
value theory using maxima in Section 2. In Section 3, we state our results for the intensity poverty whose the asymptotic theory is completely described for the Pareto and the exponential cases.

2. Basics of Extreme Values Theory Using Maxima

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with $P(Y_1 \leq x) = G(x)$ and $G(0) = 0$. The estimation of the law of the maximum $Y_{n,n} = \max(Y_1, \ldots, Y_n)$ is one of the most important questions of the extreme value theory. Recall that $Y_{n,n}$ converges in type to some random variable $Z$ with non-degenerated distribution function $H$ if and only if there exists two sequences of real numbers $(a_n > 0, n \geq 1)$ and $(b_n, n \geq 1)$ such that

$$
\lim_{n \to \infty} P(Y_{n,n} \leq a_n x + b_n) = \lim_{n \to \infty} G^n(a_n x + b_n) = H(x), \quad \forall x \in \mathbb{R}.
$$

It is then said that $G$ is in the domain of attraction of $H$, which is denoted by $G \in D(H)$. It is well-known that $H$ is necessarily one of these three types:

- **(2.1)** $H(x) = \varphi_\alpha(x) = \exp(-x^{-\alpha}) \mathbb{I}_{(x \geq 0)}$; (type I)
- **(2.2)** $H(x) = \Lambda(x) = \exp(-e^{-x})$; (type II)
- **(2.3)** $H(x) = \psi_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}) - \mathbb{I}_{(x \leq 0)} + \mathbb{I}_{(x > 0)}$; (type III)

where $\mathbb{I}_A$ stands for the indicator function of the set $A$. It should be noticed that the function $H$ has to be seen as an equivalence:

$$
(2.4) \quad H_1 \mathcal{R} H_2 \iff (\exists (a, b) \in \mathbb{R}_+^* \times \mathbb{R}, \forall x \in \mathbb{R}, H_1(x) = H_2(ax + b))
$$

In this case, any of the function $H$ in (2.2), (2.3) and (2.4) may be represented as

$$
H(x) = H_{\gamma}(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \quad 1 + \gamma x > 0,
$$

where $(1 + \gamma x)^{-1/\gamma}$ is interpreted as $e^{-x}$ for $\gamma = 0$. The values $\gamma > 0, \gamma = 0$ and $\gamma < 0$ are respectively related to the types I, II et III.

The class of the whole domain of attraction (the set of all distribution functions attracted to some $H$) is denoted by $\mathcal{D} = \cup_{\alpha > 0} \cup_{\gamma > 0} D(\varphi_\alpha) \cup D(\psi_\gamma) \cup D(\Lambda)$.

The interested reader is referred to [5], [7] or [15] for a general introduction to extreme value theory. The characterization of these three domains received very great attention in the two past decades. The frame of extreme value theory has also been widely used to find the asymptotic normality of sums of extreme values as well as Hill type statistics (see [12], mainly and [13], [16], [6], [11]).
The following representations are then used for the generalized inverse of G defined by $G^{-}(s) = \inf\{ x, G(x) \geq s \}, 0 \leq s \leq 1$. First, we have for $G \in D(\varphi_{\gamma}), z_{0} = \sup \{ x, G(x) < 1 \} = +\infty$ and

$$G^{-}(1-u) = c(1+f(u))u^{1/\gamma}\exp(\int_{u}^{1} b(t)t^{-1}dt), \quad 0 < u < 1,$$

Next, for $G \in D(\psi_{\gamma}), z_{0} = \sup \{ x, G(x) < 1 \} < +\infty$ and

$$z_{0} - G^{-}(1-u) = c(1+f(u))u^{1/\gamma}\exp(\int_{u}^{1} b(t)t^{-1}dt), 0 < u < 1$$

and, finally, for $G \in D(\Lambda), z_{0} = \sup \{ x, G(x) < 1 \} \leq +\infty$.

$$G^{-}(1-u) = d - s(u) + \int_{u}^{1} s(t)t^{-1}dt, 0 < u < 1,$$

where $s(u) = c(1+f(u))\exp(\int_{u}^{1} b(t)t^{-1}dt), 0 < u < 1$. In each of these formulae, $f$ and $b$ are functions such that $(f(u), b(u)) \to (0,0)$ when $u \to 0$, while $c > 0$ and $d$ are constants. (2.6) and (2.7) are the Karamata representations while (2.8) is the one of de Haan.

To finish, we recall the characterization on $G$ for the different domains. First, by Theorem 2.4.3 in [7], $G \in D(\Lambda)$ iff

$$\frac{1-G(t+x)}{1-G(t)} \to e^{-x}, \text{ as } x \to z_{0}$$

for any $x$ and for some positive function $r(\cdot)$. Moreover, all the functions $r(\cdot)$ in (2.9) are equivalent between them as $x \to z_{0}$ Further, by Theorem 2.4.1 in [6], $G \in D(\Lambda)$ iff $z_{0} = +\infty$ and

$$\forall \lambda > 0, \frac{(1-G(\lambda x))/(1-G(x))}{(1-G(\lambda x))/(1-G(x))} \to \lambda^{-\gamma}.$$  

In this case, $G$ admits the Karamata representation

$$1-G(x) = c(1+f_{1}(x))x^{-\gamma}\exp(\int_{1}^{x} b_{1}(t)t^{-1}dt), \quad y_{0} < x < z_{0},$$

where $(f_{1}(u), b_{1}(u)) \to (0,0)$ as $u \to 0$. The function

$$L(x) = c(1+f_{1}(u))\exp(\int_{1}^{x} b_{1}(t)t^{-1}dt)$$

is the representation of a function slowly varying at infinity that is

$L(\lambda x)/L(x) \to 1$ as $x \uparrow \infty$

for any $\lambda > 1$. Finally $G \in D(\psi_{\gamma})$ iff $z_{0} < +\infty$ and $G(z_{0} - 1/\cdot) \in D(\varphi_{\gamma})$.

Now, we must remark in our frame that $Y$ is an income variable. Then its lower endpoint $y_{0}$ is not negative. This allows us to study (1.5) via the transform $X = 1/(Y-y_{0})$. Throughout this paper, the distribution function of
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X denoted by F, whose upper endpoint is then +∞. Its extremal domain is D(Λ) or D(φγ). Hence (1.5) is transformed as

\[ P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^{q} c(n, q, j) d \left( \frac{Z - y_0 - \frac{X_{n-1}}{n+1}}{Z} \right), \]

and studied following the extremal law of X using the general representations (2.6) and (2.8).

The aim of this paper is to prepare the general asymptotic theory of the poverty measures by giving the complete study of one represent of each of the possible extremal domain: The Pareto distribution of parameter γ (for the type I) and the exponential distribution (for the type II). Only the non weighted measures are also considered.

Throughout the paper, we use the representations of the studied random variables \( X_i, i \geq 1 \), by \( F \leftarrow (1-U_i) \), where \( U_1, U_2, ... \) is a sequence of independent random variables uniformly distributed on (0, 1).

It follows that:

\[ \{ X_{n-i+1}, n, 1 \leq i \leq n ; n \geq 1 \} \overset{d}{=} \{ F^{-1}(1-U_i), 1 \leq i \leq n ; n \geq 1 \} \]

where \( U_{1,n} < ... < U_{n,n} \) are the order statistics based on \( U_1, ..., U_n \).

Our best achievement is the complete asymptotic characterization of the poverty intensity measure

\[ I_n = \frac{1}{n} \sum_{j=1}^{q} \left( \frac{Z - Y_{j,n}}{Z} \right) \]

when the exponential distribution of parameter \( \lambda > 0 \),

\[ F(x) = \begin{cases} 1 - \exp(-\lambda x), & \lambda > 0, \text{ for } x > 0 \\ 0 & \text{else} \end{cases} \]

and when the Paréto distribution of parameter \( \gamma > 0 \), i.e.,

\[ F(x) = \begin{cases} 1 - x^{-1/\gamma}, & \text{for } x > 1 \\ 0 & \text{else} \end{cases} \]

are retained for \( F(\cdot) = 1 - G(y_0 + 1/\cdot) \). Now, our results are given.

3. RESULTS

**Theorem 1.** Let \( F = 1-G(y_0 + 1/\cdot) \) be the exponential distribution function of parameter \( \lambda > 0 \), then \( \sqrt{n}(I_n - C_n) \rightarrow N(0, \sigma^2) \) where

\[ C_n = \frac{1}{nZ} q(Z - y_0) + \frac{\lambda}{Z} \int_{\frac{q}{n}}^{q/n} \frac{1}{\log(s)} ds \]

and

\[ \left( \frac{Z}{\lambda} \right)^2 \sigma^2 = \frac{-2}{\log G(Z)} \int_0^{G(Z)} (\log s)^{-2} ds + 2 \int_0^{G(Z)} (\log s)^{-3} ds - (\int_0^{G(Z)} \frac{ds}{(\log s)^2})^2 \]

with \( q/n = G_n(Z) \) and \( G(Z) = \exp(-\lambda/(Z - y_0)) \) for \( Z > y_0 \).
Theorem 2. Let \( F = 1 - G(y_0 + 1/\cdot) \) be the Pareto distribution function of parameter \( \gamma^{-1} > 0 \), then \( \sqrt{n}(I_n - D_n) \to N(0, \theta^2) \) with

\[
D_n = \frac{1}{nZ} q(Z - y_0) - \frac{1}{Z(\gamma + 1)} ((q/n)^{\gamma+1} - n^{-\gamma-1})
\]

and

\[
\theta^2 = \frac{(\gamma/Z)^2 G(Z)^{2\gamma+1}}{(\gamma + 1)(2\gamma + 1)} - \frac{G(Z)}{(\gamma + 1)^2}
\]

with \( G(Z) = ((Z - y_0)\wedge 1) \) for \( Z > y_0 \).

4. PROOFS

In the beginning we use a special representation of

\[
I_n = \frac{1}{n} \sum_{j=1}^{q} \left( \frac{Z - Y_{j,n}}{Z} \right)
\]

By using the transform \( X = 1/(Y - y_0) \), we get

\[
I_n = \frac{1}{nZ} \sum_{j=1}^{q} (Z - y_0) - \frac{1}{nZ} \sum_{j=1}^{q} Z \cdot X_{n-j+1,n}
\]

\[
I_n = \frac{1}{nZ} q(Z - y_0) - \frac{1}{nZ} \sum_{j=1}^{q} \frac{X_{n-j+1,n}}{X_{n-j,n}}
\]

\[
= \frac{1}{nZ} q(Z - y_0) - \frac{1}{nZ} \sum_{j=1}^{q} n \int_{Z_j^{-1}}^{Z_{j+1}^{-1}} \frac{1}{Q_n(s)} ds,
\]

where \( Q_n \) is the empirical quantile function based on \( X_1, X_2, \ldots, X_n \) defined by \( Q_n(s) = X_{j,n} \) for \( \frac{j-1}{n} < s \leq \frac{j}{n} \) for \( j = 1, \ldots, n \) and \( Q_n(0) = X_{1,n} \). Furthermore, we get by (2.14) the following representation of \( Q_n \) via the empirical quantile function \( V_n \) based on \( U_1, U_2, \ldots, U_n \):

\[
\{Q_n(t), 0 \leq t \leq 1\} = d \{F^-(V_n(t)), 0 \leq t \leq 1\}
\]

\[
= d \{F^-(1 - V_n(t)), 0 \leq t \leq 1\}
\]

where \( F^-(s) = \inf\{x, F(x) \geq s\}, 0 \leq s \leq 1 \), is the quantile function of \( F \). Recall that \( G_n(Z) = q/n \) when \( G_n \) denotes the empirical distribution function based on \( Y_1, Y_2, \ldots, Y_n \). Thus, the final representation of \( I_n \) is

\[
I_n = \frac{1}{nZ} q(Z - y_0) - \frac{1}{nZ F^-(1 - U_{1,n})} - \frac{1}{Z} \int_{\frac{1}{n}}^{G_n(Z)} F^-(1 - V_n(s)) ds.
\]
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Now we are ready to prove the two theorems.

Proof of theorem 1.
We have $F^{--}(s) = -\log(1-s)/\lambda$, $0 \leq s < 1$ and from (4.7),

\[ I_n = \frac{1}{nZ} q(Z - x_0) + \frac{\lambda}{nZ \log(U_{1,n})} + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{1}{\log(V_n(s))} ds. \]

Put

\[ C_n = \frac{1}{nZ} q(Z - x_0) + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{1}{\log(s)} ds. \]

By the mean value theorem

\[ I_n - C_n = \frac{\lambda}{nZ \log(U_{1,n})} - \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{V_n(s) - s}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds, \]

where $\zeta_n(s) \in [\min(s, V_n(s)); \max(s, V_n(s))]$. Thus

\[ \sqrt{n} (I_n - C_n) = \frac{\lambda}{\sqrt{nZ \log(U_{1,n})}} - \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \sqrt{n}(V_n(s) - s) \frac{\sqrt{n}(V_n(s) - s)}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds. \]

At this step, we need one version of the so-called hungarian construction. Precisely, Csörgő and al. (see [2]) have constructed a probability space holding a sequence of independent uniformly distributed random variables $U_1, U_2, \ldots$ and a sequence of brownian bridges $B_1, B_2, \ldots$ such that for each $0 < \nu < 1/2$, as $n \to \infty$,

\[ \sup_{1/n \leq s \leq 1-1/n} \frac{1}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}). \]

where \( \{\beta_n(s) = \sqrt{n}(s - V_n(s)), 0 \leq s \leq 1\} \) is the uniform quantile process. From this and (4.11), we get

\[ \sqrt{n} (I_n - C_n) = \frac{\lambda}{\sqrt{nZ \log(U_{1,n})}} + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{\beta_n(s) - B_n(s)}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{B_n(s)}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds \]

\[ = \frac{\lambda}{\sqrt{nZ \log(U_{1,n})}} + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{\beta_n(s) - B_n(s)}{s^{1/2-\nu}} \frac{1}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{B_n(s)}{s^{1/2-\nu}} \frac{1}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds \]

\[ + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{B_n(s)}{s^{1/2-\nu}} \frac{1}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds \]

(4.13)

We first claim that each of the $R_j, n$ tends to zero in probability as $n \to \infty$. 

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Claim 1.1. $R_{0,n} \rightarrow_p 0$. We have

\begin{equation}
R_{0,n} = \frac{\lambda}{\sqrt{nZ \log(U_{1,n})}} = \frac{\lambda}{\sqrt{nZ(\log(nU_{1,n}) - \log n)}}
\end{equation}

But it is well-known that $\log(nU_{1,n}) = O_p(1)$ so that $R_{0,n} = o_p(1)$.

Claim 2.1. $R_{1,n} \rightarrow_p 0$. We have

\begin{equation}
R_{1,n} = \frac{\lambda}{Z} \int_{1/2}^{G_n(Z)} \frac{\beta_n(s) - B_n(s)}{s^2 - \nu} \frac{s^\frac{1}{2} - \nu}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds
\end{equation}

\begin{equation}
= O_p(n^{-\nu}) \int_{1/2}^{G_n(Z)} \frac{s^\frac{1}{2} - \nu}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds =: O_p(n^{-\nu}) \times R'_{1,n}.
\end{equation}

Now, by Lemma 13 in [3],

\[ \lim_{n \to \infty} \inf \sup_{n \to \infty} \mathbb{P}(\rho^{-1} \leq \inf \{V_n(s)/s, 1/n \leq s \leq 1\} \leq \sup \{V_n(s)/s, 1/n \leq s \leq 1\} \leq \rho) = 1. \]

Recall again that $G_n(Z) \rightarrow G(Z) > 0$. Hence, for any $G(Z) > \varepsilon > 0$, for large values of $n$ and for large values of $\rho$, we have with probability at least greater than $1 - \varepsilon/3$ (denoted w.p. $\varepsilon/3$),

\begin{equation}
(\zeta_n(s)/s)^{-1} \leq \rho
\end{equation}

and we have w.p. $\varepsilon/3$ for large $n$.

\begin{equation}
1/n \leq s \leq G_n(Z) \leq G(Z) + \varepsilon/2 \implies 0 \leq \zeta_n(s) \leq \max(V_n(s), s)
\end{equation}

\begin{equation}
\leq \max(V_n(G(Z) + \varepsilon/2), G(Z) + \varepsilon/2)
\end{equation}

and since $V_n(G(Z) + \varepsilon/2) \rightarrow_p G(Z) + \varepsilon/2$, we also get w.p. $\varepsilon/3$ for large $n$.

\begin{equation}
0 \leq V_n(G(Z) + \varepsilon/2) \leq G(Z) + \varepsilon,
\end{equation}

and then w.p. $2\varepsilon/3$, for large $n$

\begin{equation}
1/n \leq s \leq G_n(Z) \leq G(Z) + \varepsilon \implies 0 \leq \zeta_n(s) \leq G(Z) + \varepsilon.
\end{equation}

This latter yields w.p. $2\varepsilon/3$, for large $n$

\begin{equation}
1/n \leq s \leq G_n(Z) \leq G(Z) + \varepsilon \implies 0 \leq 1/\log \zeta_n(s) \leq -1/\log(G(Z) + \varepsilon)
\end{equation}

By (4.17), (4.21), we have w.p. $\varepsilon$, for large values of $n$ and $\rho$,

\begin{equation}
|R'_{1,n}| \leq \int_{1/2}^{G(Z) + \varepsilon + \varepsilon} \frac{s^\frac{1}{2} - \nu}{(\zeta_n(s)/s)(\log(\zeta_n(s)))^2} ds \leq \frac{\rho}{(\log(G(Z) + \varepsilon))^2} (G(Z) + \varepsilon)^{1/2 - \nu}.
\end{equation}

Finally $R'_{1,n} = O_p(1)$ and then by (4.16), $R_{1,n} \rightarrow_p 0$.

Claim 3.1. $R_{2,n} \rightarrow_p 0$. We have

\begin{equation}
R_{2,n} = \frac{\lambda}{Z} \int_{1/2}^{G_n(Z)} B_n(s) \left( \frac{1}{\zeta_n(s)(\log(\zeta_n(s)))^2} - \frac{1}{s(\log s)^2} \right) ds
\end{equation}
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\begin{align}
(4.25) \quad &= \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s) \frac{(\zeta_n(s) - s)(-2 - \log(\pi_n(s)))}{\pi_n(s)^2(\log \pi_n(s))^3} \, ds.
\end{align}

where \( \pi_n(s) \in [\zeta_n(s) \wedge s, \zeta_n(s) \vee s] \). By (5) and (6) in [17], p.500, we have for any \( 0 < c < \infty \),

\begin{align}
(4.26) \quad &\sup_{c/(n+1) < s < 1-c/(n+1)} |B_n(s)/\sqrt{s(1-s)}| = O_p(b_n)
\end{align}

and

\begin{align}
(4.27) \quad &\sup_{c/(n+1) < s < 1-c/(n+1)} |\beta_n(s)/\sqrt{s(1-s)}| = O_p(b_n)
\end{align}

with \( b_n = \sqrt{2 \log \log n} \). From (4.25) and since \( \pi_n(s) \in [V_n(s) \wedge s, V_n(s) \vee s] \)

\begin{align}
|R_{2,n}| &\leq \frac{\lambda}{\sqrt{2 \log \log n}} \int_{\min(\frac{1}{n}, G_n(Z))}^{\max(\frac{1}{n}, G_n(Z))} s(1-s) \left| \frac{B_n(s)}{\sqrt{s(1-s)}} \right| \left| \frac{\beta_n(s)}{\sqrt{s(1-s)}} \right| \frac{2 + \log(\pi_n(s))}{\pi_n(s)^2 \log(\pi_n(s))^3} \, ds.
\end{align}

By the same methods that led to (4.17), we have, for any \( \varepsilon > 0 \) such that \( G(Z) + \varepsilon < 1 \), w.p. \( \varepsilon \) for large values of \( n \) and \( \rho > 0 \),

\begin{align}
|R_{2,n}| &\leq O_p(n^{-1/2} \log \log n) \rho^2 \frac{1 + 2/\log(G(Z) + \varepsilon)}{(\log(G(Z) + \varepsilon))^2} \int_{\frac{1}{n}}^{G(Z) + \varepsilon} s^{-1} \, ds.
\end{align}

which finishes the proof of the claim.

Claim 4.1. \( R_{3,n} \rightarrow_p 0 \). Since \( G_n(Z) \rightarrow_p G(Z) \) as \( n \rightarrow \infty \),

\begin{align}
sup \{s(\log s)^2, s \in [\min(G_n(Z), G(Z)), \max(G_n(Z), G(Z))]\} \rightarrow_p G(Z)(\log(G(Z))^2
\end{align}

and

\begin{align}
inf \{s(\log s)^2, s \in [\min(G_n(Z), G(Z)), \max(G_n(Z), G(Z))]\} \rightarrow_p G(Z)(\log(G(Z))^2
\end{align}

and then

\begin{align}
(4.30) \quad R_{3,n} = \frac{\lambda}{(Z \log Z)^2} (1 + o_p(1)) \int_{G(Z)}^{G_n(Z)} B_n(s) \, ds
\end{align}

But for any \( \varepsilon > 0 \), we have w.p. \( \varepsilon \), for large values of \( n \)

\begin{align}
(4.31) \quad \left| \int_{G(Z)}^{G_n(Z)} B_n(s) \, ds \right| \leq \int_{G(Z) - \varepsilon}^{G(Z) + \varepsilon} |B_n(s)| \, ds =: T_n(\varepsilon)
\end{align}
Now, as \( n \to \infty \), we conclude that for any \( \varepsilon \), we get for large values of \( n \), by Markov inequality

\[
\forall \eta > 0, \mathbb{P}\left( \int_{G(Z)}^{G(Z)+s} B_n(s) ds \geq \eta \right) \leq \frac{E(T(\varepsilon))/\eta}{\eta}
\]

that is

\[
\forall \eta > 0, \lim_{n \to \infty} \sup \mathbb{P}\left( \int_{G(Z)}^{G(Z)+s} B_n(s) ds \geq \eta \right) \leq \frac{E(T(\varepsilon))/\eta}{\eta}
\]

Now, as \( \varepsilon \to 0 \), we conclude that \( \int_{G(Z)}^{G(Z)+s} B_n(s) ds \to_p 0 \) and thus \( R_{3,n} \to_p 0 \) by (4.30) et (4.31).

Finally we achieve our proof by remarking that \( N_n \) is a Riemannian integral of a Brownian bridge and hence, is a Gaussian and centered random variable with variance

\[
\sigma_n^2 = \left( \frac{\lambda^2}{2^2} \right) \int_{G(Z)}^{G(Z)+s} \frac{1}{s(log(s)^2)} \int_{G(Z)}^{G(Z)} \min(s,t) - st \frac{1}{s(log(s)^2)t(log(t)^2)} dsdt
\]

\[
Z^2 \frac{\lambda^2}{2^2} \sigma_n^2 = \int_{G(Z)}^{G(Z)+s} \frac{1}{s(log(s)^2)} \int_{G(Z)}^{G(Z)} \frac{1}{s(log(s)^2)t(log(t)^2)} dsdt + \int_{G(Z)}^{G(Z)} \frac{1}{s(log(s)^2)} \int_{G(Z)}^{G(Z)} \frac{1}{t(log(t)^2)} ds dt
\]

\[
\int_{G(Z)}^{G(Z)+s} \frac{1}{s(log(s)^2)} \int_{G(Z)}^{G(Z)} \frac{1}{t(log(t)^2)} ds dt
\]

Now by integrating by parts the first integral with \( s^{-1}(log(s)^2)^{-2} = (-\int_s^{G(Z)} \frac{dt}{t(log(t)^2)})^2 \), we get

\[
Z^2 \frac{\lambda^2}{2^2} \sigma_n^2 = 2 \int_{G(Z)}^{G(Z)} \frac{1}{(log(s)^2)(log(s)^2)} \int_{G(Z)}^{G(Z)} \frac{1}{t(log(t)^2)} ds dt - \left( \int_{G(Z)}^{G(Z)} \frac{dt}{t(log(t)^2)^2} \right)^2
\]

Noting that \(-1/(log(s)^2)^2\) is the derivative of \( 1/(log(s)^2) \), we conclude that

\[
Z^2 \frac{\lambda^2}{2^2} \sigma_n^2 = - \frac{2}{\log G_n(Z)} \int_{1/n}^{G(Z)} (log(s))^{-2} ds + \int_{1/n}^{G(Z)} (log(s))^{-3} ds - \left( \int_{1/n}^{G(Z)} \frac{ds}{(log(s)^2)^2} \right)^2
\]

\[
Z^2 \frac{\lambda^2}{2^2} \sigma_n^2 = - \frac{2}{\log G(Z)} \int_{0}^{G(Z)} (log(s))^{-2} ds + \int_{0}^{G(Z)} (log(s))^{-3} ds - \left( \int_{0}^{G(Z)} \frac{ds}{(log(s)^2)^2} \right)^2,
\]

with \( G(Z) = exp(-\lambda/(Z - y_0)) \) for \( Z > y_0 \).
The Asymptotic Theory of the Poverty Intensity

Proof of theorem 2. We have for for \( \gamma > 0 \) and \( x \geq 1 \), \( F(x) = 1 - x^{-\gamma} \), that is \( F^-(s) = (1 - s)^{-\gamma} \) for \( 0 \leq s < 1 \). Thus by (4.7)

\[
I_n = \frac{1}{nZ} q(Z - y_0) - \frac{1}{nZ} U_{1,n}^{\gamma} - \int_{\frac{1}{n}}^{G_n(Z)} V_n(s)^\gamma \, ds.
\]

Put

\[
D_n = \frac{1}{nZ} q(Z - y_0) - \frac{1}{Z} \int_{\frac{1}{n}}^{G_n(Z)} s^\gamma \, ds
\]

By the mean value theorem, we get

\[
I_n - D_n = \frac{1}{nZ} U_{1,n}^{\gamma} - \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \beta_n(s)(\zeta_n(s))^{\gamma-1} \, ds.
\]

where \( \zeta_n(s) \in [\min(s, V_n(s)), \max(s, V_n(s))] \) and

\[
\sqrt{n} (I_n - D_n) = \frac{1}{\sqrt{nZ}} U_{1,n}^{\gamma} - \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \beta_n(s)(\zeta_n(s))^{\gamma-1} \, ds.
\]

\[
- \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s)(s)^{\gamma-1} \, ds - \frac{1}{Z} \int_{\frac{1}{n}}^{G(Z)} B_n(s)(s)^{\gamma-1} \, ds = R_{0,n} + R_{1,n} + R_{2,n} + R_{3,n} + N_n
\]

We first claim that each of the \( R_{j,n} \) tends to zero in probability as \( n \to \infty \).

Claim 1.2. \( R_{0,n} \to_p 0 \). This immediate since \( \gamma > 0 \) and \( U_{1,n} \to_p 0 \) as \( n \to \infty \).

Claim 2.2. \( R_{1,n} \to_p 0 \). Put \( 0 < \nu < 1/2 \) and \( \nu + 1/2 < 1 + \gamma > 1 \) so that \( \delta = \frac{1}{2} + \nu - \gamma < 1 \). Then by (4.12),

\[
R_{1,n} = O_p(n^{-\nu}) \int_{\frac{1}{n}}^{G_n(Z)} s^{\frac{1}{2} - \nu} \zeta_n(s)^{-1+\gamma} \, ds
\]

\[
= O_p(n^{-\nu}) \int_{\frac{1}{n}}^{G(Z)} s^{-(\frac{1}{2} + \nu - \gamma)} (\zeta_n(s)/s)^{-1+\gamma} \, ds = O_p(n^{-\nu}) R_{1,n}.
\]

We may use the same methods used in (4.23) to have for any \( \varepsilon > 0 \), for large values of \( n \) and \( \rho \),

\[
0 \leq R_{1,n} \leq \max(\rho^{-1+\gamma}, \rho^{1-\gamma}) \int_{\frac{1}{n}}^{G(Z)+\varepsilon} s^{-(\frac{1}{2} + \nu - \gamma)} \, ds
\]

\[
0 \leq R_{1,n} \leq \max(\rho^{-1+\gamma}, \rho^{1-\gamma}) \int_{\frac{1}{n}}^{G(Z)+\varepsilon} s^{-\delta} \, ds
\]
\[ (4.44) \quad \max(\rho^{-1+\gamma}, \rho^{1-\gamma}) (G(Z) + \varepsilon)^{1-\delta}. \]

The formulae (4.43) and (4.44) together achieve the proof of Claim 2.2.

Claim 3.2 \( R_{2,n} \to_p 0 \)

\[ (4.45) \quad R_{2,n} = -\frac{\gamma}{Z} \int Z B_n(s) (\zeta_n(s)^{\gamma-1} - s^{\gamma-1}) \, ds \]

\[ (4.46) \quad R_{2,n} = -\frac{\gamma(\gamma - 1)}{Z\sqrt{n}} \int Z B_n(s) \sqrt{n}(\zeta_n(s) - s) \pi_n(s)^{\gamma-2} ds \]

where \( \pi_n(\cdot) \) is defined above, so that

\[ (4.47) \quad |R_{2,n}| \leq \frac{\gamma |\gamma - 1|}{Z\sqrt{n}} \int Z \max(G_n(Z), 1/n) B_n(s) |\sqrt{n}(V_n(s) - s)| \pi_n(s)^{\gamma-2} ds. \]

By using (4.26) and (4.27), we get

\[ (4.48) \quad |R_{2,n}| \leq O_p(n^{-1/2} \log \log n) \int Z \max(G_n(Z), 1/n) s^{\gamma-1}(1-s) (\pi_n(s)/s)^{\gamma-2} ds. \]

As in what precedes, we have for any \( \varepsilon > 0 \), for large values of \( n \) and \( \rho > 0 \),

\[ (4.49) \quad |R_{2,n}| \leq O_p(n^{-1/2} \log \log n) \max(\rho^{-2+\gamma}, \rho^{2-\gamma}) \int Z^{\gamma+\varepsilon} s^{\gamma-1} ds. \]

Hence

\[ (4.50) \quad |R_{2,n}| \leq O_p(n^{-1/2} \log \log n) (G(Z) + \varepsilon)^{\gamma}. \]

This achieves the proof of Claim 3.2.

Claim 4.2. \( R_{3,n} \to_p 0 \). The proof is the same as that of Claim 4.1 with \((-1/(s \log s)^2)\) replaced by \((s^{\gamma-2})\).

It remains to remark as below that \( N_n \) is a centered Gaussian random variable with variance

\[ (4.51) \quad \theta_n^2 = \left(\frac{\gamma}{Z}\right)^2 \int Z \min(s,t) - st \, t^{\gamma-1}s^{\gamma-1} dt \, ds. \]

A direct calculation yields \( \theta_n^2 \to \theta^2 = (\gamma/Z)^2 G(Z)^{2\gamma+1} \left( \frac{2}{(\gamma+1)2^{2\gamma+1}} - \frac{G(Z)}{(\gamma+1)^2} \right) \) with \( G(Z) = ((Z - \gamma_0)^{1/2} \land 1). \)
5. Simulations of the results

We will simulate our results in the conditions of the senegalease data base ESAM (1996) related to a national survey with 3278 households. The income variable is available for ten (10) districts with a minimal size of n=172. The poor headcounts in the different areas vary from 35% to 81%. As a consequence, we make our simulations for the current estimations of γ (wich varies around 1,5) and λ (wich varies around 25000), with n=50 and for several values of the headcounts (from 35% to 85%). For each case, we will use B=100 replications and report I_n the mean value of the empirical poverty intensity, D or C the centering coefficient, E the mean quadratic error, ET the variance of the empirical intensity and the mean P-value for the normality test.

The two following tables show that convergence is very fast even for a small size since the mean quadratic error is very small. The variances will permit to have the confidence intervals of the exact poverty intensity. The high values of the p-values P are very conclusive for the asymptotic normality.

Here are our results for the Pareto law:

<table>
<thead>
<tr>
<th>Q/n(%)</th>
<th>35</th>
<th>45</th>
<th>55</th>
<th>75</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td>γ =0,8</td>
<td>I(%)</td>
<td>14,989</td>
<td>19,59</td>
<td>24,48</td>
<td>31,38</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>15,29</td>
<td>19,79</td>
<td>24,10</td>
<td>31,14</td>
</tr>
<tr>
<td></td>
<td>E(%)</td>
<td>1,7</td>
<td>2,22</td>
<td>2,46</td>
<td>2,11</td>
</tr>
<tr>
<td></td>
<td>ET</td>
<td>0,266</td>
<td>0,29</td>
<td>0,3</td>
<td>0,31</td>
</tr>
<tr>
<td></td>
<td>P(%)</td>
<td>26,38</td>
<td>20,49</td>
<td>24,94</td>
<td>28,25</td>
</tr>
</tbody>
</table>

| γ =2  | I(%)| 22,93 | 29,35 | 36,74 | 46,53 | 55,91 |
|       | C   | 22,95 | 29,79 | 36,28 | 46,92 | 55,93 |
|       | E(%)| 3,35 | 2,97  | 3,18  | 3,37  | 3,3   |
|       | ET  | 0,36 | 0,89  | 0,4   | 0,4   | 0,36  |
|       | P(%)| 23,89 | 24,62 | 23,5  | 23,88 | 20,84 |

Here are our results for the exponential law:

<table>
<thead>
<tr>
<th>Q/n(%)</th>
<th>35</th>
<th>45</th>
<th>55</th>
<th>75</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ =2</td>
<td>I(%)</td>
<td>13,46</td>
<td>19,8</td>
<td>27,38</td>
<td>42,34</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>13,6</td>
<td>19,73</td>
<td>27,55</td>
<td>42,4</td>
</tr>
<tr>
<td></td>
<td>E(%)</td>
<td>0,04</td>
<td>0,038</td>
<td>0,07</td>
<td>0,08</td>
</tr>
<tr>
<td></td>
<td>ET</td>
<td>0,476</td>
<td>0,447</td>
<td>0,55</td>
<td>0,579</td>
</tr>
<tr>
<td></td>
<td>P(%)</td>
<td>25</td>
<td>25,18</td>
<td>24,26</td>
<td>22,88</td>
</tr>
</tbody>
</table>

| λ =25000 | I(%)| 13,70 | 19,8  | 27,39 | 42,57 | 61,83 |
|          | C   | 13,6 | 19,73 | 27,4  | 42,48 | 61,18 |
|          | E(%)| 0,04 | 0,03  | 0,06  | 0,07  | 0,07  |
|          | ET  | 0,476 | 0,517 | 0,55  | 0,574 | 0,563 |
|          | P(%)| 25  | 25,93 | 25,51 | 23,68 | 24,73 |

6. Conclusion

We have been able to get the asymptotic normality of the poverty intensity for two special representatives of the two extremal domains. The results are very well supported by the simulations. This opens the way for a general
study of this poverty measure in the frame of extreme value theory first and further for the nonweighted measures. Immediate steps after this study will consist of deep comments of the results as well as simulations from the 1995 senegalese data.

REFERENCES

THE ASYMPTOTIC THEORY OF THE POVERTY INTENSITY

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