AN APPROXIMATION FOR THE POWER FUNCTION OF A SEMI-PARAMETRIC TEST OF FIT

MOHAMMED BOUKILI MAKHOUKHI

Abstract. We consider in this paper goodness of fit tests of the null hypothesis that the underlying d.f. of a sample $F(x)$, belongs to a given family of distribution functions $\mathcal{F}$. We propose a method for deriving approximate values of the power of a weighted Cramér-von Mises type test of goodness of fit. Our method relies on Karhunen-Loève [K.L] expansions on $(0,1)$ for the weighted a Brownian bridges.

1. Introduction

In this paper we investigate semi-parametric tests of fit based upon a random sample $X_1, X_2, \ldots, X_n$ with common continuous distribution function $F(x) = \mathbb{P}(X_1 \leq x)$. Here $\mathcal{F} = \{G(., \theta) : \theta \in \Theta\}$ denotes a family of all distribution function which will be specified later on, and $\Theta$ is some open set in $\mathbb{R}^k$. We seek to test the hypothesis

$$H_0 : F(\cdot) = G(\cdot, \theta) \in \mathcal{F},$$

against an alternative which will be specified later on. We will make use of the Cramér-von Mises type statistics of the form

$$\hat{W}_{n, \phi}^2 := n \int_{-\infty}^{\infty} \phi(G(x, \hat{\theta}_n)) [F_n(x) - G(x, \hat{\theta}_n)]^2 dG(x, \hat{\theta}_n),$$

with $F_n(x) = n^{-1} \sum_{i=1}^{n} \mathbb{I}_{\{X_i \leq x\}}$ denotes the usual empirical distribution function [d.f.] and $\hat{\theta}_n$ is a sequence of estimators of $\theta$ and $\phi$ is a positive and continuous function on $(0,1)$, fulfilling

$$(1.1) \quad (i) \quad \lim_{t \to 0} t^2 \phi(t) = \lim_{t \to 1} (1-t)^2 \phi(t) = 0 \quad (ii) \quad \int_{0}^{1} t(1-t) \phi(t) < \infty.$$ 

Note that, setting $Z_i = G(X_i, \hat{\theta}_n)$ for $i = 1, \ldots, n$ and letting $\hat{G}_n(t)$ denotes the empirical d.f. based upon $Z_1, \ldots, Z_n$ then, we may write, under $(H_0)$,

$$(1.2) \quad \hat{W}_{n, \phi}^2 = n \int_{0}^{1} \phi(t)(\hat{G}_n(t) - t)^2 dt,$$

with $Z_1, \ldots, Z_n$ being not independent and identically distributed [i.i.d.] uniform (0,1) r.v.'s. However, in some important cases the distribution of $Z_1, \ldots, Z_n$

2000 Mathematics Subject Classification. Primary 62G10, 62F03: Secondary 60J65.

Key words and phrases. Cramér-von Mises tests; Tests of goodness of fit; weak laws; empirical processes; Karhunen-Loève expansions; Gaussian processes; Brownian bridge; Bessel functions.
doses not depend upon \( \theta \), but only on \( F \). In this cases, the distribution of \( \hat{W}_{n,\varphi}^2 \) is parameter free. This happens if \( F \) is a location scale family and \( \hat{\theta}_n \) is an equivalent estimator, a fact noted by David and Johnson [4].

2. The empirical process with estimated parameters

A general study of the weak convergence of the estimated empirical process was carried out by Durbin [6]. We present here an approach to his main results using strong approximations.

Introduce, for each \( x \in \mathbb{R} \), the empirical process with estimated parameters

\[
\alpha_n(x, \hat{\theta}_n) = \sqrt{n}(\mathbb{F}_n(x) - G(x, \hat{\theta}_n)),
\]

where \( \hat{\theta}_n \) is a sequence of estimators of \( \theta \), and we assume that

\[
\sqrt{n}(\hat{\theta}_n - \theta) = 1 \sqrt{n} \sum_{i=1}^{n} l(X_i, \theta) + o_p(1),
\]

where \( l(X_1, \theta) = (l_1(X_1, \theta_1), \ldots, l_k(X_1, \theta_k)) \) is centered function and has finite second moments.

Suppose \( F(x) = G(x, \theta) \in \mathcal{F} \) has density \( f(x, \theta) = \frac{\partial G}{\partial \theta}(x, \theta) \). Take \( \hat{\theta}_n \) as the maximum Likelihood estimator: the maximizer of

\[
m(\theta) = \sum_{i=1}^{n} \log f(X_i, \theta).
\]

Under adequate regularity conditions \( \int \frac{\partial}{\partial \theta} \log f(x, \theta) dG(x, \theta) = 0 \) and

\[
\int \left( \frac{\partial}{\partial \theta} \log f(x, \theta) \right) \left( \frac{\partial}{\partial \theta} \log f(x, \theta) \right)^T dG(x, \theta) = - \int \frac{\partial^2}{\partial \theta^2} \log f(x, \theta) dG(x, \theta) := I(\theta).
\]

Since

\[
m'(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \log f(X_i, \theta) \quad \text{and} \quad m''(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta^2} \log f(X_i, \theta),
\]

we obtain, from the Law of Large Number, that \( \frac{1}{n} m''(\theta) \rightarrow I(\theta) \) almost surely.

Now, a Taylor expansion of \( m'(\theta) \) around \( \theta \) gives

\[
\frac{1}{\sqrt{n}} (m'(\hat{\theta}_n) - m'(\theta)) = \frac{1}{n} m''(\hat{\theta}_n) \sqrt{n}(\theta - \hat{\theta}) + o_p(1)
\]

\[
= - I(\theta) \sqrt{n}(\theta - \hat{\theta}) + o_p(1),
\]

which, taking into account that \( m'(\hat{\theta}) = 0 \), gives

\[
\sqrt{n}(\theta - \hat{\theta}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(X_i, \theta) + o_p(1),
\]
with \( l(x, \theta) = I(\theta)^{-1} \frac{\partial}{\partial \theta} \log f(x, \theta) \). Clearly \( \int l(x, \theta) dG(x, \theta) = 0 \), while

\[
\int l(x, \theta)l(x, \theta)^T dG(x, \theta) = I(\theta)^{-1} I(\theta) I(\theta)^{-1} = I(\theta)^{-1}.
\]

To obtain the null asymptotic distribution of \( \alpha_n(x, \hat{\theta}_n) \), we assume that \((H_0)\) and (2.4) and write

\[
\alpha_n(x, \hat{\theta}_n) = \sqrt{n}(\mathbb{E}_n(x) - G(x, \theta)) - \sqrt{n}(G(x, \hat{\theta}_n) - G(x, \theta))
\]

\[
= \alpha_n(G(x, \theta)) - H(G(x, \theta, \theta)^T \int_0^1 L(t, \theta) d\alpha_n(t) + o_P(1)
\]

(2.5)

\[
= \hat{\alpha}_n(G(x, \theta)) + o_P(1),
\]

where \( \alpha_n(.) \) denotes the uniform empirical process, \( H(t, \theta) = \frac{\partial G}{\partial \theta} \left(G^{-1}(t, \theta), \theta\right) \),

\( L(t, \theta) = l(G^{-1}(t, \theta), \theta) \), with \( G^{-1}(t, \theta) = \{x : G(x, \theta) \geq t\} \) denoting the quantile function of \( X_1 \), and

(2.6) \[ \hat{\alpha}_n(t) = \alpha_n(t) - H(t, \theta)^T \int_0^1 L(s, \theta) d\alpha_n(s), \text{ for } 0 < t < 1, \]

is the uniform estimated empirical process.

2.1. Some notes on stochastic integration. Equation (2.6) suggests that

\[
\hat{\alpha}_n(t) \overset{w}{\rightarrow} B(t) - H(t, \theta)^T \int_0^1 L(s, \theta) dB(s), \text{ as } n \rightarrow \infty,
\]

where \( \overset{w}{\rightarrow} \) denotes the weak convergence and \( B(.) \) is a brownian bridge (i.e., a Gaussian process with \( B(0) = B(1) = 0, \mathbb{E}(B(t)) = 0, \mathbb{E}(B(s)B(t)) = \min(s, t) - st \) for \( s, t \in [0, 1] \)).

We cannot give \( \int_0^1 L(s, \theta) dB(s) \) the meaning of a Stieltjes integral since the trajectories of \( B(.) \) are not of bounded variation. It is possible, though, to make sense of expressions like \( \int_0^1 f(s) dB(s) \), with \( f \in L^2(0, 1) \) through the following construction.

Assume first that \( f \) is simple : \( f(t) = \sum_{i=1}^n a_i \mathbb{I}_{[t_{i-1}, t_i]} \), with \( a_i \in \mathbb{R} \) and \( 0 = t_0 < t_1 < \cdots < t_n = 1 \). Then

\[
\int_0^1 f(s) dB(s) = \sum_{i=1}^n a_i (B(t_i) - B(t_{i-1})) := \sum_{i=1}^n a_i \triangle B_i,
\]

where \( \triangle B_i = B(t_i) - B(t_{i-1}) \). It can be easily checked that \( \mathbb{E}(\triangle B_i) = 0 \) and

\[
\text{Var}(\triangle B_i) = \triangle t_i (1 - \triangle t_i) \quad \text{and} \quad \text{Cov}(\triangle B_i, \triangle B_j) = -\triangle t_i \triangle t_j \quad \text{if} \quad i \neq j.
\]

The random variable is centered Gaussian with variance...
\[ \sum_{i=1}^{n} a_i^2 \text{Var}(\Delta B_i) + 2 \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(\Delta B_i, \Delta B_j) = \sum_{i=1}^{n} a_i^2 \Delta t_i - \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \Delta t_i \Delta t_j \]
\[ = \sum_{i=1}^{n} a_i^2 \Delta t_i - (\sum_{i=1}^{n} a_i \Delta t_i)^2 \]
\[ = \int_0^1 f^2(t)dt - \left( \int_0^1 f(t)dt \right)^2. \]

Thus, \( f \rightarrow \int_0^1 f(s)dB(s) \) defines an isometry between the subspace of \( L^2(0, 1) \) consisting of centered, simple functions and its range. We can therefore extend the definition to all centered function in \( L^2(0, 1) \). Finally, for a general \( f \in L^2(0, 1) \),
\[ \int_0^1 f(s)dB(s) = f \rightarrow \int_0^1 \hat{f}(s)dB(s), \]
where \( \hat{f}(s) = f(s) - \int_0^1 f(t)dt \). The stochastic integral \( \int_0^1 f(s)dB(s) \) is centered, Gaussian random variable with variance
\[ \int_0^1 f^2(t)dt - \left( \int_0^1 L(t)dt \right)^2. \]
In fact, if \( f_1, ..., f_k \in L^2(0, 1) \), then \( \left( \int_0^1 f_1(s)dB(s), \ldots, \int_0^1 f_k(s)dB(s) \right) \) has a joint centered, Gaussian law and form the isometry defining the integrals we see that
\[
(2.7) \quad \text{Cov}\left( \int_0^1 f(s)dB(s), \int_0^1 g(s)dB(s) \right) = \int_0^1 f(s)g(s)ds - \int_0^1 f(s)ds \int_0^1 g(s)ds.
\]
We can similarly check that
\[
\left( \{B(t)\}_{t \in [0, 1]}, \int_0^1 f_1(s)dB(s), \ldots, \int_0^1 f_k(s)dB(s) \right)
\]
is Gaussian and
\[ \text{Cov}\left( B(t), \int_0^1 f(s)dB(s) \right) = \int_0^t f(s)ds - t \int_0^t f(s)ds \]
(take \( g(s) = \mathbb{I}_{[0, 1]}(s) \) in (2.7) to check it).

An integration by parts formula. Suppose \( h(\cdot) \) is simple. Then
\[ \int_0^1 h(s)dB(s) = \sum_{i=1}^{n} h(t_i)\left( B(t_i) - B(t_{i-1}) \right) = - \sum_{i=0}^{n-1} B(t_i)\left( h(t_{i+1}) - h(t_i) \right) = - \int_0^1 B(t)dh(t). \]
This result can be easily extended to any $h(\cdot)$ of bounded variation and continuous on $[0, 1]$:

$$\int_0^1 h(s)dB(s) = -\int_0^1 B(t)dh(t).$$

This integration by parts formula can be used to bound the difference between stochastic integrals and the corresponding integrals with respect to the empirical process:

$$|\int_0^1 h(s)d\alpha_n(s) - \int_0^1 h(s)dB_n(s)| \leq \sup_{0 \leq t \leq 1} |\alpha_n(t) - B_n(t)| \int_0^1 |h|(s),$$

where $B_n(\cdot)$ is a sequence of brownian bridges.

We can summarize now the above arguments in the following theorem (see, e.g., [6]).

**Theorem 2.1.** Provided $H(t, \theta)$ is continuous on $[0, 1]$ and $L(s, \theta)$ is continuous and bounded variation on $[0, 1]$ we can define, on a sufficiently rich probability space, $\alpha_n(\cdot)$ and $B_n(\cdot)$ such that

$$\sup_{0 \leq t \leq 1} |\hat{\alpha}_n(t) - \hat{B}_n(t)| = O\left(\frac{\log n}{\sqrt{n}}\right)$$

almost surely [a.s.],

where $\hat{B}_n(t) = B_n(t) - H(t, \theta)^T \int_0^t L(s, \theta)dB_n(s)$ is a centered Gaussian process with function covariance

$$\hat{K}_\theta(s, t) = \min(s, t) - st - H(t, \theta)^T \int_0^s L(x, \theta)dx - H(s, \theta)^T \int_0^t L(x, \theta)dx + H(s, \theta)^T \int_0^1 L(x, \theta)L(x, \theta)^T dx H(t, \theta).$$

Note that this covariance function can be expressed as $s \wedge t - \sum_{j=1}^k \phi_j(s)\phi_j(t)$ for some real functions $\phi_j(\cdot)$. A very complete study of the Karhunen-Loève expansion of Gaussian processes with this type of covariance function was carried out in [11].

**Exemple 1.** We consider $\mathcal{F} = \{G_0(\cdot) : \theta = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^*_+\}$ is a location scale family $\left(G_0(\cdot) \text{ is a standard distribution function with density } g_0\right)$. Then

$$H(t, \theta) = -\frac{1}{\sigma}g_0\left(G_0^{-1}(t)\right) \left[\frac{1}{G_0^{-1}(t)}\right].$$
and
\[ I(\theta) = \frac{1}{\sigma^2} \left[ \int \frac{g_0(x)^2}{g_0(x)} dx \int x \frac{g_0(x)^2}{g_0(x)} dx - \int \frac{g_0(x)^2}{g_0(x)} dx \right]. \]

We can now write
\[ I(\theta) - 1 = \sigma^2 \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \]
with \( \sigma_{ij} \) depending only on \( G_0 \), but not on \( \mu \) or \( \sigma \) and
\[ \hat{K}(s, t) = \min(s, t) - st - \phi_1(s)\phi_1(t) - \phi_2(s)\phi_2(t). \]

Here
\[ \phi_1(t) = -\sqrt{\left( \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}} \right) g_0(G_0^{-1}(t))} \]
and
\[ \phi_2(t) = -\frac{\sigma_{12}}{\sqrt{\sigma_{22}}} g_0(G_0^{-1}(t)) - \sqrt{\sigma_{22}} g_0(G_0^{-1}(t)) G_0^{-1}(t). \]

If \( \mathcal{F} \) is the Gaussian family \( G_0(x) = \Phi(x), \ g_0(x) = \phi(x), \ g_0'(x) = -x\phi(x) \) and
\[ I(\theta) = \frac{1}{\sigma^2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \]
Hence, \( \sigma_{11} = 1, \ \sigma_{22} = \frac{1}{2}, \ \sigma_{12} = \sigma_{21} = 0 \) and
\[ \hat{K}(s, t) = \min(s, t) - st - \phi(\Phi^{-1}(s))\phi(\Phi^{-1}(t)) - \frac{1}{2} \phi(\Phi^{-1}(s))\Phi^{-1}(s)\phi(\Phi^{-1}(t))\Phi^{-1}(t). \]

In this Gaussian case \( L \) is not of bounded variation on \([0,1]\), but the above argument can be modified and still prove that
\[ \{\hat{\alpha}_n(t)\}_t \xrightarrow{w} \{B(t) + \phi(\Phi^{-1}(s))\int_0^1 (\Phi^{-1}(s)) dB(s) + \frac{1}{2} \phi(\Phi^{-1}(t))\Phi^{-1}(t)\int_0^1 (\Phi^{-1}(s)^2 - 1) dB(s)\}_t \]
as random variable in \( D[0,1] \) or \( L^2[0,1] \).

Theorem 2.1 provided, as an easy corollary, the asymptotic distribution of a variety of \( \hat{W}^2_{n,\phi} \) statistics under the null hypothesis. In fact, Durbin’s results also give a valuable tool for studying its asymptotic power because they include too the asymptotic distribution of the estimated empirical process under contiguous alternatives. A survey of results connected to Theorem 2.1 as well as a simple derivation of it based on Skorohod embedding can be found in [10].
3. Results (Asymptotic power of the $\hat{W}^2_{n,\varphi}$ test of fit)

Assume that (1.1) and (2.4), under the null hypothesis ($H_0$), the limiting distribution of $\hat{W}^2_{n,\varphi}$ in (1.2) coincides with the distribution of the random variable

$$\hat{W}^2_{\varphi} := \int_0^1 \varphi(t)B^2(t,\theta)dt,$$

where $B(t,\theta)$ is a Gaussian random process with zero mean and covariance function

$$(3.9) \quad K_{\varphi}(s,t) = \sqrt{\varphi(s)\varphi(t)}\hat{K}_\theta(s,t),$$

where $\hat{K}_\theta(s,t)$ has been described above in (2.8).

We chose the sequence of local alternatives which depend on the parameters $\theta = (\theta_1, \ldots, \theta_k)$ given by

$$H_a : F(.|.) = F^{(n)}(.|.,\theta),$$

where $F^{(n)}(.|.,\theta)$ is chosen as a proper distribution function such that $F^{(n)}(.|.,\theta) \to G(.|.,\theta)$, as $n \to \infty$, and with $R_n(.) := \sqrt{n}(F^{(n)}(.|.,\theta) - G(.|.,\theta)) \to R(.|.,\theta)$ in the mean square, as $n \to \infty$, and $R(.|.,\theta)$ is known and satisfies the condition $\int_{-\infty}^{+\infty} R(x,\theta)dx < \infty$.

These kinds of alternatives were proposed and discussed, in particular, by Chibisov [2]. Setting $t = G(x,\theta)$, $\delta(t,\theta) = R(G^{-1}(t,\theta),\theta)$ and assuming that

$$(3.10) \quad \int_0^1 \varphi(t)\delta^2(t,\theta)dt < \infty.$$  

Under ($H_a$), with $\delta(.|.,\theta)$ satisfies the condition (3.10), the limiting distribution (as $n \to \infty$) of statistic $\hat{W}^2_{n,\varphi}$ coincides (see, e.g., [2]) with the distribution of r.v:

$$\hat{W}^2_{\delta,\varphi} = \int_0^1 \varphi(t)\left[B(t,\theta) + \delta(t,\theta)\right]^2dt$$

$$(3.11) = \int_0^1 \varphi(t)B^2(t,\theta) + 2\int_0^1 \delta(t,\theta)\varphi(t)B(t,\theta)dt + \int_0^1 \delta(t,\theta)\varphi^2(t).$$

For a fixed parameter $\theta$ and a level of significance $\alpha \in (0,1)$, there is a threshold of confidence $t_\alpha := t_\alpha(\theta)$ satisfying the identity

$$(3.12) \quad \mathbb{P}(\int_0^1 \varphi(t)B^2(t,\theta)dt \geq t_\alpha) = \alpha.$$  

(see, e.g., [5] for a tabulation of numerical values of $t_\alpha$ for the particular cases $\varphi(t) = t^{2\beta}$, $\beta > -1$, and, $\alpha = 0.1, 0.05, 0.01, 0.005, 0.001$).
In the case above, the asymptotic power of the test of fit based upon $\hat{W}^2_{n,\varphi}$, under the sequence of local alternatives specified by $(H_a)$, is specified by

$$ (3.13) \quad P\left(\hat{W}^2_{n,\varphi} \geq t_\alpha \right) = \lim_{n \to \infty} P\left(\hat{W}^2_{n,\varphi} \geq t_\alpha | H_a \right). $$

Recalling the definitions (1.1) of $\varphi$, (3.9) of $K_{\varphi}(.,.)$ and, (3.12) of $t_\alpha$, we set

$$ (3.14) \quad g(t, \theta) := \sqrt{\varphi(t)}\delta(t, \theta), \quad x := t_\alpha - \int_0^1 K_{\varphi}(t, s)ds - \int_0^1 \varphi(t)\delta^2(t, \theta)dt, $$

$$ A := \int_0^1 K_{\varphi}^2(s, s)ds, \quad B := \int_0^1 \left[ \int_0^1 g(s, \theta)K_{\varphi}(s, t)ds \right]^2 dt, $$

$$ C := \int_0^1 \int_0^1 \left[ \int_0^1 g(u, \theta)K_{\varphi}(s, u)du \right] \int_0^1 g(v, \theta)K_{\varphi}(s, v)dv \right]^2 K_{\varphi}(s, t)dsdt, $$

$$ (3.15) \quad D^2 := \int_0^1 \int_0^1 g(s, \theta)K_{\varphi}(s, t)g(t, \theta)dsdt. $$

Let $\phi$ (resp. $\Phi$) be the probability density (resp. distribution) function of the standard normal $N(0,1)$ distribution. Namely,

$$ \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x f(u)du. $$

Then, for calculating the power function defined in (3.13), we have the following theorem. Recall the definitions (3.14)-(3.15) of $x, A, B, C$ and, $D$.

**Theorem 3.1.** Under the assumptions above, we have

$$ 1 - P(\hat{W}^2_{n,\varphi} \geq t_\alpha) = \Phi\left(\frac{x}{D}\right) + \left\{ A \int \frac{H_1(x)}{2D^2} + \frac{B}{2D^2} H_2(x) + \frac{C}{4D^4} H_3(x) + \frac{B^2}{8D^6} H_5(x) \right\} \phi\left(\frac{x}{D}\right) + \epsilon(x). $$

Here $H_j(.)$ are Hermite polynomial and, $\epsilon(.)$ is a remainder term fulfilling

$$ (3.16) \quad \sup_y |\epsilon(y)| \leq \frac{C_1}{\left(D^2 - \frac{B}{\lambda_1}\right)^\frac{3}{2}}, $$

where $C_1$ is a constant and, $\lambda_1$ is the first eigenvalue of the Fredholm transformation $h \to \int_0^1 K_{\varphi}(s, .)h(s)ds$.

**Remark 1.**
The following particular cases are of interest. If, we replace $g(., \theta)$ by $\gamma g(., \theta)$ in the alternatives of (3.10) (for some real parameter $\gamma > 0$), we obtain that

\begin{equation}
\sup_y |\varepsilon(y)| = o\left(\gamma^{-\frac{3}{2}}\right) \quad \text{as} \quad \gamma \to \infty.
\end{equation}

Proof. The proof of this theorem resembles that which was published (in the case non-parametric) in another article (see, e.g.,[1]). □

4. Numerical example

As an illustration, we will consider approximate calculation of the power of $\hat{W}_{2n, \phi}^2$ test for verifying the hypothesis of normal distribution. Here, we consider

\[ F = \{\Phi\left(\frac{y - \mu}{\sigma}\right) : (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+^*\}, \theta = (\mu, \sigma), \hat{\theta} = (\bar{X}, S^2) \]

and,

\[ H_0 : F(y) = G(y, \theta) := \Phi\left(\frac{y - \mu}{\sigma}\right). \]

We chose as an alternative,

\[ (H_a) : F(y) = F^{(n)}(y, \theta) := \Phi\left(\frac{y - \mu}{\sigma}\right) + \gamma \frac{R\left(\frac{y - \mu}{\sigma}\right)}{\sqrt{n}} + O\left(\frac{1}{n}\right), \]

where $R(x) = \frac{1}{4\sqrt{2\pi}}(3x - x^2)e^{-\frac{x^2}{2}}$ and, $\gamma$ is a real parameter positive.

Setting $t = \Phi\left(\frac{y - \mu}{\sigma}\right)$ and, $\delta(t) = R\left(\Phi^{-1}(t)\right)$, we obtain

\[ K_{\phi}(s, t) = \sqrt{\phi(s)\phi(t)}\tilde{K}_{\phi}(s, t) = \sqrt{\varphi(s)\varphi(t)}\left\{\min(s, t) - st - \left(1 + \frac{1}{2}\Phi^{-1}(s)\Phi^{-1}(t)\right)\phi\left(\Phi^{-1}(s)\right)\phi\left(\Phi^{-1}(t)\right)\right\}. \]

According to the preceding theorem, the asymptotic power of the test of fit based upon $\hat{W}_{2n, \phi}^2$, under the sequence of local alternatives specified by $(H_a)$ in the case above, is calculated for various $\gamma$ and $\alpha$. The accompanying table gives values of the power $\beta_\gamma = P\left(\hat{W}_{2n, \phi}^2 > t_\alpha\right)$ for $\phi \equiv 1$.

<table>
<thead>
<tr>
<th>$\alpha = 0.01$</th>
<th>$\gamma$</th>
<th>$\beta_\gamma$</th>
<th>$\alpha = 0.001$</th>
<th>$\gamma$</th>
<th>$\beta_\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.2</td>
<td>3</td>
<td>0.085</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.53</td>
<td>4</td>
<td>0.21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.851</td>
<td>5</td>
<td>0.532</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.98</td>
<td>6</td>
<td>0.847</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table. Approximate power for the test goodness of fit

The second column gives various values of the parameter $\gamma$. The third as well as last the columns give power values for $\beta_\gamma$. They are compared with the exact values obtained by Martynov [8].
REFERENCES


