Statistical estimate of the proportional hazard premium of loss under random censoring

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Abstract. Many insurance premium principles are defined and various estimation procedures introduced in the literature. In this paper, we focus on the estimation of the excess-of-loss reinsurance premium when the risks are randomly right-censored. The asymptotic normality of the proposed estimator is established under suitable conditions and its performance evaluated through sets of simulated data.

Résumé. Plusieurs principes de calcul de prime d'assurance sont définis et diverses procédures d'estimation sont introduites dans la littérature. Dans cet article, nous nous concentrons sur l'estimation de la prime de réassurance en excédent de sinistres lorsque les risques sont aléatoirement censurés à droite. La normalité asymptotique de l'estimateur proposé est établie sous des conditions adéquates et sa performance évaluée à travers des ensembles de données simulées.

1. Introduction

Let $X_1, ..., X_n$ be $n \geq 1$ independent copies of a non-negative random variable (rv) X, defined over some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with continuous cumulative distribution function (cdf) F. An independent sequence of independent rv's $Y_1, ..., Y_n$ with continuous cdf G censor them to the right, so that at each stage j we only can observe $Z_j := \min(X_j, Y_j)$ and the variable $\delta_j := \mathbf{1}\{X_j \leq Y_j\}$ (with $\mathbf{1}\{\cdot\}$ denoting the indicator function) informing whether or

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not there has been censorship. This model is very useful in a variety of areas where random censoring is very likely to occur such as in biostatistics, medical research, reliability analysis, actuarial science,... For more general censoring schemes and other issues involving censored data, we refer, for instance, to Cox and Oakes (1984), Kalbfleisch and Prentice (1980) and Gill (1980).

In insurance, the worst scenarios are those caused by extreme events such as natural catastrophes, human-made disasters and financial crashes. These events increase the bill of insurance and reinsurance companies. A typical requirement for actuaries is the determination of adequate premiums for such risks. Usually, the insurer's claims data do not correspond to the underlying losses, because they are censored from above, since the insurer stipulates an upper limit to the amount to be paid out and the reinsurer covers the excess over this fixed threshold. This kind of reinsurance is called excess-of-loss reinsurance (see, e.g., Rolski et al., 1999; Embrechts et al., 1997) and the upper limit has distinct designations that are specific to each insurance type. For instance, in life insurance, it is called the cedent's company retention level while in non-life insurance, it is called the deductible, where the losses should be treated separately. For a discussion on the occurrence of right-random censorship in the area of insurance, one refers to Denuit et al. (2006) in which a study on the allocated loss adjustment expenses (ALAE's) is given.

Let us assume that both F and G are heavy-tailed, that is there exist two constants $\gamma_1 > 0$ and $\gamma_2 > 0$, called tail indices or extreme value indices (EVI's), such that

$$\overline{F}(z) \sim z^{-1/\gamma_1}\ell_1(z) \text{ and } \overline{G}(z) \sim z^{-1/\gamma_2}\ell_2(z), \text{ as } z \to \infty, \tag{1}$$

where ℓ_1 and ℓ_2 are slowly varying functions at infinity, i.e. $\lim_{z\to\infty} \ell_i(xz)/\ell_i(z) = 1$ for every $x > 0$, $i = 1, 2$. Throughout the paper, we use the notation $\overline{\mathcal{S}}(x) := \mathcal{S}(\infty) - \mathcal{S}(x)$, for any function $\mathcal{S}(x)$ of $x > 0$. If relations (1) hold, then we have, for any $x > 0$

$$\lim_{z\to\infty} \frac{\overline{F}(xz)}{\overline{F}(z)} = x^{-1/\gamma_1} \text{ and } \lim_{z\to\infty} \frac{\overline{G}(xz)}{\overline{G}(z)} = x^{-1/\gamma_2}, \tag{2}$$

and we say that $\overline{F}$ and $\overline{G}$ are regularly varying at infinity as well, with respective tail indices $-1/\gamma_1$ and $-1/\gamma_2$, which we denote by $\overline{F} \in \mathcal{RV}_{-1/\gamma_1}$ and $\overline{G} \in \mathcal{RV}_{-1/\gamma_2}$. Note that, in virtue of the independence of X and Y, the cdf of the observed $Z's$, that we denote by H, is also heavy-tailed and we have $H \in \mathcal{RV}_{-1/\gamma}$ with $\gamma := \gamma_1\gamma_2/(\gamma_1 + \gamma_2)$. This class of distributions, which includes models such as Pareto, Burr, Fréchet, Lévy-stable and log-gamma, plays a prominent role in extreme value theory. Also known as Pareto-type or Pareto-like distributions, these models have important practical applications and are used rather systematically in certain branches of non-life insurance as well as in finance, telecommunications, geology and many other fields (see e.g. Resnick, 2007). The analysis of extreme values of randomly censored data is a new research topic to which Reiss and Thomas (1997) made a very brief reference, in Section 6.1, as a first step but with no asymptotic results. In the last decade, several authors started to be interested in the estimation of the tail index along with large quantiles under random censoring as one can see in Gomes and Oliveira (2003), Beirlant et al. (2007), Einmahl et al. (2008) and Worms and Worms (2014). Gomes and Neves (2011) also made a contribution to this field by providing a detailed simulation study and applying the estimation procedures on some survival data sets. Let $\{(Z_i, \delta_i), 1 \le i \le n\}$ be a sample from the couple of rv's (Z, δ) and $Z_{1:n} \le ... \le Z_{n:n}$ the order statistics pertaining to

$(Z_1, ..., Z_n)$. If we denote the concomitant of the ith order statistic by $\delta_{[i:n]}$ (i.e. $\delta_{[i:n]} = \delta_j$ if $Z_{i:n} = Z_j$), then Hill's estimator of γ_1 adapted to censored data is defined as $\widehat{\gamma}_1^{(H,c)} := \widehat{\gamma}^H / \widehat{p}$, where $\widehat{\gamma}^H := k^{-1} \sum_{i=1}^k \log\left(Z_{n-i+1:n}/Z_{n-k:n}\right)$ represents Hill's estimator (Hill, 1975) of γ, with $k = k_n$ being an integer sequence satisfying

$$1 < k < n, \ k \to \infty \text{ and } k/n \to 0 \text{ as } n \to \infty, \tag{3}$$

and $\widehat{p} := k^{-1} \sum_{i=1}^k \delta_{[n-i+1:n]}$ being the proportion of upper non-censored observations. Einmahl et al. (2008) established the asymptotic normality of $\widehat{\gamma}_1^{(H,c)}$ by assuming that cdf's are absolutely continuous. Recently, Brahimi et al. (2015) proved that $\widehat{p}$ consistently estimates $p := \gamma_2/(\gamma_1 + \gamma_2)$ leading to the consistency of $\widehat{\gamma}_1^{(H,c)}$. They also established the asymptotic normality of $\widehat{\gamma}_1^{(H,c)}$ by adopting an approach that is different from that of Einmahl et al. (2008).

In the excess-of-loss reinsurance treaty, the ceding company covers claims which do not exceed a (high) number $R \geq 0$ (called retention level), while the reinsurer pays the part $(X_i - R)_+ := \max(0, X_i - R)$ of each claim beyond R. Applying Wang's premium calculation principle (Wang, 1996), with a distortion function equal to $x^{1/\rho}$, one defines what is called the proportional hazard premium (PHP), where $\rho \geq 1$ represents the distortion parameter or the risk aversion index. Then, the PHP of loss for the layer from R to infinity is defined as follows:

$$\Pi_\rho(R) := \int_R^\infty \left(\overline{F}(x)\right)^{1/\rho} dx,$$

which may be rewritten into

$$\Pi_\rho(R) = R(\overline{F}(R))^{1/\rho} \int_1^\infty \left(\frac{\overline{F}(Rx)}{\overline{F}(R)}\right)^{1/\rho} dx.$$

By using the well-known Karamata theorem (see, for instance, de Haan and Ferreira, 2006, page 363), we get

$$\Pi_\rho(R) \sim \frac{\rho}{1/\gamma_1 - \rho} R\left(\overline{F}(R)\right)^{1/\rho}, \ 0 < \gamma_1 < 1/\rho,$$

for large R. Since $\overline{F} \in \mathcal{RV}_{-1/\gamma_1}$, then $\overline{F}(x) \sim \overline{F}(h)(x/h)^{-1/\gamma_1}$ as $x \to \infty$, where $h = h_n := H^{\leftarrow}(1 - k/n)$ with $H^{\leftarrow}(y) := \inf\{x : H(x) \geq y\}$, $0 < y < 1$, denoting the quantile function pertaining to H. This leads us to derive a Weissman-type estimator (see Weissman, 1978) for the distribution tail $\overline{F}$ for censored data as follows:

$$\widehat{\overline{F}}(x) = \left(\frac{x}{Z_{n-k:n}}\right)^{-1/\widehat{\gamma}_1^{(H,c)}} \overline{F}_n(Z_{n-k:n}).$$

In the context of randomly right censored observations, the nonparametric maximum likelihood estimator of F is given by Kaplan and Meier (1958) as the product limit estimator

$$\overline{F}_n(x) := \prod_{Z_{i:n} \leq x} \left(1 - \frac{\delta_{[i:n]}}{n - i + 1}\right) = \prod_{Z_{i:n} \leq x} \left(\frac{n - i}{n - i + 1}\right)^{\delta_{[i:n]}}, \text{ for } x < Z_{n:n},$$

which gives $\overline{F}_n(Z_{n-k:n}) = \prod_{i=1}^{n-k} \left(1 - \frac{\delta_{[i:n]}}{n-i+1}\right)$. Thus, the distribution tail estimator is of the form

$$\widehat{\overline{F}}(x) := \left(\frac{x}{Z_{n-k:n}}\right)^{-1/\widehat{\gamma}_1^{(H,c)}} \prod_{i=1}^{n-k} \left(1 - \frac{\delta_{[i:n]}}{n-i+1}\right),$$

and consequently, we define the PHP estimator as follows:

$$\widehat{\Pi}_\rho(R) := \frac{\rho R}{1/\widehat{\gamma}_1^{(H,c)} - \rho} \left(\frac{R}{Z_{n-k:n}}\right)^{-1/\left(\rho \widehat{\gamma}_1^{(H,c)}\right)} \prod_{i=1}^{n-k} \left(1 - \frac{\delta_{[i:n]}}{n-i+1}\right)^{1/\rho}.$$

The outline of the paper is as follows. In Section 2, we state our main result that consists in the asymptotic normality of the newly proposed estimator $\widehat{\Pi}_\rho(R)$, which we prove in Section 4. In Section 3, we carry out a simulation study to illustrate its finite sample behavior. Finally, some results, that are instrumental to our needs, are gathered in the Appendix.

2. Main results

It is well-known that the asymptotic normality of extreme value theory based estimators is adequately achieved within the second-order framework (see de Haan and Stadtmüller, 1996). Thus, it seems quite natural to suppose that cdf's F and G satisfy the well-known second-order condition of regular variation. That is, we assume that there exist two constants $\tau_j \leq 0$ (called second-order parameters) and two functions A_j, $j = 1, 2$, tending to zero and not changing sign near infinity, such that for any $x > 0$

$$\begin{aligned}
\lim_{t \to \infty} \frac{\overline{F}(tx)/\overline{F}(t) - x^{-1/\gamma_1}}{A_1(t)} &= x^{-1/\gamma_1} \frac{x^{\tau_1/\gamma_1} - 1}{\gamma_1 \tau_1}, \\
\lim_{t \to \infty} \frac{\overline{G}(tx)/\overline{G}(t) - x^{-1/\gamma_2}}{A_2(t)} &= x^{-1/\gamma_2} \frac{x^{\tau_2/\gamma_2} - 1}{\gamma_2 \tau_2}.
\end{aligned} \tag{4}$$

Theorem 1. *Assume that the second-order conditions of regular variation* (4) *hold, with* $0 < \gamma_1 < 1/\rho$ *and let* $k = k_n$ *be an integer sequence satisfying, in addition to* (3), $\sqrt{k} A_1(h) \to \lambda_1$. *Assume further that* $R/h \to 1$. *Then*

$$\sqrt{k} \frac{\widehat{\Pi}_\rho(R) - \Pi_\rho(R)}{(R/h)^{-1/\rho\gamma_1} R\left(\overline{F}(h)\right)^{1/\rho}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(\mu, \sigma^2\right), \text{ as } n \to \infty,$$

where

$$\mu := \frac{\rho \lambda_1}{(1 - p\tau_1)(1 - \rho\gamma_1)^2} + \frac{\lambda_1}{\rho(\gamma_1 + \tau_1 + \rho - 2)(2 - \rho - \gamma_1)},$$

and

$$\sigma^2 := \frac{\gamma_1^2}{(1 - \rho\gamma_1)^2} \left(p(2 - p) + \frac{\rho(p-1)}{(1 - \rho\gamma_1)} + \frac{\rho^2(1 - 2p)}{p(1 - \rho\gamma_1)^2}\right).$$

3. Simulation study

We carry out a simulation study to illustrate the performance of our estimator, through two sets of censored and censoring data, both drawn from the following Burr model. That is

$$\overline{F}(x) = \left(1 + x^{\eta/\gamma_1}\right)^{-1/\eta} \text{ and } \overline{G}(x) = \left(1 + x^{\eta/\gamma_2}\right)^{-1/\eta}, \ x \geq 0,$$

where $\gamma_1, \gamma_2 > 0$. We fix $\eta = 1/4$, we choose the values 0.10 and 0.25 for γ_1 and two distinct aversion index values $\rho = 1.00$ and $\rho = 1.10$. For the proportion of the really observed extreme values, we take $p = 0.40, 0.60$ and 0.80, that is, we allow the percentage of censoring in the right tail of X to be 60%, 40% and 20%. For each couple (γ_1, p), we solve the equation $p = \gamma_2/(\gamma_1 + \gamma_2)$ to get the pertaining γ_2-value. We vary the common size n of both samples $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$, then for each size, we generate 1000 independent replicates. Our overall results are taken as the empirical means of the results obtained through the 1000 repetitions. To determine the optimal number of upper order statistics (that we denote by k^*) used in the computation of $\widehat{\gamma}_1^{(H,c)}$, we apply the algorithm of Reiss and Thomas (1997), page 121. The retention level R is taken as the value of the intermediate order statistic $Z_{n-k^*:n}$. The simulation results are summarized in Table 1 for $\gamma_1 = 0.10$ and in Table 2 for $\gamma_1 = 0.25$ (where abs bias and rmse respectively stand for the absolute value of the bias and the root of the mean squared error of the estimation). On the light of these results we see that, from the point of view of the rmse, the estimation accuracy increases when the censoring percentage decreases, which seems logical. On the other hand, we note that the sample size does not have a significant effect on the estimation when the percentage of observed data is high. Moreover, the estimator performs better for the smaller value of the distortion parameter ρ.

		$p = 0.40$						
ρ		1.00				1.10		
n	$\Pi_\rho(R)$	$\widehat{\Pi}_\rho(R)$	abs bias	rmse	$\Pi_\rho(R)$	$\widehat{\Pi}_\rho(R)$	abs bias	rmse
500	0.0175	0.0274	0.0099	0.1372	0.0237	0.0429	0.0192	0.2012
1000	0.0174	0.0197	0.0023	0.0616	0.0236	0.0339	0.0102	0.0863
1500	0.0170	0.0158	0.0012	0.0146	0.0233	0.0233	0.0000	0.0203
			$p = 0.60$					
500	0.0097	0.0066	0.0032	0.0135	0.0142	0.0103	0.0039	0.0145
1000	0.0095	0.0037	0.0058	0.0065	0.0138	0.0063	0.0076	0.0091
1500	0.0095	0.0029	0.0066	0.0069	0.0137	0.0045	0.0092	0.0098
			$p = 0.80$					
500	0.0062	0.0014	0.0048	0.0049	0.0093	0.0026	0.0067	0.0074
1000	0.0062	0.0008	0.0054	0.0055	0.0092	0.0014	0.0077	0.0078
1500	0.0060	0.0006	0.0054	0.0054	0.0090	0.0010	0.0079	0.0079

Table 1. PHP estimates based on 1000 right-censored samples of size n from Burr model with tail index $\gamma_1 = 0.10$.

$p = 0.40$							
ρ	1.00				1.10		
n	$\Pi_\rho(R)$	$\widehat{\Pi}_\rho(R)$	abs bias	rmse	$\Pi_\rho(R)$	$\widehat{\Pi}_\rho(R)$	abs bias	rmse
500	0.0265	0.0767	0.0501	0.3604	0.0410	0.1148	0.0738	0.8875
1000	0.0266	0.0633	0.0368	0.1602	0.0411	0.1134	0.0723	0.4842
1500	0.0266	0.0462	0.0196	0.0664	0.0409	0.0632	0.0223	0.0941
$p = 0.60$							
500	0.0196	0.0229	0.0034	0.0965	0.0310	0.0222	0.0088	0.4596
1000	0.0197	0.0119	0.0078	0.0133	0.0317	0.0203	0.0114	0.0236
1500	0.0199	0.0093	0.0106	0.0140	0.0316	0.0152	0.0164	0.0196
$p = 0.80$							
500	0.0153	0.0056	0.0097	0.0118	0.0251	0.0091	0.0160	0.0178
1000	0.0154	0.0030	0.0125	0.0127	0.0254	0.0054	0.0200	0.0204
1500	0.0157	0.0020	0.0136	0.0137	0.0254	0.0040	0.0214	0.0215

Table 2. PHP estimates based on 1000 right-censored samples of size n from Burr model with tail index $\gamma_1 = 0.25$.

4. Proof

Before we start the proof of the theorem, let us give a brief introduction on some uniform empirical processes under random censoring. To this end, we define the functions

$$H^{(j)}(v) := \mathbb{P}(Z \leq v, \ \delta = j), \ j = 0, 1; \ v \geq 0,$$

which have a prominent role to play in the random censorship setting. Their empirical counterparts are defined by

$$H_n^{(j)}(v) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(Z_i \leq v, \delta_i = j), \ j = 0, 1; \ v \geq 0.$$

In the sequel, we will use the following two empirical processes

$$\sqrt{n} \left(\overline{H}_n^{(j)}(v) - \overline{H}^{(j)}(v) \right), \ j = 0, 1; \ v \geq 0,$$

which may be represented, almost surely, by a uniform empirical process. Indeed, let us define, for each $i = 1, ..., n$ with $\theta := H^{(1)}(\infty)$, the following rv

$$U_i := \delta_i H^{(1)}(Z_i) + (1 - \delta_i)(\theta + H^{(0)}(Z_i)).$$

From Einmahl et Koning (1992), the rv's $U_1, ..., U_n$ are independent and identically distributed according to the $(0, 1)$-uniform law. The empirical cdf and the uniform empirical process based upon $U_1, ..., U_n$ are respectively denoted by

$$\mathbb{U}_n(s) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(U_i \leq s) \text{ and } \alpha_n(s) := \sqrt{n}(\mathbb{U}_n(s) - s), \ 0 \leq s \leq 1.$$

Deheuvels and Einmahl (1996) state that almost surely

$$H_n^{(0)}(v) = \mathbb{U}_n(H^{(0)}(v) + \theta) - \mathbb{U}_n(\theta), \text{ for } 0 < H^{(0)}(v) < 1 - \theta,$$

and

$$H_n^{(1)}(v) = \mathbb{U}_n(H^{(1)}(v)), \text{ for } 0 < H^{(1)}(v) < \theta.$$

It is easy to verify that we almost surely have

$$\sqrt{n}\left(\overline{H}_n^{(1)}(v) - \overline{H}^{(1)}(v)\right) = \alpha_n(\theta) - \alpha_n\left(\theta - \overline{H}^{(1)}(v)\right), \text{ for } 0 < \overline{H}^{(1)}(v) < \theta, \quad (5)$$

and

$$\sqrt{n}\left(\overline{H}_n^{(0)}(v) - \overline{H}^{(0)}(v)\right) = -\alpha_n\left(1 - \overline{H}^{(0)}(v)\right), \text{ for } 0 < \overline{H}^{(0)}(v) < 1 - \theta. \quad (6)$$

Our methodology strongly relies on the well-known Gaussian approximation given in 1. For our needs, we use the following form:

$$\sup_{1/n \leq s \leq 1} \frac{n^\zeta |\alpha_n(1-s) - B_n(1-s)|}{s^{1/2-\zeta}} = O_\mathbb{P}(1). \quad (7)$$

For the increments $\alpha_n(\theta) - \alpha_n(\theta - s)$, we will need an approximation of the same type as (7). Following similar arguments, mutatis mutandis, as those used to in the proof of assertions (2.2) of Theorem 2.1 and (2.8) of Theorem 2.2 in Csörgő et al. (1986), we may show that, for every $0 < \theta < 1$ and $0 \leq \zeta < 1/4$, we have

$$\sup_{1/n \leq s \leq \theta} \frac{n^\zeta |\{\alpha_n(\theta) - \alpha_n(\theta - s)\} - \{B_n(\theta) - B_n(\theta - s)\}|}{s^{1/2-\zeta}} = O_\mathbb{P}(1). \quad (8)$$

The following Gaussian processes will be crucial to our needs:

$$\mathbf{B}_n(v) := B_n(\theta) - B_n\left(\theta - \overline{H}^{(1)}(v)\right), \text{ for } 0 < \overline{H}^{(1)}(v) < \theta, \quad (9)$$

and

$$\mathbf{B}_n^*(v) := \mathbf{B}_n(v) - B_n\left(1 - \overline{H}^{(0)}(v)\right), \text{ for } 0 < \overline{H}^{(0)}(v) < 1 - \theta. \quad (10)$$

4.1. *Proof of Theorem 1*

In the sequel, for two sequences of rv's, we write $V_n^{(1)} = o_\mathbb{P}\left(V_n^{(2)}\right)$ and $V_n^{(1)} \approx V_n^{(2)}$, as $n \to \infty$, to say that $V_n^{(1)}/V_n^{(2)} \to 0$ in probability and $V_n^{(1)} = V_n^{(2)}(1 + o_\mathbb{P}(1))$ respectively. With the premium

$$\Pi_\rho(R) = R(\overline{F}(R))^{1/\rho} \int_1^\infty \left(\frac{\overline{F}(Rx)}{\overline{F}(R)}\right)^{1/\rho} dx,$$

and its estimator

$$\widehat{\Pi}_\rho(R) = \frac{\rho R}{1/\widehat{\gamma}_1^{(H,c)} - \rho}\left(\frac{R}{Z_{n-k:n}}\right)^{-1/\left(\rho\widehat{\gamma}_1^{(H,c)}\right)}\left(\overline{F}_n(Z_{n-k:n})\right)^{1/\rho},$$

it is easy to verify that

$$\sqrt{k}\frac{\widehat{\Pi}_\rho(R) - \Pi_\rho(R)}{(R/h)^{-1/(\rho\gamma_1)} R\left(\overline{F}(h)\right)^{1/\rho}} = \sum_{i=1}^5 S_{ni},$$

where

$$S_{n1} := \frac{\rho}{1/\widehat{\gamma}_1^{(H,c)} - \rho} \left(\frac{\overline{F}(Z_{n-k:n})}{\overline{F}(h)} \right)^{1/\rho} \left(\frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} \right)^{1/\rho}$$

$$\times \sqrt{k} \left\{ \left(\frac{(R/Z_{n-k:n})^{-1/\widehat{\gamma}_1^{(H,c)}}}{(R/h)^{-1/\gamma_1}} \right)^{1/\rho} - 1 \right\},$$

$$S_{n2} := \left(\frac{\overline{F}(Z_{n-k:n})}{\overline{F}(h)} \right)^{1/\rho} \left(\frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} \right)^{1/\rho} \sqrt{k} \left\{ \frac{\rho}{1/\widehat{\gamma}_1^{(H,c)} - \rho} - \frac{\rho}{1/\gamma_1 - \rho} \right\},$$

$$S_{n3} := \frac{\rho}{1/\gamma_1 - \rho} \left(\frac{\overline{F}(Z_{n-k:n})}{\overline{F}(h)} \right)^{1/\rho} \sqrt{k} \left\{ \left(\frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} \right)^{1/\rho} - 1 \right\},$$

$$S_{n4} := \frac{\rho}{1/\gamma_1 - \rho} \sqrt{k} \left\{ \left(\frac{\overline{F}(Z_{n-k:n})}{\overline{F}(h)} \right)^{1/\rho} - 1 \right\},$$

and

$$S_{n5} := \sqrt{k} \left\{ \frac{\rho}{1/\gamma_1 - \rho} - \frac{\left(\overline{F}(R)/\overline{F}(h) \right)^{1/\rho}}{(R/h)^{-1/(\rho\gamma_1)}} \int_1^\infty \left(\frac{\overline{F}(Rx)}{\overline{F}(R)} \right)^{1/\rho} dx \right\}.$$

We will represent the first three terms S_{ni}, $i = 1, 2, 3$, in terms of the Gaussian processes $\mathbf{B}_n$ and $\mathbf{B}_n^*$ and we will show that $S_{n4} \overset{\mathbb{P}}{\to} 0$ while S_{n5} converges to a deterministic limit. For the first term S_{n1}, we have $\widehat{\gamma}_1^{(H,c)} \overset{\mathbb{P}}{\to} \gamma_1$ (see Brahimi et al., 2015) and $Z_{n-k:n}/h \overset{\mathbb{P}}{\to} 1$, which, in view of the regular variation of $\overline{F}$, implies that $\overline{F}(Z_{n-k:n})/\overline{F}(h) \overset{\mathbb{P}}{\to} 1$. Moreover, from (19) we have $\overline{F}_n(Z_{n-k:n})/\overline{F}(Z_{n-k:n}) \overset{\mathbb{P}}{\to} 1$. It follows that $S_{n1} = S_{n1}^{(1)} + S_{n1}^{(2)}$, where

$$S_{n1}^{(1)} := (1 + o_{\mathbb{P}}(1)) \frac{\rho\gamma_1}{1 - \rho\gamma_1}$$

$$\times \sqrt{k} \left\{ \left(\frac{Z_{n-k:n}}{h} \right)^{1/\left(\rho\widehat{\gamma}_1^{(H,c)} \right)} - 1 \right\} \left(\left(\frac{R}{h} \right)^{1/\gamma_1 - 1/\widehat{\gamma}_1^{(H,c)}} \right)^{1/\rho},$$

and

$$S_{n1}^{(2)} := (1 + o_{\mathbb{P}}(1)) \frac{\rho\gamma_1}{1 - \rho\gamma_1} \sqrt{k} \left\{ \left(\left(\frac{R}{h} \right)^{1/\gamma_1 - 1/\widehat{\gamma}_1^{(H,c)}} \right)^{1/\rho} - 1 \right\}.$$

For $S_{n1}^{(1)}$, we use the mean value theorem, the consistency of $\widehat{\gamma}_1^{(H,c)}$ and the fact that $Z_{n-k:n}/h \overset{\mathbb{P}}{\to} 1$, to have

$$S_{n1}^{(1)} = (1 + o_{\mathbb{P}}(1)) \frac{1}{1 - \rho\gamma_1} \sqrt{k} \left(\frac{Z_{n-k:n}}{h} - 1 \right).$$

Next, we apply result (2.7) of Theorem 2.1 in Brahimi et al. (2015) to get

$$S_{n1}^{(1)} = (1 + o_{\mathbb{P}}(1)) \frac{\gamma}{1 - \rho\gamma_1} \sqrt{\frac{n}{k}} \mathbf{B}_n^*(h).$$

In view of the consistency and asymptotic normality of $\widehat{\gamma}_1^{(H,c)}$ and the assumption $R/h \to 1$, we show, by applying the mean value theorem twice, that $S_{n1}^{(2)} = o_{\mathbb{P}}(1)$. Thus, we end up with

$$S_{n1} = (1 + o_{\mathbb{P}}(1)) \frac{\gamma}{1 - \rho\gamma_1} \sqrt{\frac{n}{k}} \mathbf{B}_n^*(h) + o_{\mathbb{P}}(1). \tag{11}$$

By similar arguments and using the mean value theorem once again, we easily show that

$$S_{n2} = (1 + o_{\mathbb{P}}(1)) \frac{\rho}{(1 - \rho\gamma_1)^2} \sqrt{k} \left(\widehat{\gamma}_1^{(H,c)} - \gamma_1 \right),$$

$$S_{n3} = (1 + o_{\mathbb{P}}(1)) \frac{\gamma_1}{1 - \rho\gamma_1} \sqrt{k} \left(\frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} - 1 \right),$$

and

$$S_{n4} = (1 + o_{\mathbb{P}}(1)) \frac{\gamma_1}{1 - \rho\gamma_1} \sqrt{k} \left\{ \frac{\overline{F}(Z_{n-k:n})}{\overline{F}(h)} - 1 \right\}.$$

By applying result (2.9) of Theorem 2.1 in Brahimi et al. (2015) we get, after a change of variables, that

$$S_{n2} = (1 + o_{\mathbb{P}}(1)) \frac{\rho}{(1 - \rho\gamma_1)^2} \left\{ \frac{1}{p} \sqrt{\frac{n}{k}} \int_1^\infty v^{-1} \mathbf{B}_n^*(hv)\, dv - \frac{\gamma_1}{p} \sqrt{\frac{n}{k}} \mathbf{B}_n(h) \right\}$$
$$+ (1 + o_{\mathbb{P}}(1)) \frac{\rho \sqrt{k} A_1(h)}{(1 - p\tau_1)(1 - \rho\gamma_1)^2}. \tag{12}$$

From Proposition 2, we infer that

$$S_{n3} = (1 + o_{\mathbb{P}}(1)) \frac{\gamma_1}{1 - \rho\gamma_1} \left(\sqrt{\frac{n}{k}} \mathbf{B}_n(h) + \sqrt{\frac{k}{n}} \Delta_n \right) + o_{\mathbb{P}}(1). \tag{13}$$

Now, we decompose S_{n4} into the sum of two terms

$$S_{n4}^{(1)} := (1 + o_{\mathbb{P}}(1)) \frac{\gamma_1}{1 - \rho\gamma_1} \sqrt{k} \left\{ \frac{\overline{F}(Z_{n-k:n})}{\overline{F}(h)} - \left(\frac{Z_{n-k:n}}{h} \right)^{-1/\gamma_1} \right\},$$

and

$$S_{n4}^{(2)} := (1 + o_{\mathbb{P}}(1)) \frac{\gamma_1}{1 - \rho\gamma_1} \sqrt{k} \left\{ \left(\frac{Z_{n-k:n}}{h} \right)^{-1/\gamma_1} - 1 \right\}.$$

The second-order condition (4) of $\overline{F}$ and the fact that $Z_{n-k:n}/h \overset{\mathbb{P}}{\to} 1$ yield that

$$S_{n4}^{(1)} = o_{\mathbb{P}}\left(\sqrt{k} A_1(h) \right) = o_{\mathbb{P}}(1).$$

For $S_{n4}^{(2)}$, we, once again, apply the mean value theorem (with $Z_{n-k:n}/h \xrightarrow{\mathbb{P}} 1$) then we use result (2.7) of Theorem 2.1 in Brahimi et al. (2015) to get

$$S_{n4}^{(2)} = - (1 + o_{\mathbb{P}}(1)) \frac{\gamma}{1 - \rho\gamma_1} \sqrt{\frac{n}{k}} \mathbf{B}_n^*(h).$$

Consequently, we have

$$S_{n4} = - (1 + o_{\mathbb{P}}(1)) \frac{\gamma}{1 - \rho\gamma_1} \sqrt{\frac{n}{k}} \mathbf{B}_n^*(h) + o_{\mathbb{P}}(1). \tag{14}$$

For the last term S_{n5}, we start by decomposing it into the sum of

$$S_{n5}^{(1)} := -\frac{\rho\gamma_1}{1 - \rho\gamma_1} \frac{1}{(R/h)^{-1/(\rho\gamma_1)}} \sqrt{k} \left\{ \left(\frac{\overline{F}(R)}{\overline{F}(h)} \right)^{1/\rho} - \left(\left(\frac{R}{h} \right)^{-1/\gamma_1} \right)^{1/\rho} \right\},$$

and

$$S_{n5}^{(2)} := - \left(\frac{(\overline{F}(R)/\overline{F}(h))}{(R/h)^{-1/\gamma_1}} \right)^{1/\rho} \sqrt{k} \int_1^\infty \left(\left(\frac{\overline{F}(Rx)}{\overline{F}(R)} \right)^{1/\rho} - \left(x^{-1/\gamma_1} \right)^{1/\rho} \right) dx.$$

By similar arguments as those used for $S_{n4}^{(1)}$, we show that (here we use the assumption that $R/h \to 1$)

$$S_{n5}^{(1)} = o_{\mathbb{P}} \left(\sqrt{k} A_1(h) \right) = o_{\mathbb{P}}(1).$$

For $S_{n5}^{(2)}$, we first apply the mean value theorem to have

$$S_{n5}^{(2)} = -\frac{1}{\rho} \sqrt{k} \int_1^\infty \left(\frac{\overline{F}(Rx)}{\overline{F}(R)} - x^{-1/\gamma_1} \right) \zeta^{1/\rho - 1}(x) dx,$$

where ζ lies between $\overline{F}(Rx)/\overline{F}(R)$ and x^{-1/γ_1}. Then we use Potter's inequalities, given in assertion 5 of Proposition B.1.9 in de Haan and Ferreira (2006), to get

$$S_{n5}^{(2)} = (1 + o(1)) \frac{\sqrt{k} A_1(h)}{\rho(\gamma_1 + \tau_1 + \rho - 2)(2 - \rho - \gamma_1)}.$$

Therefore

$$S_{n5} = (1 + o(1)) \frac{\sqrt{k} A_1(h)}{\rho(\gamma_1 + \tau_1 + \rho - 2)(2 - \rho - \gamma_1)} + o_{\mathbb{P}}(1). \tag{15}$$

Finally, by gathering results (11), (12), (13), (14) and (15), we obtain the following asymptotic representation to the premium estimator:

$$\sqrt{k} \frac{\widehat{\Pi}_\rho(R) - \Pi_\rho(R)}{(R/h)^{-1/\rho\gamma_1} R \left(\overline{F}(h) \right)^{1/\rho}} = o_{\mathbb{P}}(1) + \frac{\gamma_1}{1 - \rho\gamma_1} \sqrt{\frac{k}{n}} \Delta_n + \frac{1}{1 - \rho\gamma_1} \sqrt{\frac{n}{k}} \Gamma_n \tag{16}$$

$$+ \left\{ \frac{\rho\sqrt{k} A_1(h)}{(1 - p\tau_1)(1 - \rho\gamma_1)^2} + \frac{\sqrt{k} A_1(h)}{\rho(\gamma_1 + \tau_1 + \rho - 2)(2 - \rho - \gamma_1)} \right\},$$

where Δ_n is as defined in 18 and

$$\Gamma_n := \gamma_1 \left(1 - \frac{\rho}{p \left(1 - \rho \gamma_1 \right)} \right) \mathbf{B}_n \left(h \right) + \frac{\rho}{p \left(1 - \rho \gamma_1 \right)} \int_1^\infty v^{-1} \mathbf{B}_n^* \left(hv \right) dv.$$

From (16), we deduce that $\sqrt{k} \left(\widehat{\Pi}_\rho (R) - \Pi_\rho (R) \right) / \left((R/h)^{-1/\rho \gamma_1} R \left(\overline{F} \left(h \right) \right)^{1/\rho} \right)$ is asymptotically Gaussian with mean

$$\left\{ \frac{\rho}{\left(1 - p \tau_1 \right) \left(1 - \rho \gamma_1 \right)^2} + \frac{1}{\rho \left(\gamma_1 + \tau_1 + \rho - 2 \right) \left(2 - \rho - \gamma_1 \right)} \right\} \lim_{n \to \infty} \sqrt{k} A_1 \left(h \right) = \mu,$$

and variance

$$\lim_{n \to \infty} \mathbf{E} \left[\frac{\gamma_1}{1 - \rho \gamma_1} \sqrt{\frac{k}{n}} \Delta_n + \frac{1}{1 - \rho \gamma_1} \sqrt{\frac{n}{k}} \Gamma_n \right]^2.$$

Note that from the covariance structure in Csörgő (1996), page 2768, we have the following useful formulas:

$$\begin{cases} \mathbf{E} \left[\mathbf{B}_n \left(u \right) \mathbf{B}_n \left(v \right) \right] = \min \left(\overline{H}^{(1)} \left(u \right), \overline{H}^{(1)} \left(v \right) \right) - \overline{H}^{(1)} \left(u \right) \overline{H}^{(1)} \left(v \right), \\ \mathbf{E} \left[\mathbf{B}_n^* \left(u \right) \mathbf{B}_n^* \left(v \right) \right] = \min \left(\overline{H} \left(u \right), \overline{H} \left(v \right) \right) - \overline{H} \left(u \right) \overline{H} \left(v \right), \\ \mathbf{E} \left[\mathbf{B}_n \left(u \right) \mathbf{B}_n^* \left(v \right) \right] = \min \left(\overline{H}^{(1)} \left(u \right), \overline{H}^{(1)} \left(v \right) \right) - \overline{H}^{(1)} \left(u \right) \overline{H} \left(v \right). \end{cases} \quad (17)$$

After elementary but very tedious computations, using these formulas with l'Hôpital's rule, we get as $n \to \infty$,

$$\int_0^h \frac{\mathbf{E} \left[\mathbf{B}_n \left(u \right) \mathbf{B}_n \left(h \right) \right]}{\overline{H}^2 \left(u \right)} d\overline{H} \left(u \right) \to -p, \quad \int_0^h \frac{\mathbf{E} \left[\mathbf{B}_n \left(h \right) \mathbf{B}_n^* \left(u \right) \right]}{\overline{H}^2 \left(u \right)} d\overline{H}^{(1)} \left(u \right) \to -p^2,$$

$$\int_0^h \int_1^\infty \frac{\mathbf{E} \left[\mathbf{B}_n \left(v \right) \mathbf{B}_n^* \left(hu \right) \right]}{u \overline{H}^2 \left(v \right)} du \, d\overline{H} \left(v \right) \to -p\gamma,$$

$$\int_0^h \int_1^\infty \frac{\mathbf{E} \left[\mathbf{B}_n^* \left(v \right) \mathbf{B}_n^* \left(hu \right) \right]}{u \overline{H}^2 \left(v \right)} du \, d\overline{H}^{(1)} \left(v \right) \to -p\gamma,$$

$$\frac{k}{n} \int_0^h \int_0^h \frac{\mathbf{E} \left[\mathbf{B}_n \left(u \right) \mathbf{B}_n \left(v \right) \right]}{\overline{H}^2 (u) \overline{H}^2 \left(v \right)} d\overline{H}(u) d\overline{H} \left(v \right) \to 2p,$$

$$\frac{k}{n} \int_0^h \int_0^h \frac{\mathbf{E} \left[\mathbf{B}_n^* \left(u \right) \mathbf{B}_n^* \left(v \right) \right]}{\overline{H}^2 (u) \overline{H}^2 \left(v \right)} d\overline{H}^{(1)}(u) d\overline{H}^{(1)} \left(v \right) \to 2p^2,$$

and

$$\frac{k}{n} \int_0^h \int_0^h \frac{\mathbf{E} \left[\mathbf{B}_n \left(u \right) \mathbf{B}_n^* \left(v \right) \right]}{\overline{H}^2 (u) \overline{H}^2 \left(v \right)} d\overline{H}(u) d\overline{H}^{(1)} \left(v \right) \to 2p^2,$$

Using the results above with some further calculations leads to σ^2. $\square$

5. Appendix

The following proposition consists in Corollary 2.1 of Csörgő et al. (1986).

Proposition 1. *There exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with independent $(0, 1)$-uniform rv's U_1, U_2,... and a sequence of Brownian bridges $\{B_i(s);\ 0 \leq s \leq 1\}$ $(i = 1, 2, ...)$ such that, for every $0 < \lambda < \infty$, we have as $n \to \infty$*

$$\sup_{\lambda/n \leq s \leq 1} \frac{n^\zeta \left|\alpha_n(s) - B_n(s)\right|}{s^{1/2-\zeta}} = \begin{cases} O_{\mathbb{P}}(\log n) & when\ \zeta = \dfrac{1}{4}, \\ O_{\mathbb{P}}(1) & when\ 0 \leq \zeta < \dfrac{1}{4}, \end{cases}$$

$$\sup_{0 \leq s \leq 1-\lambda/n} \frac{n^\zeta \left|\alpha_n(s) - B_n(s)\right|}{(1-s)^{1/2-\zeta}} = \begin{cases} O_{\mathbb{P}}(\log n) & when\ \zeta = \dfrac{1}{4}, \\ O_{\mathbb{P}}(1) & when\ 0 \leq \zeta < \dfrac{1}{4}, \end{cases}$$

and

$$\sup_{\lambda/n \leq s \leq 1-\lambda/n} \frac{n^\zeta \left|\alpha_n(s) - B_n(s)\right|}{(s(1-s))^{1/2-\zeta}} = \begin{cases} O_{\mathbb{P}}(\log n) & when\ \zeta = \dfrac{1}{4}, \\ O_{\mathbb{P}}(1) & when\ 0 \leq \zeta < \dfrac{1}{4}. \end{cases}$$

Proof. See Csörgő et al. (1986), page 48.

In the next basic proposition, we provide an asymptotic representation to the Kaplan-Meier product limit estimator in $Z_{n-k:n}$. This result will be of prime importance in the study of the limiting behaviors of many statistics based on censored data exhibiting extreme values.

Proposition 2. *Assume that all second-order conditions* (4) *hold. Let $k = k_n$ be an integer sequence satisfying, in addition to* (3) $\sqrt{k}A_j(h) = O(1)$, *for $j = 1, 2$, as $n \to \infty$. Then there exists a sequence of Brownian bridges $\{B_n(s);\ 0 \leq s \leq 1\}$ such that*

$$\sqrt{k}\left\{\frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} - 1\right\} = \sqrt{\frac{n}{k}}\mathbf{B}_n(h) + \sqrt{\frac{k}{n}}\Delta_n + o_{\mathbb{P}}(1),$$

where

$$\Delta_n := \int_0^h \frac{\mathbf{B}_n(v)}{\overline{H}^2(v)}d\overline{H}(v) - \int_0^h \frac{\mathbf{B}_n^*(v)}{\overline{H}^2(v)}d\overline{H}^{(1)}(v), \tag{18}$$

with $\mathbf{B}_n(v)$ and $\mathbf{B}_n^*(v)$ respectively defined in (9) *and* (10). *Consequently,*

$$\sqrt{k}\left\{\frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} - 1\right\} \xrightarrow{d} \mathcal{N}(0, p(1-p)), \ as\ n \to \infty, \tag{19}$$

Proof. In view of Proposition 5 of Csörgő (1996), combined with equation (4.9) in the same reference, we have for any $x \leq Z_{n-k:n}$,

$$\frac{\overline{F}_n(x) - \overline{F}(x)}{\overline{F}(x)}$$
$$= \int_0^x \frac{d\left(\overline{H}_n^{(1)}(v) - \overline{H}^{(1)}(v)\right)}{\overline{H}(v)} - \int_0^x \frac{\overline{H}_n(v) - \overline{H}(v)}{\overline{H}^2(v)}d\overline{H}^{(1)}(v) + O_{\mathbb{P}}\left(\frac{1}{k}\right).$$

Upon integrating the first integral by parts, we get

$$\frac{\overline{F}_n(x) - \overline{F}(x)}{\overline{F}(x)} \tag{20}$$

$$= -\left(\overline{H}_n^{(1)}(0) - \overline{H}^{(1)}(0)\right) + \frac{\overline{H}_n^{(1)}(x) - \overline{H}^{(1)}(x)}{\overline{H}(x)}$$

$$+ \int_0^x \frac{\overline{H}_n^{(1)}(v) - \overline{H}^{(1)}(v)}{\overline{H}^2(v)} d\overline{H}(v) - \int_0^x \frac{\overline{H}_n(v) - \overline{H}(v)}{\overline{H}^2(v)} d\overline{H}^{(1)}(v) + O_{\mathbb{P}}\left(\frac{1}{k}\right).$$

Recall that

$$\sqrt{n}\left(\overline{H}_n(v) - \overline{H}(v)\right) = \sqrt{n}\left(\overline{H}_n^1(v) - \overline{H}^1(v)\right) + \sqrt{n}\left(\overline{H}_n^0(v) - \overline{H}^0(v)\right),$$

which by representations (5) and (6) becomes

$$\sqrt{n}\left(\overline{H}_n(v) - \overline{H}(v)\right) = \left(\alpha_n(\theta) - \alpha_n\left(\theta - \overline{H}^{(1)}(v)\right)\right) - \alpha_n\left(1 - \overline{H}^{(0)}(v)\right).$$

On the other hand, by the classical central limit theorem, we have $\overline{H}_n^{(1)}(0) - \overline{H}^{(1)}(0) = O_{\mathbb{P}}\left(n^{-1/2}\right)$. Using these results in (20) and then multiplying by $\sqrt{k}$, we get

$$\sqrt{k}\frac{\overline{F}_n(x) - \overline{F}(x)}{\overline{F}(x)}$$

$$= O_{\mathbb{P}}\left(\sqrt{\frac{k}{n}}\right) + O_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\right) + \sqrt{\frac{k}{n}}\frac{\alpha_n(\theta) - \alpha_n\left(\theta - \overline{H}^{(1)}(x)\right)}{\overline{H}(x)}$$

$$+ \sqrt{\frac{k}{n}}\int_0^x \frac{\alpha_n(\theta) - \alpha_n\left(\theta - \overline{H}^{(1)}(v)\right)}{\overline{H}^2(v)} d\overline{H}(v)$$

$$- \sqrt{\frac{k}{n}}\int_0^x \frac{\alpha_n(\theta) - \alpha_n\left(\theta - \overline{H}^{(1)}(v)\right) - \alpha_n\left(1 - \overline{H}^{(0)}(v)\right)}{\overline{H}^2(v)} d\overline{H}^{(1)}(v).$$

The Gaussian approximations (7) and (8), in $x = Z_{n-k:n}$, and the facts that $\sqrt{k/n}$ and $1/\sqrt{k}$ tend to zero as $n \to \infty$, lead to

$$\sqrt{k}\frac{\overline{F}_n(Z_{n-k:n}) - \overline{F}(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} = \sqrt{\frac{n}{k}}\mathbf{B}_n(Z_{n-k:n})$$

$$+ \sqrt{\frac{k}{n}}\int_0^{Z_{n-k:n}} \frac{\mathbf{B}_n(v)}{\overline{H}^2(v)} d\overline{H}(v) - \sqrt{\frac{k}{n}}\int_0^{Z_{n-k:n}} \frac{\mathbf{B}_n^*(v)}{\overline{H}^2(v)} d\overline{H}^{(1)}(v) + o_{\mathbb{P}}(1).$$

Applying Lemma 1 completes the proof. The asymptotic normality property is straightforward. For the variance computation, we use the covariance formulas (17) and the results at the end of Section 4.

Lemma 1. *Assume that the second-order conditions of regular variation* (4) *and let* $k := k_n$ *be an integer sequence satisfying* (3). *Then*

$$(i) \quad \sqrt{\frac{k}{n}} \int_h^{Z_{n-k:n}} \frac{\mathbf{B}_n(v)}{\overline{H}^2(v)} d\overline{H}(v) = o_{\mathbb{P}}(1).$$

$$(ii) \quad \sqrt{\frac{k}{n}} \int_h^{Z_{n-k:n}} \frac{\mathbf{B}_n^*(v)}{\overline{H}^2(v)} d\overline{H}^{(1)}(v) = o_{\mathbb{P}}(1).$$

$$(iii) \quad \sqrt{\frac{n}{k}} \left\{ \mathbf{B}_n(Z_{n-k:n}) - \mathbf{B}_n(h) \right\} = o_{\mathbb{P}}(1)$$

$$(iv) \quad \sqrt{\frac{n}{k}} \left\{ \mathbf{B}_n^*(Z_{n-k:n}) - \mathbf{B}_n^*(h) \right\} = o_{\mathbb{P}}(1).$$

Proof. We begin by proving the first assertion. For fixed $0 < \eta, \varepsilon < 1$, we have

$$\mathbb{P}\left(\left| \sqrt{\frac{k}{n}} \int_h^{Z_{n-k:n}} \mathbf{B}_n(v) \frac{d\overline{H}(v)}{\overline{H}^2(v)} \right| > \eta \right)$$
$$\leq \mathbb{P}\left(\left| \frac{Z_{n-k:n}}{h} - 1 \right| > \varepsilon \right) + \mathbb{P}\left(\left| \sqrt{\frac{k}{n}} \int_h^{(1+\varepsilon)h} \mathbf{B}_n(v) \frac{d\overline{H}(v)}{\overline{H}^2(v)} \right| > \eta \right).$$

It is clear that the first term the right-hand side tends to zero as $n \to \infty$. Then, it remains to show that the second one goes to zero as well. Indeed, observe that

$$\mathbf{E} \left| \sqrt{\frac{k}{n}} \int_h^{(1+\varepsilon)h} \mathbf{B}_n(v) \frac{d\overline{H}(v)}{\overline{H}^2(v)} \right| \leq -\sqrt{\frac{k}{n}} \int_h^{(1+\varepsilon)h} \mathbf{E} \left| \mathbf{B}_n(v) \right| \frac{d\overline{H}(v)}{\overline{H}^2(v)}.$$

From the first result of (17), we have $\mathbf{E} \left| \mathbf{B}_n(v) \right| \leq \sqrt{\overline{H}^1(v)}$. Then

$$\mathbf{E} \left| \sqrt{\frac{k}{n}} \int_h^{(1+\varepsilon)h} \mathbf{B}_n(v) \frac{d\overline{H}(v)}{\overline{H}^2(v)} \right| \leq -\sqrt{\frac{k}{n}} \int_h^{(1+\varepsilon)h} \sqrt{\overline{H}^1(v)} \frac{d\overline{H}(v)}{\overline{H}^2(v)},$$

which, in turn, is less than or equal to

$$\sqrt{\frac{k}{n}} \sqrt{\overline{H}^{(1)}(h)} \left(\frac{1}{\overline{H}((1+\varepsilon)h)} - \frac{1}{\overline{H}(h)} \right).$$

Since $\overline{H}(h) = k/n$, then this may be rewritten into

$$\sqrt{\frac{\overline{H}^{(1)}(h)}{\overline{H}(h)}} \left(\frac{\overline{H}(h)}{\overline{H}((1+\varepsilon)h)} - 1 \right).$$

Since $\overline{H}^{(1)}(h) \sim p\overline{H}(h)$ and $\overline{H} \in \mathcal{RV}_{(-1/\gamma)}$, then the previous quantity tends to $p^{1/2} \left((1+\varepsilon)^{1/\gamma} - 1 \right)$ as $n \to \infty$. Being arbitrary, ε may be chosen small enough so that this limit be zero. By similar arguments, we also show assertion (ii), therefore we omit the

details. The last two assertions are shown following the same technique, that we use to prove (iv). Notice that, from the definition of $\mathbf{B}_n^*(v)$ and the second covariance formula in (17), we have

$$\{\mathbf{B}_n^*(v); \ v \geq 0\} \overset{d}{=} \{\mathcal{B}_n(\overline{H}(v)); \ v \geq 0\},$$

where $\{\mathcal{B}_n(s); \ 0 \leq s \leq 1\}$ is a sequence of standard Brownian bridges. Hence

$$\sqrt{\frac{n}{k}} \{\mathbf{B}_n^*(Z_{n-k:n}) - \mathbf{B}_n^*(h)\} \overset{d}{=} \sqrt{\frac{n}{k}} \{\mathcal{B}_n(\overline{H}(Z_{n-k:n})) - \mathcal{B}_n(\overline{H}(h))\}.$$

Let $\{\mathcal{W}_n(t); \ 0 \leq s \leq 1\}$ be a sequence of standard Wiener processes such that $\mathcal{B}_n(t) = \mathcal{W}_n(t) - t\mathcal{W}_n(1)$. Then $\sqrt{n/k}\{\mathbf{B}_n^*(Z_{n-k:n}) - \mathbf{B}_n^*(h)\}$ is equal in distribution to

$$\sqrt{\frac{n}{k}} \left(\{\mathcal{W}_n(\overline{H}(Z_{n-k:n})) - \mathcal{W}_n(\overline{H}(h))\} - \{\overline{H}(Z_{n-k:n}) - \overline{H}(h)\} \mathcal{W}_n(1) \right).$$

By using the facts that $\overline{H}(h) = k/n$ and $\overline{H}(Z_{n-k:n})/\overline{H}(h) \approx 1$, we get

$$\sqrt{\frac{n}{k}} \left(\overline{H}(Z_{n-k:n}) - \overline{H}(h) \right) = \sqrt{\frac{k}{n}} \left(\frac{\overline{H}(Z_{n-k:n})}{\overline{H}(h)} - 1 \right) = o_{\mathbb{P}}(1).$$

Now, we prove that

$$\vartheta_n := \sqrt{\frac{n}{k}} \{\mathcal{W}_n(\overline{H}(Z_{n-k:n})) - \mathcal{W}_n(\overline{H}(h))\} = o_{\mathbb{P}}(1).$$

To this end, we show that $\mathbb{P}(|\vartheta_n| > \eta) \to 0$, for any fixed real number $\eta > 0$. Since $Z_{n-k:n}/h \overset{\mathbb{P}}{\to} 1$, then for arbitrary $\epsilon > 0$ and sufficiently large n, the probability of $A_n(\epsilon) := \{|Z_{n-k:n}/h - 1| \leq \epsilon\}$ is close to 1. Next, we use the following useful inequality: $\mathbb{P}(|\vartheta_n| > \eta) \leq \mathbb{P}\{|\vartheta_n| > \eta, A_n(\epsilon)\} + \mathbb{P}\{A_n^c(\epsilon)\}$, with $A_n^c(\epsilon)$ denoting the complement set of $A_n(\epsilon)$. It is easy to verify that ϑ_n may be rewritten into

$$\frac{\mathcal{W}_n(\overline{H}(h)\xi_n + \overline{H}(h)) - \mathcal{W}_n(\overline{H}(h))}{\sqrt{\overline{H}(h)}},$$

where $\xi_n := \overline{H}(Z_{n-k:n})/\overline{H}(h) - 1$. Thereby $\mathbb{P}(|\vartheta_n| > \eta) \leq I_n + \mathbb{P}\{A_n^c(\epsilon)\}$, where

$$I_n := \mathbb{P}\left(\sup_{0 \leq t \leq \overline{H}(h)\xi_n} |\mathcal{W}_n(t + \overline{H}(h)) - \mathcal{W}_n(\overline{H}(h))| > \eta\sqrt{\overline{H}(h)}, A_n(\epsilon) \right).$$

Since $\overline{H}$ is regularly varying at infinity, then we readily show that, in the set $A_n(\epsilon)$, we have $|\xi_n| \leq \epsilon$. Note that, for a fixed $0 \leq s \leq 1$, we have

$$\{\mathcal{W}_n(t + s) - \mathcal{W}_n(s); \ 0 \leq t \leq 1 - s\} \overset{d}{=} \{\mathcal{W}_n(t); \ 0 \leq t \leq 1 - s\}.$$

It follows that $I_n = \mathbb{P}\left(\sup_{0 \leq t \leq \epsilon\overline{H}(h)} |\mathcal{W}_n(t)| > \eta\sqrt{\overline{H}(h)} \right)$ which, by Doob's martingale inequality, satisfies

$$I_n \leq \frac{\mathbf{E}|\mathcal{W}_n(\epsilon\overline{H}(h))|}{\eta\sqrt{\overline{H}(h)}}.$$

Since $\mathbf{E}\left|\mathcal{W}_n\left(\epsilon\overline{H}\left(h\right)\right)\right| \le \sqrt{\epsilon\overline{H}\left(h\right)}$ and $\mathbb{P}\left\{A_n^c\left(\epsilon\right)\right\} < \epsilon$, then $\mathbb{P}\left(\left|\vartheta_n\right| > \eta\right) \le \eta^{-1}\epsilon^{1/2} + \epsilon$ which tends to zero as $\epsilon \downarrow 0$, as sought.

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