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The Ristić and Balakrishnan Lindley-Poisson Distribution: Model, Theory and Application

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Abstract. A new distribution called Ristić and Balakrishnan Lindley-Poisson (RB-LP) distribution is introduced and its properties are explored. This new distribution contains several new and well known sub-models, including Lindley-Poisson, RB-Lindley and Lindley distributions. Some statistical properties of the proposed distribution including hazard rate function, moments and conditional moments are presented. Mean deviations, Lorenz and Bonferroni curves, Rényi entropy and distribution of the order statistics are given. Maximum likelihood estimation technique is used to estimate the model parameters. Finally, application of the model to a real dataset is presented to illustrate the usefulness of the proposed distribution.

Key words: Generalized Distribution; Ristić and Balakrishnan, Gamma Distribution, Lindley Distribution, Maximum Likelihood Estimation
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Résumé. Une nouvelle distribution appelée distribution Ristić et Balakrishnan Lindley-Poisson (RB-LP) est introduite et ses propriétés sont explorées. Cette nouvelle distribution contient plusieurs nouveaux sous-modèles bien connus, notamment Distributions de Lindley-Poisson, RB-Lindley et Lindley. Quelques propriétés statistiques de la distribution proposéey compris la fonction de taux de risque, les moments et les moments conditionnels sont présentés. Écarts moyens,Les courbes de Lorenz et Bonferroni, l'entropie de R enyi et la distribution des statistiques d'ordre sont données. Maximum La technique d'estimation de la vraisemblance est utilisée pour estimer les paramètres du modèle. Enfin, l'application du modèle à un jeu de données réel est présenté pour illustrer l'utilité de la distribution proposée.

1. Introduction

Lindley distribution is a very useful model for analysis of lifetime and reliability data. Properties of Lindley distribution have been examined and studied (see Ghitany *et al.* (2008)). The authors showed that Lindley distribution outperforms the exponential distribution. Lindley distribution has an increasing, decreasing and constant hazard rate function. However, in reliability analysis and related areas, models with complex failure rate shapes such as bathtub, unimodal are desired (Oluyede and Yang (2015)). There are several new generalizations that have been introduced, that constitutes flexible families of distributions in terms of the varieties of shape and hazard function, and some of the generalizations introduced include those by Pararai *et al.* (2015); Ghitany *et al.* (2013); Nadarajah *et al.* (2015); Zakerzadeh and Dolati (2009).

There are many techniques of generating new distributions from classical ones, these techniques include compounding. It is common to derive distributions by compounding Poisson, geometric and binomial distributions with other continuous distributions. Some compounded distributions in the literature include: exponential Geometric (EG) distribution by Adamidis and Loukas (1998) which was obtained by compounding the exponential and geometric distributions, exponential Poisson (EP) distribution presented by Kus (2007) and exponential logarithmic (EL) distribution by Tahmasbi and Rezaei (2008), and Weibull-Poisson distribution given by Lu and Shi (2012). These distributions have various failure rate shapes including increasing, bathtub, decreasing and unimodal.

Lindley distribution was introduced 1958 to demonstrate the difference between fiducial and posterior distributions (Lindley (1958)). Lindley distribution is a mixture of exponential and length-biased exponential distributions with *pdf*

$$f(x;\theta) = (1-p)f_G(x,\theta) + p(f_E(x;\theta)) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x},$$

where p=$\frac{1}{1+\theta}$, is the mixing proportion, $f_G(x;\theta) = \theta^2 x e^{-\theta x}$, $f_E(x;\theta) = \theta e^{-\theta x}$, for $x \geq 0$ and $\theta > 0$. Lindley-Poisson (LP) distribution belongs to the compound class of distributions and was studied by Gui *et al.* (2014). The authors showed that LP distribution has a decreasing and bath-tub shaped failure rate function. The *pdf* and *cdf* of the LP distribution are given by

$$g(x) = \frac{\lambda\theta^2(1+x)e^{-\theta x} exp\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)}{(1+\theta)(e^{\lambda}-1)}, x>0, \lambda, \theta>0, \tag{1}$$

and

$$G(x) = \frac{1-exp\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)}{1-e^{\lambda}}, \tag{2}$$

respectively.

1.1. *Zografos-Balakrishnan and Ristić-Balakrishnan families of distributions*

Recently, Zografos and Balakrishnan (Zografos and Balakrishnan (2009)), and Ristić and Balakrishnan (Ristić and Balakrishnan (2011)) proposed two classes of distributions. These classes of distributions are generated by gamma random variables, with an extra positive shape parameter. Nadarajah *et al.* (2015) derived some mathematical properties of the gamma generator given by Zografos and Balakrishnan (Nadarajah *et al.* (2011)). Ramos *et al.* (2013) and Pinho *et al.* (2012) introduced and studied the Zografos-Balakrishnan log-logistic distribution and gamma exponentiated Weibull distribution, respectively. They showed that the Zografos-Balakrishnan log-logistic distribution and the gamma exponentiated Weibull distribution perform better than their parent distributions.

1.2. *Zografos and Balakrishnan Model*

The *pdf* $f(x)$ and *cdf* $F(x)$ of the family of distributions proposed byZografos and Balakrishnan (2009) is given as:

$$f(x) = \frac{1}{\Gamma(\delta)\psi^{\delta}}\left[-log(1-G(x))\right]^{\delta-1}(1-G(x))^{\frac{1}{\psi}-1}g(x), x\in R, \delta>0, \tag{3}$$

and

$$F(x) = \frac{1}{\Gamma(\delta)\psi^{\delta}}\int_0^{-log(1-G(x))} t^{\delta-1}e^{-t/\psi}dt = \frac{\gamma\left(\delta, -\psi^{-1}log(1-G(x))\right)}{\Gamma(\delta)}, \tag{4}$$

respectively, where $\gamma(x,\delta) = \int_0^x t^{\delta-1}e^{-t}dt$ is the incomplete gamma function. The corresponding hazard rate function is

$$h_G(x) = \frac{f(x)}{1-F(x)}\frac{\left[-log(1-G(x))\right]^{\delta-1}f(x)(1-G(x))^{1/\psi-1}}{\psi^{\delta}\left(\Gamma(\delta)-\gamma\left(-\psi^{-1}log(1-G(x)),\delta\right)\right)}.$$

When ψ=1, the distribution becomes what is called the "ZB-G family of distributions".

1.3. Ristić and Balakrishnan Model

As an alternative, a gamma-generator was proposed by Ristić and Balakrishnan (2011). They defined the *pdf* and *cdf* by

$$f_2(x) = \frac{1}{\Gamma(\delta)\psi^{\delta}} \left[-log(G(x))\right]^{\delta-1} G(x)^{1/\psi-1} g(x), \tag{5}$$

and

$$F_2(x) = 1 - \frac{1}{\Gamma(\delta)\psi^{\delta}} \int_0^{-log(G(x))} t^{\delta-1} e^{t/\psi} dt, x \in R, \delta > 0, \tag{6}$$

respectively. When ψ=1, this distribution is referred to as the Ristić and Balakrishnan-G (RB-G) family of distributions.

In this paper, a generalization of the Lindley-Poisson distribution motivated by the results of Ristić and Balakrishnan family of distributions in equation (6) when $\psi = 1$ is developed and presented. We present a generalization of the Lindley-Poisson distribution that stems from the Ristić and Balakrishnan family of distributions Ristić and Balakrishnan (2011). Statistical properties of the model are derived, the maximum likelihood estimates and an application of the model is presented. In addition to the motivations provided by Ristić and Balakrishnan, its is also the case that generalization of the Lindley-Poisson distribution via the gamma-generator also establishes the relationship between the distributions in equations (4) and (6), and weighted distributions in general. See Oluyede (1999) and Oluyede *et al.* (2014) for additional details. Ristić and Balakrishnan provided motivations for the new family of distributions given in equation (6), that is, for $n \in N$, equation (6) given above is the *cdf* of the n^{th} lower record value of a sequence of i.i.d. variables from a population with density g(x) (Ristić and Balakrishnan (2011)). They used the exponentiated exponential (EE) distribution with *cdf* $F(x) = (1 - e^{-\beta x})^{\alpha}$, where $\alpha > 0$ and $\beta > 0$; in equation (6) to obtained and study the gamma-exponentiated exponential (GEE) model.

The results in this paper are organized in the following manner. The model, its sub-models, quantile function, and hazard function are given in section 2. In section 3, moments, moment generating function and conditional moments are presented. Mean deviations, Lorenz and Bonferroni curves are given in section 4. Section 5 contain results on Rényi entropy and density of the order statistics. Maximum likelihood estimates of the model parameters are given in section 6. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates are presented in section 7. Section 8 contain an application of the new model to real dataset. A short conclusion is given in section 9.

2. Ristić and Balakrishnan-Lindley-Poisson Model

When ψ=1, the Ristić and Balakrishnan-G (RBG) *pdf* and *cdf* can be writen as

$$f_2(x) = \frac{1}{\Gamma(\delta)}[-log(G(x))]^{\delta-1} g(x), \tag{7}$$

and

$$F_2(x) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-log(G(x))} t^{\delta-1} e^{-t} dt, x \in R, \delta > 0, \tag{8}$$

respectively.
Substitution of equations (1) and (2) in equations (7) and (8) yields the *pdf* and *cdf* of RB-LP distribution given by

$$\begin{aligned} f_2(x) &= \frac{1}{\Gamma(\delta)} \left[-log \left(\frac{1 - exp(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{1 - e^{\lambda}} \right) \right]^{\delta-1} \\ &\times \frac{\lambda\theta^2(1+x)e^{-\theta x} exp(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{(1+\theta)(e^{\lambda}-1)}, x > 0, \theta, \lambda, \delta > 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} F_2(x) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-log\left[\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right]} t^{\delta-1} e^{-t} dt \\ &= \frac{\Gamma(\delta) - \gamma\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}} \right), \delta \right]}{\Gamma(\delta)}, \end{aligned} \tag{10}$$

respectively, where θ and λ are scale parameters and δ is the shape parameter of the distribution. Plots for the *pdf* of the RB-LP distribution for selected parameter values is given in figure 2. The plots show that RB-LP can be decreasing, increasing and right skewed among other possible shapes.

2.1. Sub-models

In this sub-section, the sub-models of the RB-LP distribution are presented.

- When $\delta = 1$; Lindley-Poisson (LP) distribution is obtained.
- When $\theta \to 0^+$ and setting $\delta = 1$; Lindley (L) distribution is the limiting form of the RB-LP distribution.
- If $\theta \to 0^+$; Ristić and Balakrishnan-Lindley (RB-L) distribution is obtained.

2.2. Series Expansion of the Density

A series expansion of the RB-LP *pdf* is presented in this subsection. Note that the RB-LP *pdf* in equation (7) can be written as

$$\begin{aligned} f_2(x) &= \frac{1}{\Gamma(\delta)} \left[-log \left(1 - \left(1 - \frac{1 - exp(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{1 - e^{\lambda}} \right) \right) \right]^{\delta-1} \\ &\times \frac{\lambda\theta^2(1+x)e^{-\theta x} exp(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{(1+\theta)(e^{\lambda}-1)}, x > 0, \theta, \lambda, \delta > 0. \end{aligned}$$

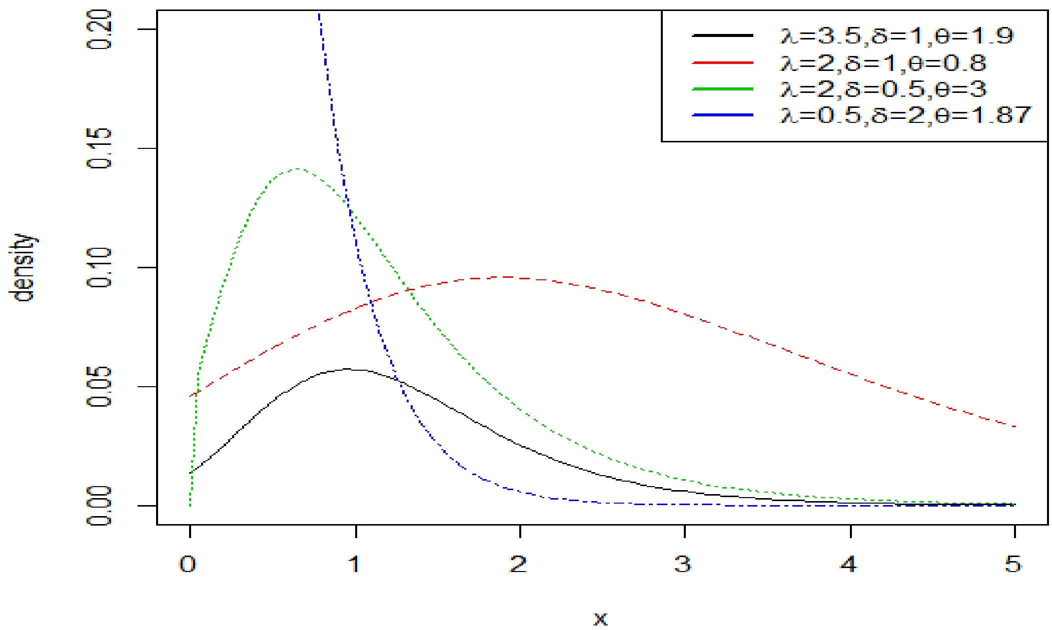

Fig. 1. Plots of the *pdf* of the RB-LP distribution for selected values of the parameters

Let $y = 1 - \frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}, 0 < y < 1$, then using the series representation

$$-\,log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}, \tag{11}$$

we have

$$[-log(1-y)]^{\delta-1} = y^{\delta-1} \sum_{m=0}^{\infty} \binom{\delta-1}{m} y^m \left[\sum_{s=0}^{\infty} \frac{y^s}{s+2} \right]^m . \tag{12}$$

Applying the result on a power series raised to a positive integer, with $a_s = (s+2)^{-1}$, we obtain

$$\left(\sum_{s=0}^{\infty} a_s y^s \right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{13}$$

where $b_{s,m} = (sa_o)^{-1}\sum_{i=1}^{s}[m(l+i)-s]a_l b_{s-l,m}$ and $b_{0,m} = a_0^m$ (Gradshteyn and Ryzhik (2007)), the RB-LP distribution can be written as

$$\begin{aligned} f_2(x) &= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1}\frac{\lambda\theta^2(1+x)e^{-\theta x} exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{(1+\theta)(e^{\lambda}-1)} \\ &= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m} b_{s,m}\left(1-\frac{1-exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{1-e^{\lambda}}\right)^{m+s+\delta-1} \\ &\times \frac{\lambda\theta^2(1+x)e^{-\theta x} exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{(1+\theta)(e^{\lambda}-1)}. \end{aligned}$$

Now, we apply the generalization of the binomial theorem,

$$(1-z)^{b-1} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(b)}{\Gamma(b-k)k!}z^k, \tag{14}$$

for a positive real integer b and $|z|<1$, to

$$\begin{aligned} \left(1-\frac{1-exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{1-e^{\lambda}}\right)^{m+s+\delta-1} &= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-k)k!} \\ &\times \left(\frac{1-exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{1-e^{\lambda}}\right)^k, \end{aligned}$$

and

$$\begin{aligned} \left[1-exp\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)\right]^k &= \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k+1)}{\Gamma(k+1-j)j!} \\ &\times \left(exp\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)\right)^j \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k+1)}{\Gamma(k+1-j)j!}e^{\lambda j(1-\frac{1+\theta+\theta x}{1+\theta})e^{-\theta x}}. \end{aligned}$$

Therefore, the *pdf* of the RB-LP distribution can be written as

$$\begin{aligned}
f_2(x) &= \frac{1}{\Gamma(\delta)}\sum_{m,s,k=0}^{\infty}\frac{(-1)^{k+j}b_{s,m}\Gamma(m+s+\delta)\Gamma(k+1)}{\Gamma(m+s+\delta-k)k!\Gamma(k+1-j)j!}\\
&\times \frac{e^{\lambda(j+1)}-1}{e^{\lambda(j+1)}-1}\frac{e^{\lambda j(1-\frac{1+\theta+\theta x}{1+\theta})e^{-\theta x}}}{(1-e^{\lambda})^k}\binom{\delta-1}{m}\frac{\lambda\theta^2(1+x)e^{-\theta x}exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{(1+\theta)(e^{\lambda}-1)}\\
&= \frac{1}{\Gamma(\delta)}\sum_{m,s,k=0}^{\infty}\frac{(-1)^{k+j}b_{s,m}\Gamma(m+s+\delta)\Gamma(k+1)}{\Gamma(m+s+\delta-k)k!\Gamma(k+1-j)j!}\\
&\times \frac{e^{\lambda(j+1)}-1}{(1-e^{\lambda})^k(e^{\lambda}-1)}\binom{\delta-1}{m}\frac{\lambda\theta^2(1+x)e^{-\theta x}(exp\lambda(j+1)(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{(1+\theta)(e^{\lambda(j+1)}-1)}\\
&= \sum_{m,s,k,j=0}^{\infty}\omega_{m,s,k,j}\, g(x;\theta,\lambda(j+1)),
\end{aligned}$$

where

$$\omega_{m,s,k,j} = \frac{1}{\Gamma(\delta)}\frac{(-1)^{k+j}b_{s,m}\Gamma(m+s+\delta)\Gamma(k+1)}{\Gamma(m+s+\delta-k)k!\Gamma(k+1-j)j!}\frac{e^{\lambda(j+1)}-1}{(1-e^{\lambda})^k(e^{\lambda}-1)}\binom{\delta-1}{m} \tag{15}$$

are the weights of the RB-LP distribution and $g(x;\theta,\lambda(j+1))$ is the Lindley-Poisson *pdf* with parameters $\theta>0$ and $\lambda(j+1)>0$. It follows that the RB-LP distribution can be expressed as a linear combination of Lindley-Poisson density functions. Therefore, the mathematical and statistical properties of the RB-LP distributions follows directly from those of the LP distribution.

2.3. Survival and Hazard Rate Functions

The survival, hazard and reverse hazard functions of the RB-LP distribution are given by

$$S(x) = \frac{\gamma\left[-log\left(\frac{1-exp(1-\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right),\delta\right]}{\Gamma(\delta)},$$

$$h(x) = \frac{\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{\delta-1}\left[\frac{\lambda\theta^2(1+x)e^{-\theta x}exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{(1+\theta)(e^{\lambda}-1)}\right]}{\gamma\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{1-e^{\lambda}}\right),\delta\right]},$$

and

$$\tau(x) = \frac{\left[-log\left(\frac{1-exp\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{1-e^{\lambda}}\right)\right]^{\delta-1}\left[\frac{\lambda\theta^2(1+x)e^{-\theta x}exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta})e^{-\theta x})}{(1+\theta)(e^{\lambda}-1)}\right]}{\Gamma(\delta)-\gamma\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right),\delta\right]},$$

respectively. The plots of the hazard function are given by figure 2. The graphs of the hazard function show that the RB-LP distribution has various shapes including monotonically increasing, bathtub, and upside down bathtub shapes amongst other possible shapes.

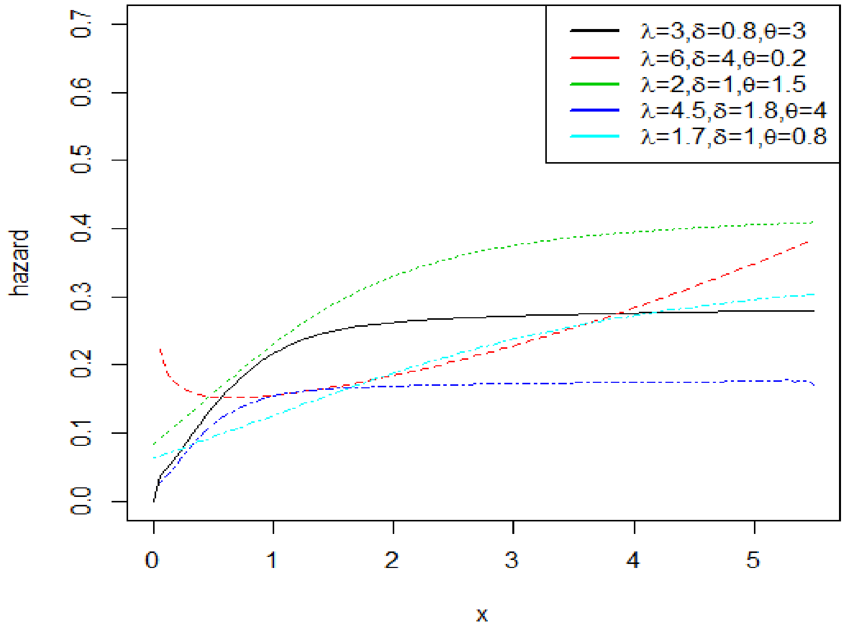

Fig. 2. Plots of hazard function for selected values of the parameters of the RB-LP distribution hazard

2.4. Quantile Function

To obtain the quantile function of RB-LP distribution, the equation $F(Q(p)) = p$, where $0 < p < 1$ is solved. That is, we solved the equation below:

$$\frac{\gamma\left(-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta Q(p)}{1+\theta}e^{-\theta Q(p)}))}{1-e^{\lambda}}\right),\delta\right)}{\Gamma(\delta)} = 1-p.$$

Let $Z(p) = -1-\theta-\theta Q(p)$, then

$$\frac{1}{\Gamma(\delta)}\gamma\left(-log\left(\frac{1-exp(\lambda(1+\frac{Z(p)}{1+\theta}exp(Z(p)+1+\theta)))}{1-e^{\lambda}}\right),\delta\right) = 1-p.$$

That is,

$$-log\left(\frac{1-exp(\lambda(1+\frac{Z(p)}{1+\theta}exp(Z(p)+1+\theta)))}{1-e^{\lambda}}\right)=\gamma^{-1}\left((1-p)\Gamma(\delta),\delta\right),$$

and

$$\frac{1-exp\left(\lambda\left(1+\frac{Z(p)}{1+\theta}exp(Z(p)+1+\theta)\right)\right)}{1-e^{\lambda}}=exp\left(-\gamma^{-1}\left((1-p)\Gamma(\delta)\right),\delta\right),$$

so that

$$Z(p)exp(Z(p))=\frac{\theta+1}{exp(1+\theta)}\left[\frac{log\left(1-\left[exp\left(-\gamma^{-1}\left((1-p)\Gamma(\delta)\right),\delta\right)\right]\left[1-e^{\lambda}\right]\right)}{\lambda}-1\right].$$

The solution for Z(p) is

$$Z(p)=W\left(\frac{\theta+1}{exp(1+\theta)}\left[\frac{log\left(1-\left[exp\left(-\gamma^{-1}\left((1-p)\Gamma(\delta)\right),\delta\right)\right]\left[1-e^{\lambda}\right]\right)}{\lambda}-1\right]\right),$$

for $0<p<1$, where W is the Lambert function (See Oluyede and Yang (2015) and references therein). It follows that

$$-1-\theta-\theta Q(p)=W\left(\frac{\theta+1}{exp(1+\theta)}\left[\frac{log\left(1-\left[exp\left(-\gamma^{-1}\left((1-p)\Gamma(\delta)\right),\delta\right)\right]\left[1-e^{\lambda}\right]\right)}{\lambda}-1\right]\right),$$

and

$$\theta Q(p)=-1-\theta-W\left(\frac{\theta+1}{exp(1+\theta)}\left[\frac{log\left(1-\left[exp\left(-\gamma^{-1}\left((1-p)\Gamma(\delta)\right),\delta\right)\right]\left[1-e^{\lambda}\right]\right)}{\lambda}-1\right]\right).$$

Consequently, the quantile function of the RB-LP distribution is given by

$$Q(p)=-1-\frac{1}{\theta}-\frac{1}{\theta}W\left(\frac{\theta+1}{exp(1+\theta)}\left[\frac{log\left(1-\left[exp\left(-\gamma^{-1}\left((1-p)\Gamma(\delta)\right),\delta\right)\right]\left[1-e^{\lambda}\right]\right)}{\lambda}-1\right]\right).$$

Table 1 presents the quantiles of the RB-LP distribution for selected values of the model parameters.

3. Moments

Moments are necessary and important in any statistical analysis, especially in applications. They are used to study the most important features and characteristics of a distribution (e.g. measures of central tendency, dispersion, skewness and kurtosis).

Table 1. RB-LP Table of Quantiles for selected parameter values

	$(\lambda, \delta, \theta)$			
p	(5.0,1.0,1.0)	(0.5,2.0,5.0)	(4.0,2.0,3.0)	(1.0,2.0,1.0)
0.1	0.9008661	0.006407772	0.08996364	0.06912791
0.2	1.3730307	0.015835345	0.17192703	0.16576898
0.3	1.7553994	0.027928001	0.24622738	0.28259488
0.4	2.1100205	0.043052117	0.31785724	0.41986109
0.5	2.4666326	0.061991047	0.39077899	0.58109881
0.6	2.8501531	0.086167142	0.46913455	0.77396693
0.7	3.2938001	0.118315614	0.55891771	1.01394379
0.8	3.8624008	0.164614333	0.67202380	1.33634823
0.9	4.7521532	0.244714974	0.84419509	1.85266522

3.1. Moments and Related Measures

In this subsection, we present the moments and related measures for the RB-LP distribution. The r^{th} moment of the RB-LP distribution, denoted μ_r is

$$\mu_r = E(X^r) = \sum_{m,s,k,j=0}^{\infty} \frac{\omega_{m,s,k,j}\lambda\theta^2}{(\theta+1)(e^{\lambda(j+1)}-1)} \int_0^{\infty} x^r(1+x)e^{-\theta x}$$
$$\times\ exp\left(\lambda(j+1)(1-\frac{1+\theta+\theta x)}{1+\theta}e^{-\theta x}\right) dx,$$

where $\omega_{m,s,k,j}$ is defined in equation (15).

Lemma 1. Let

$$L_1(\theta,\lambda(k+1),r) = \int_0^{\infty} x^r(1+x)e^{-\theta x} exp\left(\lambda(k+1)\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right) dx,$$

then

$$L_1(\theta,\lambda(k+1),r) = \sum_{w=0}^{\infty}\sum_{p=0}^{w}\sum_{q=0}^{p}\sum_{t=0}^{q+1}\binom{w}{p}\binom{p}{q}\binom{q+1}{t}\frac{(-1)^p\lambda^w(k+1)^w\theta^q}{w!(\theta+1)^p}$$
$$\times \frac{\Gamma(r+t+1)}{\theta(p+1)^{r+t+1}}.$$

Proof Using the series expansion, $e^z = \sum_{p=0}^{\infty}\frac{z^p}{p!}$,

$$exp\left(\lambda(k+1)\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right) = \sum_{w=0}^{\infty}\frac{\left(\lambda(k+1)(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)^w}{w!}.$$

The above integral can be written as

$$L_1(\theta,\lambda(k+1),r) = \sum_{w=0}^{\infty}\frac{\lambda^w(k+1)^w}{w!}\int_0^{\infty}x^r(1+x)e^{-\theta x}\left[1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right]^w dx.$$

Note that

$$\left[1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right]^w = \sum_{p=0}^{\infty}(-1)^p\binom{w}{p}\frac{\sum_{q=0}^{p}\binom{p}{q}\theta^q(1+x)^q}{(1+\theta)^p}exp(-\theta px),$$

and

$$(1+x)^{q+1} = \sum_{t=0}^{q+1}\binom{q+1}{t}x^t.$$

Therefore,

$$L_1(\theta,\lambda(k+1),r) = \sum_{w=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{t=0}^{q+1}\binom{w}{p}\binom{p}{q}\binom{q+1}{t}\frac{(-1)^p\lambda^w(k+1)^w\theta^q}{w!(\theta+1)^p} \times \int_0^{\infty}x^{r+t}e^{-\theta(p+1)x}dx.$$

Now consider the integral $\int_0^{\infty}x^{r+t}e^{-\theta(p+1)x}dx$. By letting $u=\theta(p+1)x$, then $x=\frac{u}{\theta(p+1)}$ and dx=$\frac{du}{\theta(p+1)}$. Thus

$$\int_0^{\infty}x^{r+t}e^{-\theta(p+1)}dx = \frac{\Gamma(r+t+1)}{[\theta(p+1)]^{r+t+1}}.$$

Consequently,

$$L_1(\theta,\lambda(k+1),r) = \sum_{w=0}^{\infty}\sum_{p=0}^{w}\sum_{q=0}^{p}\sum_{t=0}^{q+1}\binom{w}{p}\binom{p}{q}\binom{q+1}{t}\frac{(-1)^p\lambda^w(k+1)^w\theta^q}{w!(\theta+1)^p} \times \frac{\Gamma(r+t+1)}{\theta(p+1)^{r+t+1}}.$$

From Lemma 1, the RB-LP distribution r^{th} is given by

$$\mu_r = \sum_{k,s,m,j=0}^{\infty}\frac{\lambda\theta^2\omega_{k,s,m,j}}{(\theta+1)(e^{\lambda(j+1)}-1)}L_1(\theta,\lambda(j+1),r).$$

The mean, variance, coefficient of variation, coefficient of skewness, and coefficient of kurtosis are given by

$$\mu_1 = \sum_{k,s,m,j=0}^{\infty}\frac{\lambda\theta^2\omega_{k,s,m,j}}{(\theta+1)(e^{\lambda(j+1)}-1)}L_1(\theta,\lambda(j+1),1),$$

$$\sigma^2=\mu_2^{'}-\mu^2, \quad CV=\frac{\sqrt{\mu_2^{'}-\mu^2}}{\mu}, \quad CS=\frac{\mu_2^{'}-3\mu\mu_2^{'}+\mu^3}{(\mu_2^{'}-\mu^2)^{3/2}},$$

Table 2. RB-LP Moments for selected values

	(λ,δ,θ)			
$\mu_r^{'}$	(1.0,1.0,1.0)	(0.5,2.0,2.0)	(2.0,3.0,0.8)	(0.5,2.0,3.0)
$\mu_1^{'}$	1.124884	0.2106208	0.12408893	4.883846e+00
$\mu_2^{'}$	3.348918	0.1334097	0.04939768	4.651672e+01
$\mu_3^{'}$	13.343497	0.1258957	0.02969964	6.419803e+02
$\mu_4^{'}$	66.253570	0.1555039	0.02358454	1.151092e+04
$\mu_5^{'}$	393.841333	0.2356265	0.02312313	2.526103e+05
$\mu_6^{'}$	2729.008522	0.4213159	0.02688122	6.533111e+06
SD	1.443452	0.2984101	0.18438984	4.760753e+00
CV	1.283200	1.4168118	1.48594914	9.747960e-01
CS	1.625544	2.2686874	2.41370967	1.792538e+00
CK	6.181750	9.9682061	10.98244977	7.630899e+00

and

$$CK = \frac{\mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 3\mu^4}{(\mu_2^{'} - \mu^2)^2},$$

respectively. Table 2 shows the first six moments for selected parameter values of the RB-LP distribution.

The plots of skeweness and kurtosis are presented in figure 3 and figure 4. Plots of CS and CK against the shape parameter δ shows the dependence of the kurtosis and skewness measures on the shape parameter δ.

3.2. Conditional Moments

We apply Lemma 2 to obtain the conditional moments of the RB-LP distribution.

Lemma 2 Let

$$L_2(\theta,\lambda(k+1),r,t) = \int_t^{\infty} x^r(1+x)e^{-\theta x} exp\left(\lambda(k+1)\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)dx,$$

then

$$L_2(\theta,\lambda(k+1),r,t) = \sum_{w=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{v=0}^{q+1}\binom{w}{p}\binom{p}{q}\binom{q+1}{v}\frac{(-1)^p\lambda^w(k+1)^w\theta^q}{w!(\theta+1)^p} \times \frac{\Gamma(r+v+1,\theta(p+1)t)}{\theta(p+1)^{r+v+1}},$$

where $\Gamma(r,t)=\int_t^{\infty} t^{r-1}e^t dt$ is the upper incomplete gamma function.

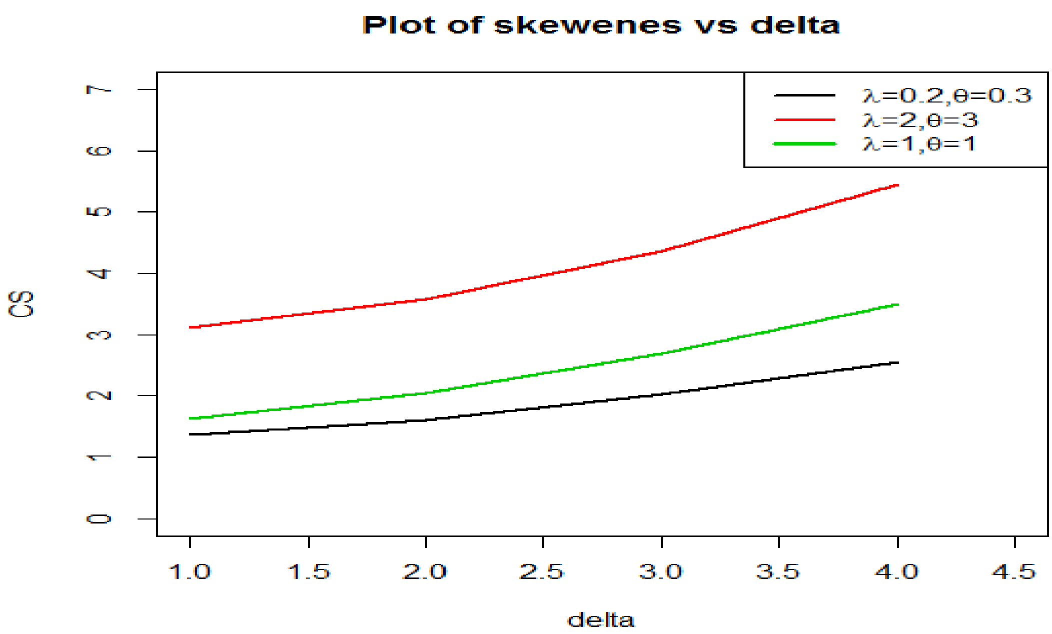

Fig. 3. Plots of skeweness for selected values of the parameters-RB-LP distribution

Proof: Following the same procedure used in Lemma 1, the simplified form of $L_2(\theta, \lambda(k+1), r, t)$ is

$$
\begin{aligned}
L_2(\theta, \lambda(k+1), r, t) &= \sum_{w=0}^{\infty} \frac{\lambda^w (k+1)^w}{w!} \int_t^{\infty} x^r (1+x) e^{-\theta x} \left[1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}\right]^w dx \\
&= \sum_{w=0}^{\infty} \sum_{p=0}^{w} \sum_{q=0}^{p} \sum_{v=0}^{q+1} \binom{w}{p} \binom{p}{q} \binom{q+1}{v} \frac{(-1)^p \lambda^w (k+1)^w \theta^q}{w!(\theta+1)^p} \\
&\times \int_t^{\infty} x^{r+v} e^{-\theta(p+1)x} dx.
\end{aligned}
$$

Now consider the integral $\int_t^{\infty} x^{r+v} e^{-\theta(p+1)x} dx$, and let $u = \theta(p+1)x$, then $x = \frac{u}{\theta(p+1)}$ and dx=$\frac{du}{\theta(p+1)}$. The above integral can written by using the incomplete gamma function as

$$
\int_{\theta(p+1)t}^{\infty} \left(\frac{u}{\theta(p+1)}\right)^{r+v} e^{-u} \frac{du}{[\theta(p+1)]} = \frac{\Gamma(r+v+1, \theta(p+1)t}{[\theta(p+1)]^{r+v+1}}.
$$

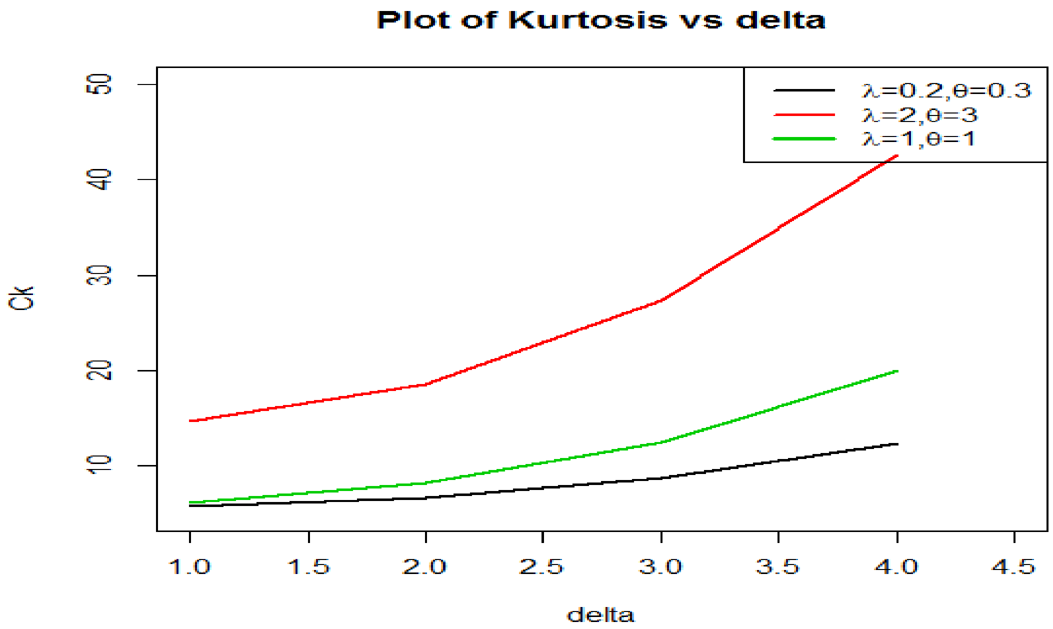

Fig. 4. Plots of Kurtosis for selected values of the parameters-RB-LP distribution

Consequently,

$$L_2(\theta, \lambda(k+1), r, t) = \sum_{w=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{v=0}^{q+1}\binom{w}{p}\binom{p}{q}\binom{q+1}{v}\frac{(-1)^p\lambda^w(k+1)^w\theta^q}{w!(\theta+1)^p} \times \frac{\Gamma(r+v+1, \theta(p+1)t)}{\theta(p+1)^{r+v+1}}.$$

Lemma 2 is used to calculate the conditional moments, that is, the r^{th} conditional moment of the RB-LP distribution is given by

$$E(X^r|X>t) = \frac{1}{\overline{F}_{RB-LP}(t)}\sum_{m,s,k,j=0}^{\infty}\frac{\omega_{m,s,k,j}\lambda\theta^2}{(\theta+1)(e^{\theta(j+1)}-1)}L_2(\theta, \lambda(j+1), r, t).$$

4. Mean Deviations, Lorenz and Bonferroni Curves

We apply Lemma 2 to obtain the mean deviations for the RB-LP distribution. The mean deviation about the mean and the mean deviation about the median of the RB-LP distribution are given by

$$D(\mu) = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} xf(x)dx = 2\mu F(\mu) - 2\sum_{m,s,k,j=0}^{\infty}\frac{\omega_{m,s,k,j}\lambda\theta^2}{(\theta+1)(e^{\lambda(j+1)}-1)} \times L_2(\theta, \lambda(j+1), 1, \mu),$$

and

$$D(M) = -\mu + 2 \sum_{m,s,k,j=0}^{\infty} \frac{\omega_{m,s,k,j}\lambda\theta^2}{(\theta+1)(e^{\lambda(j+1)}-1)} L_2(\theta, \lambda(j+1), 1, M),$$

respectively. The Lorenz and Bonferroni curves are given by

$$L(p) = \frac{1}{\mu}\int_0^q t f_{RB-LP}(t)dt, \quad \text{and} \quad B(p) = \frac{1}{p\mu}\int_0^q t f_{RB-LP}(t)dt,$$

respectively, where $q = F^{-1}_{RB-LP}(t)$. Also using Lemma 2, we note that Bonferroni and Lorenz curves of the RB-LP distribution can be written as

$$\begin{aligned} B(p) &= \frac{1}{p\mu}\left[\int_0^\infty x f_{RB-LP}(x)dx - \int_q^\infty x f_{RB-LP}(x)dx\right] \\ &= \frac{1}{p\mu}\left[\mu - \sum_{m,s,k,j=0}^{\infty} \frac{\omega_{m,s,k,j}\lambda\theta^2}{(\theta+1)(e^{\lambda(k+1)}-1)} L_2(\theta, \lambda(j+1), 1, q)\right], \end{aligned}$$

and

$$\begin{aligned} L(p) &= \frac{1}{\mu}\left[\int_0^\infty x f_{RB-LP}(x)dx - \int_q^\infty x f_{RB-LP}(x)dx\right] \\ &= \frac{1}{\mu}\left[\mu - \sum_{m,s,k,j=0}^{\infty} \frac{\omega_{m,s,k,j}\lambda\theta^2}{(\theta+1)(e^{\lambda(k+1)}-1)} L_2(\theta, \lambda(j+1), 1, q)\right], \end{aligned}$$

respectively.

5. Distribution of Order Statistics and Rényi Entropy

Order statistics are of tremendous practical importance in probability and statistics, particularly in lifetime data analysis. Rényi entropy is an extension of Shanon entropy. Entropy is a measure of uncertainty or randomness. In this section, the distribution of the i^{th} order statistics and Rényi entropy for the RB-LP distribution are presented.

5.1. Distribution of Order Statistics

The *pdf* of i^{th} order statistics from the RB-LP distribution is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[F(x)]^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}(-1)^p\binom{n-i}{p}[F(x)]^{i+p-1} \\ &\times \frac{1}{\Gamma(\delta)}\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{\delta-1} \\ &\times \frac{\lambda\theta^2(1+x)e^{-\theta x}(exp\lambda(1-\frac{1+\theta+\theta x)}{1+\theta}e^{-\theta x}))}{(1+\theta)(e^{\lambda}-1)}. \end{aligned}$$

Note that, the *cdf* of RB-LP distribution can be written as

$$
\begin{aligned}
F_2(x) &= 1-\frac{1}{\Gamma(\delta)}\int_0^{-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)} t^{\delta-1}e^{-t}dt\\
&= 1-\sum_{k=0}^{\infty}(-1)^k\frac{\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{\delta+k}}{\Gamma(\delta)k!(\delta+k)}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
[F(x)]^{i+p-1} &= \left[1-\frac{\sum_{k=0}^{\infty}(-1)^k\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{\delta+k}}{\Gamma(\delta)k!(\delta+k)}\right]^{i+p-1}\\
&= \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(i+p)}{\Gamma(i+p-j)j!}\left[\frac{\sum_{k=0}^{\infty}(-1)^k\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{\delta+k}}{\Gamma(\delta)k!(\delta+k)}\right]^{j}\\
&= \sum_{j,k=0}^{\infty}\frac{(-1)^j\Gamma(i+p)d_{\delta+k,j}}{\Gamma(i+p-j)j!\,[\Gamma(\delta)]^j}\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{\delta+k},
\end{aligned}
$$

where $d_{\delta+k,j}=((\delta+k)c_0)^{-1}\sum_{l=1}^{\delta+k}[(j)l-\delta-k+l]\,c_l d_{\delta+k-1,j}$ and $d_0=c_0^j$. Therefore, the *pdf* of the i^{th} order statistic can be written as:

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n!}{[\Gamma(\delta)]^{j+1}(i-1)!(n-i)!}\sum_{p=0}^{n-i}\sum_{j,k=0}^{\infty}\frac{(-1)^{p+k+j}\Gamma(i+p)d_{\delta+k,j}}{\Gamma(i+p-j)j!}\binom{n-i}{p}\\
&\times\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{2\delta+k-1}\\
&\times\frac{\lambda\theta^2(1+x)e^{-\theta x}(exp\lambda(1-\frac{1+\theta+\theta x)}{1+\theta}e^{-\theta x}))}{(1+\theta)(e^{\lambda}-1)}.
\end{aligned}
$$

Using series representation defined earlier in equations (11), (12), and (13), we have

$$
\begin{aligned}
\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]^{2\delta+k-1} &= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{2\delta+k-1}{m}b_{s,m}\\
&\times\left(1-\frac{1-e^{\left(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}}{1-e^{\lambda}}\right)^{m+s+2\delta+k-1}.
\end{aligned}
$$

Now using the generalised binomial theorem in equation (14), for a positive integer b and $|z| < 1$, we have

$$\left[1 - \frac{1 - exp\left(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{1 - e^{\lambda}}\right]^{m+s+2\delta+k-1} = \sum_{v=0}^{\infty} \frac{(-1)^v \Gamma(m+s+2\delta+k)}{\Gamma(m+s+\delta+k-v)v!} \times \left(\frac{1 - exp\left(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)}{1 - e^{\lambda}}\right)^v$$

and

$$\begin{aligned}
\left[1 - exp\left(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta})e^{-\theta x}\right)\right]^v &= \sum_{w=0}^{\infty} \frac{(-1)^w \Gamma(v+1)}{\Gamma(v+1-w)w!}\left(exp\left(\lambda(1 - \frac{1+\theta+\theta x}{1+\theta})e^{-\theta x}\right)\right)^w \\
&= \sum_{w=0}^{\infty} \frac{(-1)^w \Gamma(v+1)}{\Gamma(v+1-w)w!} e^{\lambda(w(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{[\Gamma(\delta)]^{j+1}(i-1)!(n-i)!} \sum_{p=0}^{n-i} \sum_{j,k,m=0}^{\infty} \sum_{s,v,w=0}^{\infty} (-1)^{p+k+j+v+w} \\
&\times \binom{n-i}{p}\binom{2\delta+k-1}{m} b_{s,m} d_{\delta+k,j} \frac{e^{\lambda(w+1)} - 1}{e^{\lambda(w+1)} - 1} \\
&\times \frac{\Gamma(m+s+2\delta+k)\Gamma(v+1)}{\Gamma(m+s+\delta+k-v)v!\Gamma(v+1-w)w!} \frac{e^{\lambda w(1-\frac{1+\theta+\theta x}{1+\theta})e^{-\theta x}}}{(1-e^{\lambda})^v} \\
&\times \frac{\lambda\theta^2(1+x)e^{-\theta x}(exp[\lambda(1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})])}{(1+\theta)(e^{\lambda} - 1)} \\
&= \frac{n!}{[\Gamma(\delta)]^{j+1}(i-1)!(n-i)!} \sum_{p=0}^{n-i} \sum_{j,k,m=0}^{\infty} \sum_{s,v,w=0}^{\infty} (-1)^{p+k+j+v+w} \\
&\times \binom{n-i}{p}\binom{2\delta+k-1}{m} b_{s,m} d_{\delta+k,j} \\
&\times \frac{e^{\lambda(w+1)} - 1}{(1-e^{\lambda})^v(e^{\lambda} - 1)} \frac{\Gamma(m+s+2\delta+k)\Gamma(v+1)}{\Gamma(m+s+\delta+k-v)v!\Gamma(v+1-w)w!} \\
&\times \frac{\lambda\theta^2(1+x)e^{-\theta x}(exp[\lambda(w+1)(1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})])}{(1+\theta)(e^{\lambda(w+1)} - 1)} \\
&= \frac{n!}{[\Gamma(\delta)]^{j+1}(i-1)!(n-i)!} \sum_{p=0}^{n-i} \sum_{j,k,m=0}^{\infty} \sum_{s,v,w=0}^{\infty} (-1)^{p+k+j+v+w} \\
&\times \binom{n-i}{p}\binom{2\delta+k-1}{m} b_{s,m} d_{\delta+k,j} \\
&\times \frac{e^{\lambda(w+1)} - 1}{(1-e^{\lambda})^v(e^{\lambda} - 1)} \frac{\Gamma(m+s+2\delta+k)\Gamma(v+1)}{\Gamma(m+s+\delta+k-v)v!\Gamma(v+1-w)w!} f_{LP}(x;\theta,\lambda(w+1)),
\end{aligned}$$

where

$$f_{LP}(x;\theta,\lambda(j+1)) = \frac{\lambda\theta^2(1+x)e^{-\theta x}(exp[\lambda(j+1)(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})])}{(1+\theta)(e^{\lambda(w+1)}-1)}$$

is the Lindley-Poisson *pdf* with parameters $\theta > 0$ and $\lambda(w+1) > 0$. That is, the distribution of the i^{th} order statistic from RB-LP distribution is a linear combination of Lindley-Poisson with parameters θ and $\lambda(w+1) > 0$.

5.2. Rényi Entropy

Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} log\left(\int_0^\infty [f_{RB-LP}(x)]^v\right) dx, v \neq 1, v > 0.$$

Note that

$$\begin{aligned}\int_0^\infty [f_{RB-LP}(x)]^v\, dx &= \frac{\lambda^v\theta^{2v}}{[\Gamma(\delta)(1+\theta)(e^\lambda-1)]^v} \\ &\times \int_0^\infty \left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^\lambda}\right)\right]^{\delta v - v} \\ &\times (1+x)^v exp\left(-\theta v x\right) e^{\left(\lambda v\left(1-\frac{1+\theta+\theta x}{1-e^\lambda}e^{-\theta x}\right)\right)} dx.\end{aligned}$$

Using the series representations in equations (11), (12), and (13), we get

$$\begin{aligned}\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^\lambda}\right)\right]^{\delta v - v} &= \sum_{s=0}^\infty\sum_{m=0}^\infty \binom{\delta v - v}{m}\frac{b_{s,m}}{(1-e^\lambda)^{m+s+\delta v - v}} \\ &\times \left[1-\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^\lambda}\right]^{m+s+\delta v - v}.\end{aligned}$$

Now using the generalization of binomial theorem in equation (14), we have

$$\begin{aligned}\left[1-\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^\lambda}\right]^{m+s+\delta v - v} &= \sum_{k=0}^\infty \frac{(-1)^k\Gamma(m+s+\delta v - v+1)}{\Gamma(m+s+\delta v - v+1-k)k!} \\ &\times \left[\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^\lambda}\right]^k\end{aligned}$$

and

$$\left[1-exp\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)\right]^k = \sum_{j=0}^\infty \frac{(-1)^j\Gamma(k+1)}{\Gamma(k+1-j)j!}e^{\left(\lambda j\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)}.$$

Therefore,

$$\begin{aligned}\int_0^{\infty}\left[f_{RB-LP}(x)\right]^v dx &= \frac{\lambda^v\theta^{2v}}{[\Gamma(\delta)(1+\theta)(e^{\lambda}-1)]^v}\sum_{k,s,m,j=0}^{\infty}\binom{\delta v-v}{m}b_{s,m}\frac{1}{(1-e^{\lambda})^k}\\ &\times \frac{(-1)^{k+j}\Gamma(m+s+\delta v-v+1)\Gamma(k+1)}{\Gamma(m+s\delta v-v+1-k)k!\Gamma(k+1-j)j!}\\ &\times \int_0^{\infty}(1+x)^v e^{-\theta v x}exp\left(\lambda(j+v)\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)dx.\end{aligned}$$

Now consider the integral

$$\int_0^{\infty}(1+x)^v e^{-\theta v x}exp\left(\lambda(j+v)\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)dx.$$

Note that

$$exp\left(\lambda(j+v)(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)=\sum_{w=0}^{\infty}\frac{\left(\lambda(j+v)(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x})\right)^w}{w!},$$

$$\left[1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right]^w=\sum_{p=0}^{\infty}(-1)^p\binom{w}{p}\frac{\sum_{q=0}^{p}\binom{p}{q}\theta^q(1+x)^q}{1+\theta)^q}exp(-\theta p x),$$

$$(1+x)^{v+q}=\sum_{r=0}^{v+q}\binom{v+q}{r}x^r,\quad \text{and}\quad \int_0^{\infty}x^r e^{\theta(p+v)x}dx=\frac{\Gamma(r+1)}{[\theta(p+v)]^{r+1}}.$$

Therefore,

$$\begin{aligned}\int_0^{\infty}\left[f_{RB-LP}(x)\right]^v dx &= \frac{\lambda^v\theta^{2v}}{[\Gamma(\delta)(1+\theta)(e^{\lambda}-1)]^v}\sum_{k,s,m,j,w,p=0}^{\infty}\sum_{r=0}^{v+q}\binom{\delta v-v}{m}b_{s,m}\frac{1}{(1-e^{\lambda})^k}\\ &\times \frac{(-1)^{k+j+p}\Gamma(m+s+\delta v-v+1)\Gamma(k+1)}{\Gamma(m+s\delta v-v+1-k)k!\Gamma(k+1-j)j!}\\ &\times \frac{\lambda^w(j+v)^w}{w!}\binom{w}{p}\binom{v+q}{r}\frac{\sum_{q=0}^{p}\theta^q}{(1+\theta)^q}\frac{\Gamma(r+1)}{[\theta(p+v)]^{r+1}}.\end{aligned}$$

Consequently, Rényi entropy of the RB-LP distribution is given by

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}log\Bigg[\frac{\lambda^v\theta^{2v}}{[\Gamma(\delta)(1+\theta)(e^{\lambda}-1)]^v}\sum_{k,s,m,j,w,p=0}^{\infty}\sum_{r=0}^{v+q}\binom{\delta v-v}{m}b_{s,m}\frac{1}{(1-e^{\lambda})^k}\\ &\times \frac{(-1)^{k+j+p}\Gamma(m+s+\delta v-v+1)\Gamma(k+1)}{\Gamma(m+s+\delta v-v+1-k)k!\Gamma(k+1-j)j!}\\ &\times \frac{\lambda^w(j+v)^w}{w!}\binom{w}{p}\binom{v+q}{r}\frac{\sum_{q=0}^{p}\theta^q}{(1+\theta)^q}\frac{\Gamma(r+1)}{[\theta(p+v)]^{r+1}}\Bigg], v\neq 1, v>0.\end{aligned}$$

6. Estimation and Inference

6.1. Maximum Likelihood Estimation

Let $X \sim RB{-}LP(\lambda,\delta,\theta)$ and $\Delta = (\lambda,\delta,\theta)^T$ be the parameter vector. The log-likelihood function for a single observation x of X is given by

$$\begin{aligned} L &= (\delta-1)log\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right] + log(\lambda) + 2log(\theta) + log(1+x) \\ &- \theta x + \left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right) - log(1+\theta) - log(e^{\lambda}-1) - log(\Gamma(\delta)). \end{aligned} \tag{16}$$

The first derivatives of the log-likelihood function with respect to the parameters $\Delta = (\lambda,\delta,\theta)^T$ are given by

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= -\frac{(\delta-1)}{\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]\left[\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right]} \\ &\times \frac{\left[e^{\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)}\right]\left[1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right]\left[1-e^{\lambda}\right] + \left[1-e^{(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}\right]e^{\lambda}}{(1-e^{\lambda})^2} \\ &+ \frac{1}{\lambda} - \frac{e^{\lambda}}{e^{\lambda}-1} + \left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right), \end{aligned}$$

$$\frac{\partial L}{\partial \delta} = log\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right] - \frac{\Gamma'(\delta)}{\Gamma(\delta)},$$

and

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -\frac{(\delta-1)}{\left[-log\left(\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right)\right]\left[\frac{1-exp(\lambda(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}))}{1-e^{\lambda}}\right]} \\ &\times \left[exp\left(\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)\right)\right]\left[\frac{1}{1+\theta}-1+\theta+\theta x\right]\frac{\lambda x e^{-\theta x}}{1+\theta} \\ &+ \frac{2}{\theta} - x - \frac{1}{1+\theta} + \left[\frac{1}{1+\theta}-1+\theta+\theta x\right]\frac{\lambda x e^{-\theta x}}{1+\theta}, \end{aligned}$$

respectively. Solving the nonlinear equations $\frac{\partial \ell}{\partial \lambda}=0$, $\frac{\partial \ell}{\partial \delta}=0$, $\frac{\partial \ell}{\partial \theta}=0$, where $\ell = \sum_{i=1}^{n} L_i$ yields the maximum likelihood estimates. Since the equations are not in closed form, they must be solved by using iterative methods, such as Newton-Raphson method.

6.2. Fisher Information Matrix and Asymptotic Confidence Intervals

The Fishers Information Matrix (FIM) of the RB-LP distribution is a (3×3) matrix given by

$$I(\Delta) = \begin{pmatrix} I_{\lambda\lambda} & I_{\lambda\delta} & I_{\lambda\theta} \\ I_{\delta\lambda} & I_{\delta\delta} & I_{\delta\theta} \\ I_{\theta\lambda} & I_{\theta\delta} & I_{\theta\theta} \end{pmatrix}.$$

The elements of the FIM are obtained by considering the second order partial derivatives of equation (16) with respect to parameters $\Delta = (\lambda, \delta, \theta)^T$. Let $\hat{\Delta} = (\hat{\lambda}, \hat{\delta}, \hat{\theta})$ be the maximum likelihood estimates of $\Delta = (\lambda, \delta, \theta)^T$. When the conditions for parameters in the interior of parameter space, but on the boundary are met, the asymptotic distribution of $\hat{\Delta}$ is given by

$$\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_3(\underline{0}, I^{-1}(\Delta)),$$

where $I(\Delta)$ is the expected Fisher Information matrix. The multivariate normal distribution $N_3(\underline{0}, I^{-1}(\Delta))$ can be used to construct confidence regions and confidence intervals for the parameter models. A large sample $100(1-\alpha)\%$ confidence for λ, δ and θ are given by

$$\hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\Delta})}, \hat{\delta} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Delta})}, \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\Delta})},$$

respectively, where $I_{\lambda\lambda}^{-1}, I_{\delta\delta}^{-1}$ and $I_{\theta\theta}^{-1}$ are the diagonal elements of $I_n^{-1}(\hat{\Delta})$, and $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha^{th}}{2}$ percentile of the standard normal distribution.

To compare the RB-LP distribution with its sub-models, the Likelihood ratio (LR) test is applied. For example, the LR test statistic for testing the null hypothesis $\delta = 1$ is

$$\omega = 2[ln(L(\hat{\lambda}, \hat{\delta}, \hat{\theta} - ln(L(\tilde{\lambda}, 1, \tilde{\theta})),]$$

where $\hat{\lambda}, \hat{\delta}, \hat{\theta}$ are the unrestricted estimates and $\tilde{\lambda}, \tilde{\theta}$ are the restricted estimates. When $\omega > \chi^2_\alpha$, the null hypothesis is rejected, where χ^2_α denotes the upper $100\alpha\%$ point of the chi-square distribution with 1 degree of freedom.

7. Monte Carlos Simulation Study

In this section, the performance of the RB-LP distribution is studied. Various simulations are conducted for different parameter values and sample sizes. The simulations were repeated N=5000 times each with sample sizes $n = 25, 50, 100, 200, 400, 800$, the true parameters values (two sets) are $\lambda = 5, \delta = 1, \theta = 0.3$ and $\lambda = 7, \delta = 2, \theta = 1$. Five quantities were computed in this simulation study: Five quantities were computed in this simulation study: the mean, average bias, Root Mean Squared Error (RMSE), Coverage Probability (CP) and Average Width (AW) of the 95% confidence interval, that is,

1. Mean of the MLE $\widehat{\Delta}$ of the parameters $\Delta = (\lambda, \delta, \theta)$:

$$\frac{\sum_{i=1}^{N} \widehat{\Delta_i}}{N}.$$

2. Average bias of the MLE $\widehat{\Delta}$:

$$\frac{\sum_{i=1}^{N}(\widehat{\Delta}_i - \Delta)}{N}.$$

3. Root Mean Squared Error (RMSE) of the MLE $\widehat{\Delta}$:

$$\sqrt{\frac{\sum_{i=1}^{N}(\widehat{\Delta}_i - \Delta)^2}{N}}.$$

4. Coverage Probability (CP) of 95% confidence intervals of the parameters.
5. Average Width (AW) of the 95% confidence intervals.

Table 3 shows the mean MLEs of the three model parameters along with their respective RMSE and bias for different sample sizes. From the results it can be verified that as the sample size n increases, the mean converges to the true value, the RMSE decays toward zero, the biases decrease and the average width of the confidence intervals decreases for all the parameter values.

Table 3. Monte Carlo Simulation Results:Mean, Average Bias, RMSE, CP and AW

		$\lambda=5.0, \delta=1.0, \theta=0.3$					$\lambda=7.0, \delta=2.0, \theta=1.0$				
Parameter	n	Mean	Average Bias	RMSE	CP	AW	Mean	Average Bias	RMSE	CP	AW
λ	25	11.5573	6.5573	17.1962	0.9956	72.5470	12.7936	5.7936	14.1007	0.9902	55.2821
	50	7.5153	2.5153	5.9086	0.9896	19.2009	9.0182	2.0182	6.8854	0.9988	21.9189
	100	5.9561	0.9561	2.1722	0.9852	7.1225	8.0533	1.0533	3.1787	0.9586	10.6152
	200	5.4182	0.4182	1.0576	0.9796	3.7908	7.5449	0.5449	1.8567	0.955	6.5760
	400	5.1914	0.1914	0.6307	0.9674	9.6302	7.2436	0.2436	1.1524	0.9538	4.2674
	800	5.0830	0.0830	0.3988	0.9558	1.5506	7.1355	0.1355	0.7624	0.9532	2.9290
δ	25	3.4601	2.4601	8.1622	0.9318	27.9168	5.9097	3.9097	10.8987	0.8958	42.4436
	50	1.7418	0.7418	2.6472	0.9052	6.8577	3.7463	1.7463	5.2278	0.8912	16.0136
	100	1.2661	0.2661	0.9516	0.9144	2.8563	2.6482	0.6482	2.2799	0.8972	7.4272
	200	1.1101	0.1101	0.4830	0.9318	1.6117	2.3430	0.3430	1.2782	0.9300	4.5052
	400	1.0534	0.0534	0.2825	0.942	1.0362	2.1496	0.1496	0.7819	0.9372	2.8759
	800	1.0234	0.0234	0.1780	0.9562	0.6941	2.0844	0.0844	0.5079	0.9472	1.9577
θ	25	0.3062	0.0062	0.1082	0.8966	0.4949	1.2603	0.2603	1.5234	0.8986	6.5335
	50	0.3049	0.0049	0.0825	0.9244	0.3275	1.2121	0.2121	1.1842	0.907	4.3808
	100	0.3045	0.0045	0.0560	0.9422	0.2132	1.1588	0.1588	0.8301	0.9300	2.9611
	200	0.3024	0.0024	0.0380	0.9434	0.1454	1.0496	0.0496	0.5042	0.937	1.9351
	400	0.3007	0.0007	0.0177	0.9432	0.1010	1.0338	0.0338	0.3482	0.9402	1.3487
	800	0.3004	0.0004	0.0147	0.9498	0.0708	1.0097	0.0097	0.2388	0.9432	0.9371

Table 4. Failure times of 50 Components data

0.036	0.058	0.061	0.074	0.078	0.086	0.102	0.103	0.114	0.116
0.148	0.183	0.192	0.254	0.262	0.379	0.381	0.538	0.570	0.574
0.590	0.618	0.645	0.961	1.228	1.600	2.006	2.054	2.804	3.058
3.076	3.147	3.625	3.704	3.931	4.073	4.393	4.534	4.893	6.274
6.816	7.896	7.904	8.022	9.337	10.940	11.020	13.880	14.730	15.080

8. Application

In this section, an application of RB-LP distribution to a real dataset is presented to demonstrate the flexibility and applicability of the RB-LP distribution. The R-B Gamma Lindley-Poisson (RB-LP), Lindley Poisson (LP) and Lindley (L) distributions were fitted to the dataset. Another three parameter model, the beta log-Logistic (BLLoG) distribution is fitted to the dataset for comparison purposes. The data set was studied by Murthy *et al.* (2004), and it represents the failure times of 50 components (per 1000h) of an engineering system and is presented in Table 4. The *cdf* of the beta log-logistic (BLLoG) distribution is given by

$$\begin{aligned} F_{BLLoG}(x;\alpha,\lambda,\delta) &= \frac{1}{B(\lambda,\theta}\int_0^{G_{LLoG}(x;\delta)} t^{\lambda-1}(1-t)^{\theta-1}dt \\ &= I_{G_{LLoG}(x;\delta)}(\lambda,\theta), \end{aligned}$$

where $G_{LLoG}(x;\delta) = 1-(1+x^{\delta})^{-1}$ is the log-logistic with shape parameter δ.

We maximized the likelihood function using *NLmixed* in SAS as well as the function *nlm* in R (R Development Core Team (2011)). These functions were applied and executed for wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum. The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions (see Seregin (2010); Santos Silva and Tenreyro (2010); Zhou (2009); Xia *et al.* (2009)) and references therein for additional details.

The estimates of the parameters of the distributions, standard errors (in parentheses), -2loglikelihood statistic (-2log(L)), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (AICC) and Hannan-Quinn information criterion (HQIC) are given in table 5. The Likelihood ratio (LR) is used test to compare the RB-LP distribution with its nested models. Table 5 presents the results obtained from RB-LP, LP, L and *BLLoG* distributions to the dataset.
The estimated variance-covariance matrix of the RB-LP distribution is given by

$$\begin{pmatrix} 1.478616\times 10^{-04} & 0.0001719267 & -9.609033\times 10^{-06} \\ 1.719267\times 10^{-04} & 0.5215083086 & -4.286563\times 10^{-02} \\ -9.60903\times 10^{-06} & -0.042865632 & 3.87013500\times 10^{-03} \end{pmatrix}$$

Table 5. Estimates of Models for Failure times of 50 Components data

	Estimates			Statistics				
Model	λ	δ	θ	$-2\log L$	AIC	$AICC$	BIC	$HOIC$
RB-LP	2.7415×10^{-07}	2.80030	0.158290	216.9950	222.9952	223.5169	228.7313	225.1795
	(0.01216)	(0.72216)	(0.06221)					
LP	6.3103×10^{-10}	1	0.422069	278.0679	282.0679	282.3232	285.8919	283.5241
	(0.008245)	-	(0.045636)					
L	-	1	0.498729	240.3559	242.3559	242.4393	244.2679	243.084
	-	-	(0.051314)					
BLLoG	0.495752	0.594450	1.101594	320.9639	326.9639	327.4856	332.6999	329.1482
	(0.056299)	(0.109952)	(0.138123)					

and the 95% confidence intervals for λ, δ and θ are given by $\lambda \in 2.7415 \times 10^{-07} \pm 1.96 \times \sqrt{1.478616 \times 10^{-04}}$, $\delta \in 2.8003 \pm 1.96 \times \sqrt{0.5215083086}$, and $\theta \in 1.5829 \times 10^{-01} \pm 1.96 \times \sqrt{3.870135 \times 10^{-03}}$, respectively.

To test if the RB-LP distribution is significantly different from LP and L distributions, the LR test is used and the following hypotheses are tested. H_0 :LP versus H_a :RB-LP and H_0 :L versus H_a :RB-LP, the test statistics and the p-values are 119.9 (p-value<0.0001) and 63.2 (p-value<0.0001), respectively. Smaller p-values suggests that there are significant differences between RB-LP and LP distributions as well as between the RB-LP and L distributions. We conclude from the results that the RB-LP distribution provides a better fit, as it has the smallest values of the goodness-of-fit statistics. The plot of the fitted densities shown in figure 5 also suggests that the RB-LP distribution provides a better fit.

9. Concluding Remarks

A new and generalized distribution called the Ristić and Balakrishnan gamma Lindley-Poisson distribution (RB-LP) is proposed and studied. The RB-LP distribution has the gamma Lindley, Lindley-Poisson and Lindley distributions as special cases. The RB-LP distribution possesses hazard function with very flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, Bonferroni and Lorenz curves, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique is used to estimate the model parameters. Finally, the RB-LP distribution is fitted to a real dataset to illustrate its applicability and usefulness. The model can be applied in the same vein in different fields of study including survival analysis, reliability analysis and engineering.

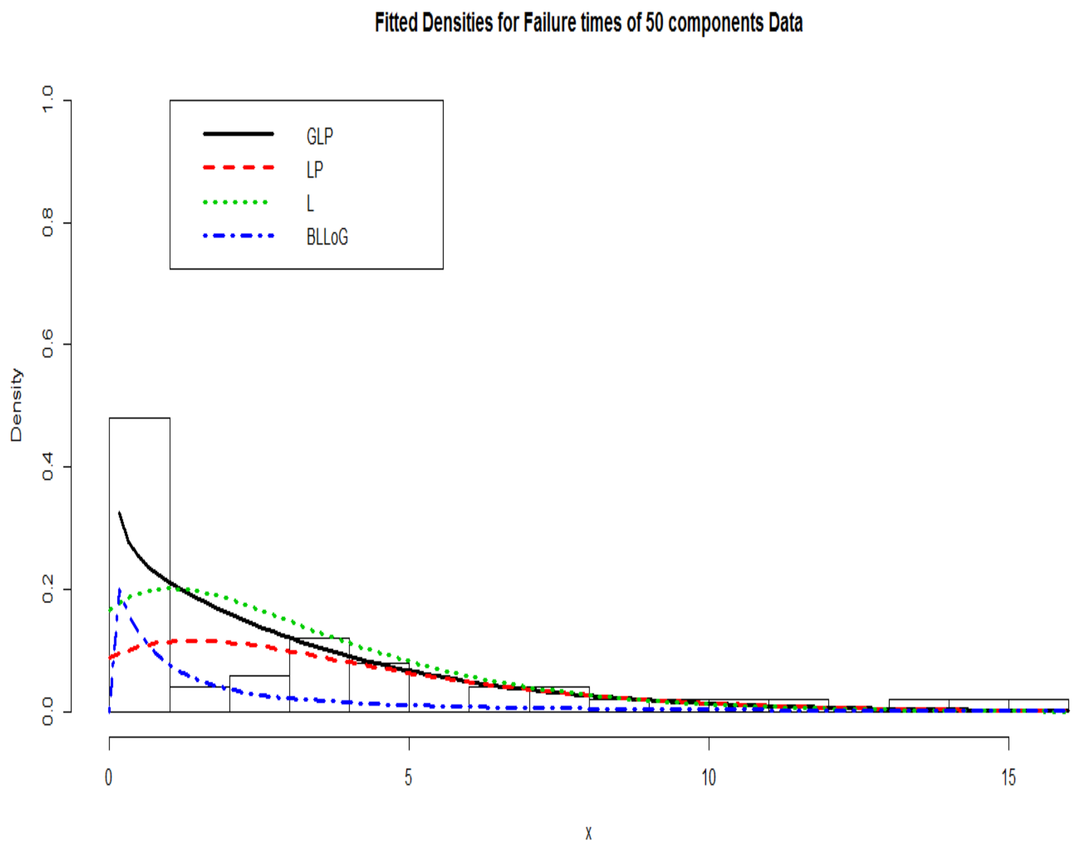

Fig. 5. Fitted densities for Failure times of 50 Components data

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