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Bivariate copulas parameters estimation using the trimmed L-moments method

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Abstract. The main purpose of this paper is to use the trimmed L-moments method for the introduction of a new estimator of multi-parametric copulas in the case where the mean does not exist. The consistency and asymptotic normality of this estimator is established. An extended simulation study shows the performance of the new estimator is carried.

Résumé. Le but principal de ce papier est d'utiliser la méthode des trimmed L-moments pour l'introduction d'un nouvel estimateur des copules multi-paramétriques dans le cas où la moyenne n'existe pas. La consistance et la normalité asymptotique de cet estimateur sont établies. Une étude de simulation étendue, qui montre la performance du nouvel estimateur, est menée.

Key words: Copulas; Dependence; Bivariate L-moments; Trimmed L-moments; Trimmed L-comoments.
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1. Introduction and motivation

Let (X_1, X_2) be a 2-dimensional vector with joint distribution function (df) $H(x_1, x_2)$ and margins $F_j(x_j)$, $j = 1, 2$. Under the Theorem of Sklar (1959) we can link F and the F_j's by a function C called copula, which is defined from $[0,1]^2$ to $[0,1]$ as follows

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)),$$

where the Copula C is the joint df with uniform margins $U = (U_1, U_2)$ with $U_j = F_j(X_j)$, defined by

$$C(u_1, u_2) = H\left((F_1^{-1}(u_1), F_2^{-1}(u_2)\right),$$

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where $F_j^{-1}(s) := \inf\{x : F_j(x) \geq s\}$ is the generalized inverse function of $F_j, j = 1, 2$.

The copula function also describes and models the dependence structure of a multivariate data set. It characterizes many properties as the symmetry and the invariance transform. The importance of these two properties appears in measuring of association such as Kendall's tau and Spearman's rho written in terms of copula, by

$$\tau = 4 \int_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1,$$
$$\rho = 12 \int_{[0,1]^2} u_1 u_2 dC(u_1, u_2) - 3. \tag{1}$$

Many parametric copula families have been introduced and applied in different fields such as insurance, medical science, hydrology and survival analysis (see, e.g., Frees and Valdez, 1998, Cui and Sun, 2004 and Genest and Favre, 2007). Among these families, we have the Archimedean copula class which has been named by Ling (1965), they have found many successful applications like the actuarial and survey actuarial applications, in finance (Clayton, 1978, Oakes, 1982, Cook and Johnson, 1981). This class of copulas has a nice properties, as: the ease with which it can be constructed; the great variety of families of copulas which it contains (see, Nelsen, 2006, p.109). A bivariate Archimedean copula is defined by

$$C(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2)),$$

where φ is a positive continuous function, strictly decreasing on $[0, 1]$ called generator with pseudo inverse φ^{-1}.

Many families of Archimedean copulas are cited in (Nelsen, 2006, Table 4.1, p.116-p119) such as Gumbel copula, defined by

$$C_\beta(u_1, u_2) = \exp(-\left[(-\ln u_1)^\beta + (-\ln u_2)^\beta\right]^{1/\beta}), \ \beta \geq 1, \tag{2}$$

with generator

$$\varphi(s) = (-\ln s)^\alpha \text{ and } \varphi^{-1}(s) = \exp\left(-s\right)^{1/\alpha}.$$

Also, a very popular applied in engineering and medical fields called the FGM copulas (see Blischke and Prabhakar, 2000). For a dependence parameter α with $|\alpha| \leq 1$ it is defined by

$$C_\alpha(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 \bar{u}_1 \bar{u}_2, \tag{3}$$

where $\bar{u}_j = 1 - u_j, j = 1, 2$, Johnson *et al.* (1979) introduced the $(r-1)$-iterated FGM family with r-dimensional parameter $\alpha = (\alpha_1, ..., \alpha_r)$ as

$$C_\alpha(u_1, u_2) = u_1 u_2 + \sum_{j=1}^{r} \alpha_j (u_1 u_2)^{[j/2]+1} (\bar{u}_1 \bar{u}_2)^{[j/2+1/2]} \tag{4}$$

where $[t]$ denotes the integer part of t, for $r = 2$, we obtain one iterated FGM copula with two parameters, given by

$$C_{\alpha_1,\alpha_2}(u_1, u_2) = u_1 u_2 (1 + \alpha_1 \bar{u}_1 \bar{u}_2 + \alpha_2 u_1 u_2 \bar{u}_1 \bar{u}_2), \tag{5}$$

and the range of parameters (α_1, α_2) is given by the region

$$\mathcal{R} = \left\{ (\alpha_1, \alpha_2), |\alpha_1| \leq 1, \alpha_1 + \alpha_2 \geq -1, \alpha_2 \leq \frac{1}{2} \left[3 - \alpha_1 + (9 - 6\alpha_1 - 3\alpha_1^2)^{1/2} \right] \right\}.$$

Suppose that the parametric copula C belongs to a class $\mathcal{C}$ where $\mathcal{C} := \{C_\theta : \theta \in \mathcal{O}\}$ and $\mathcal{O}$ is an open subset of $\mathbb{R}^r$ for $r \geq 1$. The problem of estimating θ under this assumption has already been the object of much work, beginning with classical methods: fully maximum likelihood (ML), Pseudo maximum likelihood (PML) and Inference function of margins (IFM) (see Genest, 1987, Joe, 2005). (τ, ρ)-inversion methods Oakes (1982), Genest et al. (1995). Minimum distance (MD) (see Tsukahara, 2005, Biau and Wegkamp, 2005) which is based on: the empirical copula process, Kendall's dependence function which is proposed by Genest et al. (2006) and Rosenblatt's probability integral transform proposed by Rosenblatt (1952). Many comparative studies between these methods were discussed in the literature such as in Kim et al., 2007) and Gregor (2009). Semi parametric estimation methods for multi-parametric copulas were also discussed by Brahimi and Necir (2012), Benatia et al. (2011) and Brahimi et al. (2014) based on moments (CM) and copula L-moments (CLM). They noted that these methods are quick and dos not use the density function and therefore no boundary problems arise. In a comparative simulation study, they concluded that the PML and the CM based estimation perform better than the(τ, ρ)-inversion method and the main feature of CM and CLM methods is that they provide estimators with explicit forms.

The aim of this paper is to estimate the dependence and the marginals parameters using a new representation of TL-moments. This method is analogous to bivariate L-comoment method where the largest value is removed from the conceptual sample to study its influence on bias and root mean squared error (RMSE).

The outline of the paper is as follows. In Section 2 we present a brief introduction of L-moments and L-comoments, and we discuss the representation of bivariate L-moments in terms of copula and by analogy we presents bivariate Trimmed L-moments. Section 3 consecrated to the parameter estimation procedures, and an illustrative examples with simulation study. Consistency and asymptotic normality is relegated to Section 4.

2. Bivariate Trimmed L-comoments

The L-moments play an important role to describe the characteristics of a probability distribution as: location, scale and shape. They are related to expected values of order statistics and was first introduced and defined by Hosking (1990). For $Y_{1:r}, ...Y_{r,r}$ denoting the ordered observations for a sample of size r from a univariate distribution, the rth L-moment λ_r is defined as

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathbb{E}(Y_{r-k:r}), \tag{6}$$

where

$$\mathbb{E}(Y_{r-k:r}) = \frac{r!}{(r-k-1)!k!} \int_{[0,1]} F_Y^{-1}(u) u^{r-k-1} (1-u)^k du. \tag{7}$$

By Substitution into (6) of a standard expression for the expected value of an order statistic (7) yields the follow representation of L-moments

$$\lambda_r = \int_{[0,1]} F_Y^{-1}(u) P_{r-1}(u) dFu, \tag{8}$$

where $P_r(u) := \sum_{k=0}^{r} p_{r,k} u^k$, with $p_{r,k} = (-1)^{r+k}(r+k)!/[(k^2)!(r-k)!]$, presents the shifted Legendre polynomials. The orthogonality of P_{k-1} and using $P_0 \equiv 1$ leads to a representation of (8) in terms of covariance

$$\lambda_r = \begin{cases} \mathbb{E}[Y], & r = 1 \\ Cov(Y, P_{r-1}(F_Y(y))), & r \geq 2 \end{cases} \tag{9}$$

Hosking (2007) showed that L-moments have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. However, there are some applications for which these advantages are insufficient. Some kinds of data, such as loss distributions in insurance and traffic volumes on computer networks, involve distributions with very heavy tails, such that there may be doubts about whether even the first moment exists. For these applications, it would be useful to have measures analogous to L-moments that remain meaningful for distributions that have no mean. This measure is the TL-moments, witch is defined by Elamir and Seheult (2003) as a generalization of L-moments where they replace the expected value $\mathbb{E}[Y_{r-k:r}]$ by $\mathbb{E}[Y_{r+t_1-k:r+t_1+t_2}]$. Thus TL-moments noted, $\lambda_r^{(t_1,t_2)}$ are given as follows

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \frac{(r-1)!}{k!(r-k-1)!}\mathbb{E}[Y_{r+t_1-k:r+t_1+t_2}], r = 1, 2, ... \tag{10}$$

where t_1 and t_2 are positive integers. The case $t_1 = t_2 = 0$ yields the original L-moments. An analogous result for trimmed L-moments presented by Hosking (2007), by using shifted Jacobi polynomials, so (10) may be written as

$$\lambda_r^{(t_1,t_2)} = \int_{[0,1]} F_Y^{-1}(u) P_{r-1}^{*(t_1,t_2)}(u) du, \tag{11}$$

where

$$P_{r-1}^{*(t_1,t_2)}(u) = \sum_{k=0}^{r-1}(-1)^k \frac{(r-1)!(r+t_1+t_2)!}{rk!(r-k-1)!(r+t_1-k-1)!(t_2+k)!} u^{r+t_1-k-1}\{1-u\}^{t_2+k},$$

also shifted Jacobi polynomials are orthogonal on $[0,1]$ with weight function $u^{t_1}(1-u)^{t_2}$.

For $r = 1, 2$, 3 and $t_1 = 0, t_2 = 1$, we get

$$\begin{aligned}
P_0^{*(0,1)}(u) &= 2(1-u), \\
P_1^{*(0,1)}(u) &= (3/2)(4u - 3u^2 - 1), \\
P_2^{*(0,1)}(u) &= \frac{4}{3}(-10u^3 + 18u^2 - 9u + 1), \\
P_3^{*(0,1)}(u) &= \frac{5}{4}(-35u^4 + 80u^3 - 60u^2 + 16u - 1),
\end{aligned}$$

and the first TL-moments are given as follows

$$\begin{aligned}
\lambda_1^{(0,1)} &= 2\int_{[0,1]} F_Y^{-1}(u)(1-u)du, \\
\lambda_2^{(0,1)} &= (3/2)\int_{[0,1]} F_Y^{-1}(u)(4u-3u^2-1)du \\
\lambda_3^{(0,1)} &= (4/3)\int_{[0,1]} F_Y^{-1}(u)(-10u^3+18u^2-9u+1)du
\end{aligned} \tag{12}$$

The multivariate L-moments are discussed by (Serfling and Xiao, 2007) and a new representation in terms of copula are given by Brahimi et al. (2014). We present in the following bivariate case and all results are considered for bivariate random variables.

Let $\left(X^{(1)}, X^{(2)}\right)$ have a joint distribution function F and marginals F_1, F_2, with finite mean and L-moment sequences $\left\{\lambda_r^{(1)}\right\}$ and $\left\{\lambda_r^{(2)}\right\}$. Using the covariance representation for L-moments in (9) the rth L-comoment of $X^{(1)}$ with respect to $X^{(2)}$ is defined by

$$\lambda_{r[12]} = Cov(X^{(1)}, P_r(F_2(X^{(2)})) \tag{13}$$

and by the same the rth L-comoment of $X^{(2)}$ with respect to $X^{(1)}$ is defined by

$$\lambda_{r[21]} = Cov(X^{(2)}, P_r(F_1(X^{(1)}).$$

For $X^{(1)} = X^{(2)}$, yields the L-moments λ_r and when $X^{(1)}, X^{(2)}$ are independent, $\lambda_{r[21]} = 0$, all $r \geq 2$. Brahimi et al. (2014) showed that if F belongs to a parametric family of dfs then the rth L-comoment $\lambda_{r[12]}$ will depend on the parameters relies the marginals and the dependence structure between $X^{(1)}$ and $X^{(2)}$, they gave a new representation of L-comoments, depend only on the estimation of parameter dependence, so they defined the L-comoments in terms of marginals distribution function $F(X^{(j)}), j = 1, 2$. Then the rth L-comoments $\delta_{r[12]}$ of $F(X^{(1)})$ with respect to $X^{(2)}$ is given as follows

$$\delta_{r[12]} = Cov(F(X^{(1)}), P_r(F_2(X^{(2)})), \qquad r = 1, 2, ...$$

and the rth L-comoments $\delta_{r[21]}$ of $F(X^{(2)})$ with respect to $X^{(1)}$ is given as follows

$$\delta_{r[21]} = Cov(F(X^{(2)}), P_r(F_2(X^{(1)})), \qquad r = 1, 2, ...$$

This representation leads to the following representation in terms of copula C, according (Theorem 3.1 Brahimi et al., 2014) where

$$\delta_{r[12]} = \int_{\mathbb{I}^2} (C(u_1, u_2) - u_1 u_2) du_1 P_r(u_2), \qquad r = 1, 2, ... \tag{14}$$

As example, the first bivariate copula L-comoments of $X^{(1)}$ with respect to $X^{(2)}$ for copula C_θ with three parameters $(\theta_1, \theta_2, \theta_3)$ are

$$\begin{aligned}
\delta_{1[12]} &= 2\int_{\mathbb{I}^2} C_\theta(u_1, u_2)du_1 du_2 - \frac{1}{2} \\
\delta_{2[12]} &= 6\int_{\mathbb{I}^2} (2u_2 - 1)C_\theta(u_1, u_2)du_1 du_2 - \frac{1}{2} \\
\delta_{3[12]} &= \int_{\mathbb{I}^2} (60u_2^2 - 60u_2 + 12)C_\theta(u_1, u_2)du_1 du_2 - \frac{1}{2}
\end{aligned} \tag{15}$$

observe that the $\lambda_{r[12]}$, $\lambda_{r[21]}$ exist under two conditions as (Serfling and Xiao, 2007) showed: the first one is the existence of mean ($\mathbb{E}(X^{(j)} < \infty$) and the second the existence of all L-moments so in that case we can not apply it on distributions with infinite means as Cauchy distribution where its mean is not finite and the L-moments λ_r exist for $r \geq 2$. In this paper we propose an alternative modification of (13) in which $P_r(F_j(X^{(j)})$ is replaced by $P^{*(t_1,t_2)}_{r-1}(F_j(X^{(j)})$, $j = 1, 2$. Then the representation of $\lambda^{(t_1,t_2)}_{r[12]}$, $\lambda^{(t_1,t_2)}_{r[21]}$ defined in the following proposition.

Proposition 1. *Let $\left(X^{(1)}, X^{(2)}\right)$ have a joint distribution function F belongs to a parametric family of df's and marginals F_1, F_2. Using the covariance representation for TL-moments in (9), so the rth TL-comoment of $X^{(1)}$ with respect to $X^{(2)}$ is defined by*

$$\lambda^{(t_1,t_2)}_{r[12]} = \int_{\mathbb{I}^2} F_1^{-1}(u_1) P^{*(t_1,t_2)}_{r-1}(u_2) dC(u_1, u_2). \tag{16}$$

and the rth L-comoment $\lambda^{(t_1,t_2)}_{r[21]}$ of $F(X^{(2)})$ with respect to $X^{(1)}$ is defined as follows

$$\lambda^{(t_1,t_2)}_{r[21]} = \int_{\mathbb{I}^2} F_2^{-1}(u_2) P^{*(t_1,t_2)}_{r-1}(u_1) dC(u_1, u_2). \tag{17}$$

Calculating the first TL-comoments for $t_1 = 0, t_2 = 1$, we obtain

$$\begin{aligned}
\lambda^{(0,1)}_{1[12]} &= 2 \int_{\mathbb{I}^2} F^{-1}(u_1)(1 - 2u_2) dC(u_1, u_2), \\
\lambda^{(0,1)}_{2[12]} &= (3/2) \int_{\mathbb{I}^2} F^{-1}(u_1)(4u_2 - 3u_2^2 - 1) dC(u_1, u_2) \\
\lambda^{(0,1)}_{3[12]} &= (4/3) \int_{\mathbb{I}^2} F^{-1}(u_1)(-10u_2^3 + 18u_2^2 - 9u_2 + 1) dC(u_1, u_2)
\end{aligned} \tag{18}$$

3. Illustrative example and simulation study

As an illustrative example, we choose the FGM copula (3) and (4) and Archimedean copulas (Gumbel copula given in (2)) with two different marginals: Cauchy and Pareto laws.

3.1. FGM-copulas with Cauchy marginals

Let X, Y two random variables with two parameters Cauchy distribution μ and σ, then their distribution functions are defined by

$$F_{\mu_1,\sigma_1}(x) = \frac{1}{\pi} \arctan\left(\frac{x - \mu_1}{\sigma_1}\right) + \frac{1}{2}, \quad F_{\mu_2,\sigma_2}(y) = \frac{1}{\pi} \arctan\left(\frac{x - \mu_2}{\sigma_2}\right) + \frac{1}{2},$$

with quantile function

$$Q(u_1) = \mu_1 + \sigma_1 \tan\left(\pi\left(u_1 - \frac{1}{2}\right)\right), \quad Q(u_2) = \mu_2 + \sigma_2 \tan\left(\pi\left(u_2 - \frac{1}{2}\right)\right),$$

then we have a joint distribution function with three parameters as follows

$$F_{\alpha,\mu,\sigma}(x, y) = C_\alpha(F_{\mu_1,\sigma_1}(x), F_{\mu_2,\sigma_2}(y)),$$

where C_α in that case is the one-parameter FGM copula (3), so by using (18) and (14), we get the following results

$$\begin{aligned}
\lambda^{(0,1)}_{1[12]} &= \mu_1,\ \lambda^{(0,1)}_{1[21]} = \mu_2,\\
\lambda^{(0,1)}_{2[12]} &= \frac{1}{24}\pi\alpha\sigma_1,\ \lambda^{(0,1)}_{2[21]} = \frac{1}{24}\pi\alpha\sigma_2 \\
\delta_{1[12]} &= \frac{1}{18}\alpha,
\end{aligned} \tag{19}$$

the parameters may be written as

$$\begin{aligned}
\mu_1 &= \lambda^{(0,1)}_{1[12]}, \mu_2 = \lambda^{(0,1)}_{1[21]}\\
\sigma_1 &= \frac{4}{3\pi\delta_{1[12]}}\lambda^{(0,1)}_{2[12]}\\
\sigma_2 &= \frac{4}{3\pi\delta_{1[12]}}\lambda^{(0,1)}_{2[12]}\\
\alpha &= 18\delta_{1[12]}
\end{aligned} \tag{20}$$

when taking the same marginals and FGM copula with two parameters (5), and using the same system, we obtain

$$\begin{aligned}
\lambda^{(0,1)}_{1[12]} &= (4.43\times 10^7)\sigma_1(\alpha_1+\alpha_2)+\mu_1,\\
\lambda^{(0,1)}_{1[21]} &= (4.43\times 10^7)\sigma_2(\alpha_1+\alpha_2)+\mu_2\\
\lambda^{(0,1)}_{2[12]} &= \frac{\pi\sigma_1\alpha_1}{24}+\frac{\pi\sigma_1\alpha_2}{240},\\
\lambda^{(0,1)}_{2[21]} &= \frac{\pi\sigma_2\alpha_1}{24}+\frac{\pi\sigma_2\alpha_2}{240}\\
\delta_{1[12]} &= \frac{\alpha_1}{18}+\frac{\alpha_2}{72}\\
\delta_{2[12]} &= \frac{\alpha_2}{120}.
\end{aligned} \tag{21}$$

so, we have an explicit form

$$\begin{aligned}
\mu_1 &= \lambda^{(0,1)}_{1[12]} - (4.43\times 10^7)\frac{\lambda^{(0,1)}_{2[12]}(18\delta_{1[12]}-\frac{59}{3}\delta_{2[12]}+120\delta_{2[12]})}{(\pi\frac{3}{4}\delta_{1[12]}-\pi\frac{59}{72}\delta_{2[12]}+\pi\frac{1}{2}\delta_{2[12]})}\\
\mu_2 &= \lambda^{(0,1)}_{1[21]} - (4.43\times 10^7)\frac{\lambda^{(0,1)}_{2[21]}(18\delta_{1[12]}-\frac{59}{3}\delta_{2[12]}+120\delta_{2[12]})}{(\pi\frac{3}{4}\delta_{1[12]}-\pi\frac{59}{72}\delta_{2[12]}+\pi\frac{1}{2}\delta_{2[12]})}\\
\sigma_1 &= \lambda^{(0,1)}_{2[12]}/\left(\pi\frac{3}{4}\delta_{1[12]}-\pi\frac{59}{72}\delta_{2[12]}+\pi\frac{1}{2}\delta_{2[12]}\right),\\
\sigma_2 &= \lambda^{(0,1)}_{2[21]}/\left(\pi\frac{3}{4}\delta_{1[12]}-\pi\frac{59}{72}\delta_{2[12]}+\pi\frac{1}{2}\delta_{2[12]}\right)\\
\alpha_1 &= 18\delta_{1[12]}-\frac{59}{3}\delta_{2[12]}\\
\alpha_2 &= 120\delta_{2[12]}.
\end{aligned} \tag{22}$$

3.2. FGM copulas with Pareto marginals

The distribution function of two random variables X,Y of Generalized Pareto law is defined by

$$F_{\gamma_1,\kappa_1}(x) = 1 - \left\{1 - \kappa_1\left(\frac{x}{\gamma_1}\right)\right\}^{\left(\frac{1}{\kappa_1}\right)}, F_{\gamma_2,\kappa_2}(x) = 1 - \left\{1 - \kappa_2\left(\frac{x}{\gamma_2}\right)\right\}^{\left(\frac{1}{\kappa_2}\right)},$$

with quantile functions

$$Q(u_1) = \frac{\gamma_1}{\kappa_1}\left(1 - (1-u_1)^{\kappa_1}\right),\ Q(u_2) = \frac{\gamma_2}{\kappa_2}(1 - (1-u_2)^{\kappa}).$$

where $\gamma_j, \kappa_j, j = 1,2$ are scale and shape parameters, so, we obtain a joint distribution function as follows

$$F_{\alpha,\gamma_1,k_1}(x,y) = C_\alpha(F_{\gamma_1,\kappa_1}(x), F_{\gamma_2,\kappa_2}(y)),$$

and using the system (18), we get

$$\begin{aligned}
\lambda_{1[12]}^{(0,1)} &= \left(\frac{\gamma_1}{\kappa_1+1}\right) + \frac{1}{3}\frac{-\alpha\gamma_1}{\kappa_1^2+3\kappa_1+2}, \\
\lambda_{1[21]}^{(0,1)} &= \left(\frac{\gamma_2}{\kappa_2+1}\right) + \frac{1}{3}\frac{-\alpha\gamma_2}{\kappa_2^2+3\kappa_2+2} \\
\lambda_{2[12]}^{(0,1)} &= \frac{\alpha\gamma_1}{4(\kappa_1^2+3\kappa_1+2)}, \\
\lambda_{2[21]}^{(0,1)} &= \frac{\alpha\gamma_2}{4(\kappa_2^2+3\kappa_2+2)}, \\
\delta_{1[12]} &= \frac{1}{18}\alpha.
\end{aligned} \tag{23}$$

then, we have

$$\begin{aligned}
\kappa_1 &= \frac{9\delta_{1[12]}\left((\lambda_{1[12]}^{(0,1)} + 12\lambda_{2[12]}^{(0,1)})\right) - 4\lambda_{2[12]}^{(0,1)}}{2\lambda_{2[12]}^{(0,1)}}, \\
\gamma_1 &= \left(\frac{\lambda_{1[12]}^{(0,1)} + 12\lambda_{2[12]}^{(0,1)}}{\lambda_{2[12]}^{(0,1)}} - 1\right)\left(\frac{\lambda_{1[12]}^{(0,1)} + 12\lambda_{2[12]}^{(0,1)}}{\lambda_{2[12]}^{(0,1)}}\right) \\
\kappa_2 &= \frac{9\delta_{1[21]}\left((\lambda_{1[21]}^{(0,1)} + 12\lambda_{2[21]}^{(0,1)})\right) - 4\lambda_{2[21]}^{(0,1)}}{2\lambda_{2[21]}^{(0,1)}}, \\
\gamma_2 &= \left(\frac{\lambda_{1[21]}^{(0,1)} + 12\lambda_{2[21]}^{(0,1)}}{\lambda_{2[2]}^{(0,1)}} - 1\right)\left(\frac{\lambda_{1[21]}^{(0,1)} + 12\lambda_{2[21]}^{(0,1)}}{\lambda_{2[21]}^{(0,1)}}\right) \\
\delta_{1[12]} &= \frac{1}{18}\alpha.
\end{aligned} \tag{24}$$

3.3. Semi parametric TL-moment estimation

Let $\left(X_i^{(1)}, X_i^{(2)}\right)$ a random sample of r.v $X = (X^{(1)}, X^{(2)})$,with empirical marginal distribution functions

$$F_{j:n}(x_j) = n^{-1} \sum_{i=1}^{n} 1 \left\{ X_i^{(j)} \leq x_j \right\}, j = 1, 2$$

and $F^*_{j:n}(X_i^{(1)}) = nF_{j:n}/(n+1)$.

The estimation procedure consists of two steps:

1. Estimating the dependence parameters by solving the system

$$\begin{cases} \delta_{1[12]}(\theta_1, ..., \theta_l) &= \hat{\delta}_{1[12]} \\ \delta_{2[12]}(\theta_1, ..., \theta_l) &= \hat{\delta}_{1[12]} \\ \qquad \vdots & \\ \delta_{l[12]}(\theta_1, ..., \theta_l) &= \hat{\delta}_{l[12]} \end{cases} \tag{25}$$

   where

$$\hat{\delta}_{r[12]} = n^{-1} \sum_{i=1}^{n} F^*_{1:n}(X_i^{(1)}) P_r(F^*_{2:n}(X_i^{(2)})). \tag{26}$$

2. Estimating the marginals parameters by substitution the value of $\hat{\theta}$ in terms of $\hat{\delta}_{r[12]}$ in $\hat{\lambda}^{(t_1,t_2)}_{r[1,2]}$,where

$$\hat{\lambda}^{(t_1,t_2)}_{r[12]} = n^{-1} \sum_{i=1}^{n} X_i^{(1)} P^*_{r-1}(F^*_{2:n}(X_i^{(2)})) \tag{27}$$

   Such example, for FGM copula with Cauchy margins, we have

$$\hat{\alpha} = 18\hat{\delta}_{1[12]}, \hat{\mu}_1 = \hat{\lambda}^{(0,1)}_{1[12]}, \hat{\sigma}_1 = \frac{4}{3\pi}\hat{\delta}_{1[12]}$$
$$\hat{\alpha} = 18\hat{\delta}_{1[12]}, \hat{\mu}_2 = \hat{\lambda}^{(0,1)}_{1[12]}, \hat{\sigma}_2 = \frac{4}{3\pi}\hat{\delta}_{2[12]}$$

Remark 1. In some cases, it is not easy to find an explicit formulas for parameters estimation then we solve the system of equation by numerical method.

3.4. Simulation study

In our simulation study we select many different sample sizes with $n = 30, 50$ and 200 to assess their influence on the bias and RMSE of the estimators and we choose different values of dependence parameters, according the degree of dependence calculated by Spearman's rho (1), that is consider three cases, corresponding to weak, moderate and strong dependence Table (1) and marginal's parameters. For each choice we make $N = 1000$ repetitions and we compute the estimation bias and RMSE:

$$Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta),\ RMSE = \left(\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2\right)^{1/2}.$$

We can summarize the procedure of simulation as follows:

1. Determine the value of the parameters, sample sizes n and the number of simulated Samples N.
2. Simulate a sample $(u_1, ...u_n)$ of size n from the copula C_α (FGM and Gumbel copulas).
3. Compute the parameter estimates by solving the system (22, 24).
4. Compare the parameter estimates with the true parameters (presented in Tables 2 and 3) by computing the biases and RMSE.

	FGM		Gumbel	
Sparman ρ	α_1	α_2	ρ	β
0.001	0.1	0	0.01	1.01
0.208	0.4	0.9	0.5	1.6
0.427	0.941	1.445	0.88	3.45

Table 1. True parameters of FGM copula and Gumbel copula used for the simulation study.

			$\rho = 0.001$					
n	$\alpha_1 = 0.1$		$\alpha_2 = 0$		$\mu = -1$		$\sigma = 0.5$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
30	1.600	0.620	1.005	0.552	−0.820	0.510	0.340	0.200
50	1.320	0.430	0.956	0.462	0.610	0.420	0.250	0.105
200	0.507	0.306	0.596	0.382	0.412	0.201	0.210	0.100
			$\rho = 0.208$					
n	$\alpha_1 = 0.4$		$\alpha_2 = 0.9$		$\mu = -1$		$\sigma = 0.5$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
30	0.650	0.340	0.493	0.330	0.613	0.312	0.260	0.201
50	0.520	0.250	0.402	0.250	0.512	0.212	0.120	0.098
200	0.430	0.200	0.360	0.335	0.200	0.100	0.101	0.055
			$\rho = 0.941$					
n	$\alpha_1 = 0.941$		$\alpha_2 = 1.445$		$\mu = -1$		$\sigma = 0.5$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
30	0.420	0.301	−1.111	0.420	−0.111	0.076	0.200	0.121
50	0.340	0.210	−0.487	0.357	−0.0780	0.069	0.192	0.098
200	0.201	0.140	−0.250	0.220	0.065	0.032	0.079	0.052

Table 2. Bias, RMSE of the dependence and margins estimator of FGM copula with Cauchy margins.

4. Consistency and asymptotic normality

To study the asymptotic normality of the TL-moments estimator noted $\hat{\theta}^{CTL}$, we put

$$\mathcal{K}_r(u;\theta) = F_i^{-1}(u_i) P_{r-1}^{*(0,1)}(u_j) - \lambda_{r[12]}^{(0,1)}, i, j = 1, 2, i \neq j. \tag{28}$$

	$\rho = 0.01$					
n	$\beta = 1.01$		$\gamma = 1.5$		$\kappa = 3$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE
30	-0.339	0.276	0.760	0.520	0.811	0.650
50	0.205	0.110	0.494	0.320	0.540	0.320
200	-0.167	0.108	0.201	0.212	0.254	0.214
	$\rho = 0.5$					
n	$\beta = 1.6$		$\gamma = 1.5$		$\kappa = 3$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE
30	-0.320	0.280	0.420	0.213	0.413	0.376
50	-0.212	0.210	0.315	0.279	0.325	0.301
200	-0.120	0.100	0.202	0.201	0.119	0.260
	$\rho = 0.8$					
n	$\beta = 3.45$		$\gamma = 1.5$		$\kappa = 3$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE
30	0.116	0.250	0.210	0.450	0.222	0.310
50	0.109	0.131	0.111	0.320	0.09	0.150
200	0.06	0.110	0.054	0.215	0.06	0.115

Table 3. Bias, RMSE of the dependence and margins estimator of Gumbel copula with Pareto margins.

and

$$\mathcal{K}_r(u;\theta) = (\mathcal{K}_1(u;\theta), ..., \mathcal{K}_l(u;\theta)).$$

Let θ_0 be the true value of θ and assume that the following assumptions [$\mathcal{A}1$]-[$\mathcal{A}3$] hold.

- [$\mathcal{A}1$] $\theta_0 \in \mathcal{O} \subset \mathbb{R}^r$ is the unique zero of the mapping $\theta \to \int_{[0,1]^d} \mathcal{K}(u;\theta) dC\theta_0(u)$ which is defined from $\mathcal{O}$ to $\mathbb{R}^r$
- [$\mathcal{A}2$] $\mathcal{K}(.;\theta)$ is differentiable with respect to θ such that the Jacobian matrix denoted by $\dot{\mathcal{K}}(u;\theta) = [\partial \mathcal{K}_r(u;\theta)/\partial\theta_k]_{l\times l}$ and $\dot{\mathcal{K}}(u;\theta)$ is continuous both in u and θ, and the Euclidean norm $\left|\dot{\mathcal{K}}(u;\theta)\right|$ is dominated by a $dC_\theta-$integrable function.
- [$\mathcal{A}3$] The $r \times r$ matrix $\mathcal{B}_0 := \int_{[0,1]^d} \dot{\mathcal{K}}(u;\theta) dC\theta_0(u)$ is nonsingular.

Theorem 1. *Assume that the concordance ordering condition (9) and assumptions [$\mathcal{A}1$] - [$\mathcal{A}3$] hold. Then, there exists a solution $\hat{\theta}^{CTL}$ to the system (11) which converges in probability to θ_0. Moreover*

$$\sqrt{n}(\hat{\theta}^{CTL} - \theta_0) \overset{D}{\to} \mathcal{N}(0, \mathcal{B}_0^{-1}\mathcal{D}_0(\mathcal{B}_0^{-1})^T), \text{ as } n \to \infty,$$

where

$$\mathcal{D}_0 := var(\{\mathcal{K}_r(\vartheta;\theta_0) + \mathcal{V}(\vartheta;\theta_0)\},$$
$$\mathcal{V}(\vartheta;\theta_0) = (\mathcal{V}_1(\vartheta;\theta_0), ..., \mathcal{V}_r(\vartheta;\theta_0)),$$

with

$$\mathcal{V}_r(\vartheta;\theta_0) = \sum_{j=1}^{2} \int_{[0,1]^2} \frac{\partial(C_\theta(u)P_{r-1}^{*(0,1)}(K_\theta(C_\theta)))}{\partial u_j} (1\{\vartheta_j \le u_j\} - u_j) dC_{\theta_0}(u),$$

where ϑ *is a* $(0,1)$ *−uniform rv.*

Remark 2. Following Genest et al. (1995) and Tsukahara (2005) in the case of PML estimator and Z-estimator, one may consistently estimate the asymptotic variance $\mathcal{B}_0^{-1}\mathcal{D}_0(\mathcal{B}_0^{-1})^T$ by the sample variance of the sequence of rv's

$$\left\{\hat{\mathcal{B}}_i^{-1}\hat{\mathcal{D}}_i(\hat{\mathcal{B}}_i^{-1})^T,\ i=1,...,n\right\},$$

where

$$\hat{\mathcal{B}}_i := \int_{[0,1]^2} \dot{\mathcal{K}}(u,\hat{\theta}^{CTL})dC_{\hat{\theta}^{CTL}}(u),$$

and

$$\hat{\mathcal{D}}_i = \dot{\mathcal{K}}(\hat{U}_i,\hat{\theta}^{CTL}) + V(\hat{U}_i,\hat{\theta}^{CTL}).$$

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