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Half-Logistic Odd Weibull-Topp-Leone-G Family of Distributions: Model, Properties and Applications

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Abstract. A new flexible and versatile generalized family of distributions, namely, half logistic odd Weibull-Topp-Leone-G (HLOW-TL-G) distribution is presented. The distribution can be traced back to the exponentiated-G distribution. We derive the statistical properties of the proposed family of distributions. Maximum likelihood estimates of the HLOW-TL-G family of distributions are also presented. Five special cases of the proposed family are presented. A simulation study and real data applications on one of the special cases are also presented.

Key words: half-logistic-G; Topp-Leone-G; Odd Weibull-G; generalized distribution; maximum likelihood estimation
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Résum'e (French Abstract) Une généralisation de la famille de lois Exponentiated-G est proposée. Quelques propriétes statistiques et l'estimation du modèle par la méthode du maximum de vraisemblance sont étudiées. Des applications à des données réelles sont faites..

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1. Introduction

Gurvich *et al.*(1997) introduced a general family of univariate distributions from the Weibull distribution. The cumulative distribution function (*cdf*) and probability density function (*pdf*) of the generalized family are given by

$$G(x;\xi) = 1 - \exp[-\alpha H(x;\xi)] \tag{1}$$

and

$$g(x;\xi) = \alpha \exp[-\alpha H(x;\xi)]h(x;\xi), \tag{2}$$

respectively, where $H(x;\xi)$ is a non-negative and differentiable monotonically increasing function which depends of the parameter vector ξ. The choices of $H(x;\xi)$ is infinitely many and for example $H(x;\xi)$ can be x, x^2, $\frac{K(x;\xi)}{\overline{K}(x;\xi)}$, $\frac{1-\overline{K}^{\alpha}(x;\xi)}{\overline{K}^{\alpha}(x;\xi)}$, where K is a *cdf*, to mention a few.

The generalization by Gurvich has led to the advent of many generalized distributions, for example the Weibull-G by Bourguignon *et al.*(2014). There are several generalizations in the literature including beta-G by Eugene *et al.*(2002), beta odd Lindley-G by Chipepa *et al.*(2019a), Half Logistic-G by Cordeiro *et al.*(2016), Topp-Leone-G by Al-Shomrani *et al.*(2016), Topp-Leone-Marshall-Olkin-G by Chipepa *et al.*(2020), Kumaraswamy-G by Cordeiro *et al.*(2010), Kumaraswamy odd Lindley-G by Chipepa *et al.*(2019b), to mention a few. All these generalizations provide versatile and flexible models in data fitting.

Balakrishnan(1985) also developed the half logistic distribution which received much attention from physicists and hydrologists. The distribution by Balakrishnan(1985) exhibit monotonic hazard rate shapes. Generalizations of the

half logistic distribution also produced very flexible distributions in data modeling. Available in the literature are the exponentiated half-logistic family of distributions by Cordeiro *et al.*(2014), type I half-logistic family of distributions by Cordeiro *et al.*(2016), Kumaraswamy type 1 half logistic family of distributions by El-Sayed *et al.*(2019), odd exponentiated half-logistic-G family by Afify *et al.*(2017), exponentiated Half-Logistic Exponential distribution by Abdullah *et al.*(2018), to mention a few.
In this paper, we propose a new model referred to as the half logistic odd Weibull-Topp-Leone-G (HLOW-TL-G) family of distributions. We use the generalization by Cordeiro *et al.*(2016) with *cdf* given by

$$F_{HL-G}(x;\xi)=\frac{G(x;\xi)}{1+\overline{G}(x;\xi)} \tag{3}$$

and our recently proposed generalized distribution referred to as the odd Weibull-Topp-Leone-G (OW-TL-G) family of distributions with *cdf* given by

$$F(x;b,\beta,\xi)=1-\exp\left\{-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{[1-(1-\overline{G}^2(x;\xi))^b]}\right]^\beta\right\}, \tag{4}$$

for $b,\beta>0$ and parameter vector ξ, where $\overline{G}(x;\xi)=1-G(x;\xi)$ and $G(x;\xi)$ is the baseline *cdf*. Note that $F(x;b,\beta,\xi)$ is a *cdf* since it is non-decreasing, right continuous and bounded. We develop a new family of distributions that

- is versatile and flexible in data fitting;
- can model both monotonic and non-monotonic hazard rate functions;
- has good tractability property and can be traced back to the exponentiated-G distribution.

We hope the new distribution will attract the attention of more researchers in data analysis.
The rest of the paper is arranged as follows; the proposed model, expansion of the density is presented in Section 2. In Section 3, we present statistical properties. We do estimation in Section 4. Special cases are presented in Section 5. A simulation study is conducted in Section 6. Real data examples are given in Section 7, followed by concluding remarks.

2. The Model

We develop the HLOW-TL-G family of distributions using equations 3 and 4. The *cdf* and *pdf* of the HLOW-TL-G family of distributions are given by

$$F_{HLOW-TL-G}(x;b,\beta,\xi)=\frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)} \tag{5}$$

and

$$f_{HLOW-TL-G}(x;b,\beta,\xi) = \frac{4b\beta g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta-1}\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)}{(1-[1-\overline{G}^2(x;\xi)]^b)^{\beta+1}\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)\right)^2}, \tag{6}$$

respectively, for $b, \beta > 0$ and ξ is a vector of parameters from the baseline distribution function G(.). Clearly, $0 \leq F_{HLOW-TL-G}(x;b,\beta,\xi) \leq 1$ for all $x > 0$ and equation (5) is a *cdf* based on the generalizations of Cordeiro et al. Cordeiro *et al.*(2016) in equation (3) and the OW-TL-G family of distributions in equation (4).

2.1. Hazard Rate and Reverse Hazard Rate Functions

The hazard rate and reverse hazard rate functions of the HLOW-TL-G family of distributions are given by

$$\begin{aligned} h_F(x;b,\beta,\xi) &= \frac{4b\beta g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta-1}\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)}{(1-[1-\overline{G}^2(x;\xi)]^b)^{\beta+1}\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)\right)^2} \\ &\times \left[1-\frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)}\right]^{-1} \end{aligned}$$

and

$$\begin{aligned} \tau_F(x;b,\beta,\xi) &= \frac{4b\beta g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta-1}\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)}{(1-[1-\overline{G}^2(x;\xi)]^b)^{\beta+1}\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)\right)} \\ &\times \left[1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^\beta\right)\right]^{-1}, \end{aligned}$$

respectively.

2.2. Some Sub-Families of Distributions

(1) When $\beta = 1$, we obtain the half logistic odd exponential-Topp-Leone-G (HLOW-TL-G) family of distributions.

(2) We obtain the half logistic odd Rayleigh-Topp-Leone-G (*HLOR-TL-G*) family of distributions by setting $\beta = 2$.

(3) By letting $b = 1$, we obtain a new family of distributions with *cdf* given by

$$F(x;\beta,\xi) = \frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]^{\beta}\right)}.$$

(4) We also obtain other new families of distributions with *cdf*s given by

$$F(x;\xi) = \frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]\right)}$$

and

$$F(x;\xi) = \frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]^{2}\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]}{\overline{G}^2(x;\xi)}\right]^{2}\right)},$$

setting $b = \beta = 1$, and $b = 1$ and $\beta = 2$, respectively.

2.3. Expansion of Density Function

In this section, we provide series expansion of density of the HLOW-TL-G family of distributions. By applying the series expansion $(1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}$, get

$$\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right)^{-2} = \sum_{n=1}^{\infty} n(-1)^{n-1} \times \exp\left(-(n-1)\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)$$

such that

$$
\begin{aligned}
f_{HLOW-TL-G}(x;b,\beta,\xi) &= \sum_{n=1}^{\infty} n(-1)^{n-1} \frac{4b\beta g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta-1}}{(1-[1-\overline{G}^2(x;\xi)]^b)^{\beta+1}} \\
&\times \exp\left(-n\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right).
\end{aligned}
$$

Furthermore, using the following series expansions

$$
\exp\left(-n\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right) = \sum_{z=0}^{\infty} \frac{(-1)^z n^z}{z!}\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta z},
$$

$$
(1-[1-\overline{G}^2(x;\xi)]^b)^{-(\beta(z+1)+1)} = \sum_{w=0}^{\infty}(-1)^w \binom{-(\beta(z+1)+1)}{w}[1-\overline{G}^2(x;\xi)]^{bw},
$$

$$
[1-\overline{G}^2(x;\xi)]^{b(\beta z+\beta+w)-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b(\beta z+\beta+w)-1}{i}\overline{G}^{2i}(x;\xi)
$$

and

$$
\overline{G}^{2i+1}(x;\xi) = \sum_{j=0}^{\infty}(-1)^j \binom{2i+1}{j} G^j(x;\xi)
$$

yields

$$
\begin{aligned}
f_{HLOW-TL-G}(x;b,\beta,\xi) &= \sum_{z,w,i,j=0}^{\infty}\sum_{n=1}^{\infty} \frac{4b\beta n^{z+1}(-1)^{z+w+i+j+n-1}}{z!}\binom{-(\beta(z+1)+1)}{w} \\
&\times \binom{b(\beta z+\beta+w)-1}{i}\binom{2i+1}{j} g(x;\xi)G^j(x;\xi) \\
&= \sum_{j=0}^{\infty}\psi_j h_j(x;\xi), \qquad (7)
\end{aligned}
$$

where $h_j(x;\xi) = (j+1)g(x;\xi)G^j(x;\xi)$ is the exponentiated-G (Exp-G) distribution with power parameter j and

$$
\begin{aligned}
\psi_j &= \sum_{z,w,i=0}^{\infty}\sum_{n=1}^{\infty} \frac{4b\beta n^{z+1}(-1)^{z+w+i+j+n-1}}{z!(j+1)}\binom{-(\beta(z+1)+1)}{w} \\
&\times \binom{b(\beta z+\beta+w)-1}{i}\binom{2i+1}{j}. \qquad (8)
\end{aligned}
$$

Therefore, the HLOW-TL-G family of distributions is an infinite linear combination of Exp-G distribution. Other statistical properties such as moments, generating function, and probability-weighted moments of the HLOW-TL-G family of distributions can be derived directly from the Exp-G distribution.

3. Statistical Properties

In this section, we present some statistical properties of the HLOW-TL-G family of distributions. The statistical properties considered are the quantile function, moments, generating functions, distribution of order statistics, probability-weighted moments, and entropy.

3.1. Quantile Function

The quantile function for the HLOW-TL-G family of distributions is obtained by solving the non-linear equation:

$$\frac{1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)}=u$$

for $0\leq u\leq 1$. By simplifying the equation we get

$$\ln\left[\frac{1-u}{1+u}\right]=-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}$$

which simplifies to

$$1-\overline{G}^2(x;\xi)=\left[\frac{\left(-\ln\left[\frac{1-u}{1+u}\right]\right)^{1/\beta}}{1+\left(-\ln\left[\frac{1-u}{1+u}\right]\right)^{1/\beta}}\right]^{1/b}.$$

The equation can be further simplified to

$$G(x;\xi)=1-\left(1-\left[\frac{\left(-\ln\left[\frac{1-u}{1+u}\right]\right)^{1/\beta}}{1+\left(-\ln\left[\frac{1-u}{1+u}\right]\right)^{1/\beta}}\right]^{1/b}\right)^{1/2}.$$

Therefore, we obtain quantiles for HLOW-TL-G family of distributions by solving the equation

$$x(u) = G^{-1}\left[1-\left(1-\left[\frac{\left(-\ln\left[\frac{1-u}{1+u}\right]\right)^{1/\beta}}{1+\left(-\ln\left[\frac{1-u}{1+u}\right]\right)^{1/\beta}}\right]^{1/b}\right)^{1/2}\right] \tag{9}$$

via iterative methods in R or Matlab software. We present some quantiles for selected parameters values for the half logistic odd Weibull-Topp-Leone-log-logistic (HLOW-TL-LLoG) distribution in Table 1.

u	(0.5,1,1)	(0.8,1,1.5)	(1.1,1.5,0.5)	(0.5,0.9,0.5)	(1,1,0.9)
0.1	0.0143	0.1501	0.0346	0.0001	0.0738
0.2	0.0444	0.2515	0.0780	0.0015	0.1539
0.3	0.0822	0.3372	0.1259	0.0058	0.2358
0.4	0.1254	0.4151	0.1784	0.0150	0.3205
0.5	0.1737	0.4897	0.2371	0.0309	0.4104
0.6	0.2286	0.5648	0.3046	0.0568	0.5092
0.7	0.2936	0.6451	0.3864	0.0983	0.6235
0.8	0.3766	0.7384	0.4949	0.1693	0.7675
0.9	0.5028	0.8670	0.6684	0.3168	0.9845

Table 1: Table of Quantiles for Selected Parameters of the HLOW-TL-LLoG Distribution

3.2. Moments and Generating Functions

The HLOW-TL-G family of distributions is an infinite linear combination of the Exp-G distribution, we, therefore, derive the properties of the HLOW-TL-G family of distributions directly from the Exp-G distribution. Let H_j be an Exp-G distribution with power parameter j, then the r^{th} ordinary moment of the HLOW-TL-G is given by

$$\mu'_r = E(X^r) = \sum_{j=0}^{\infty} \psi_j E(H_j^r), \tag{10}$$

where ψ_j is the linear component and is given by equation (8) and $E(H_j^r)$ is the r^{th} moment of the Exp-G distribution. Also, the s^{th} central moment of X is given by

$$\mu_s = \sum_{r=0}^{s}\binom{s}{r}(-\mu'_1)^{s-r}E(X^r) = \sum_{r=0}^{s}\sum_{j=0}^{\infty}\psi_j\binom{s}{r}(-\mu'_1)^{s-r}E(H_j^r).$$

Furthermore, the r^{th} incomplete moment of X is given by

$$\phi_r(z)=\int_{-\infty}^{z}x^r f(x)dx=\sum_{j=0}^{\infty}\psi_j\int_{-\infty}^{z}x^r h_j(x;\xi)dx, \tag{11}$$

where $h_j(x;\xi)=(j+1)g(x;\xi)G^j(x;\xi)$ is the Exp-G distribution with power parameter j and $\int_{-\infty}^{z}x^r h_j(x;\xi)dx$ is the r^{th} incomplete moment of the Exp-G distribution. The incomplete moment is used to estimate Bonferroni and Lorenz curves, which are very useful in reliability, medicine, economics, demography, and insurance.

A table of moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for the HLOW-TL-LLoG distribution for selected parameters values are given in Table 2.

	(1,0.5,1.5)	(1.5,1,0.5)	(0.5,0.5,1.5)	(1.5,1.5,0.5)	(0.5,0.5,1)
$E(X)$	0.2922	0.2529	0.2746	0.3507	0.2032
$E(X^2)$	0.1820	0.1464	0.1602	0.2059	0.1113
$E(X^3)$	0.1313	0.1016	0.1113	0.1415	0.0754
$E(X^4)$	0.1023	0.0774	0.0846	0.1062	0.0566
$E(X^5)$	0.0836	0.0623	0.0679	0.0844	0.0451
SD	0.3108	0.2871	0.2912	0.2879	0.2646
CV	1.0637	1.1350	1.0602	0.8210	1.3019
CS	0.7195	0.9663	0.8398	0.4650	1.3108
CK	2.1675	2.7209	2.4669	2.0857	3.6237

Table 2: Moments of the HLOW-TL-LLoG distribution for some parameter values

The moment generating function (mgf) of the HLOW-TL-G family of distributions is given by

$$M_X(t)=E(e^{tX})=\sum_{j=0}^{\infty}\psi_j M_j(t),$$

where $M_j(t)$ is the mgf of the Exp-G distribution with power parameter j.

3.3. Distribution of Order Statistics

The *pdf* of the i^{th} order statistics can be obtained using the formula given by

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-j}\binom{n-i}{j}F(x)^{j+i-1}, \tag{12}$$

where $B(.,.)$ is the beta function. Substituting the *cdf* and *pdf* of the HLOW-TL-G family of distributions into equation (12) and considering $f(x)F(x)^{j+i-1}$, we have

$$f(x)F(x)^{j+i-1} = \frac{4b\beta g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta-1}\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)}{(1-[1-\overline{G}^2(x;\xi)]^b)^{\beta+1}} \times \frac{\left[1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right]^{j+i-1}}{\left[1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right]^{j+i+1}}.$$

Considering the following expansions

$$\left[1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right]^{-(j+i+1)} = \sum_{z=0}^{\infty}\binom{-(j+i+1)}{z} \times \exp\left(-z\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right),$$

$$\left[1-\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right]^{j+i-1} = \sum_{w=0}^{\infty}(-1)^w\binom{j+i-1}{w} \times \exp\left(-w\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)$$

and

$$\exp\left(-(z+w+1)\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right) = \sum_{l=0}^{\infty}\frac{(-1)^l(z+w+1)^l}{l!} \times \left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta l},$$

we get

$$f(x)F(x)^{j+i-1} = \sum_{z,w,l=0}^{\infty}\frac{4b\beta(-1)^{w+l}(z+w+1)^l}{l!}\binom{-(j+i+1)}{z}\binom{j+i-1}{w} \times \frac{g(x;\xi)\overline{G}(x;\xi)[1-\overline{G}^2(x;\xi)]^{b\beta(l+1)-1}}{(1-[1-\overline{G}^2(x;\xi)]^b)^{\beta(l+1)+1}}.$$

Also, considering the following generalized binomial series expansions

$$(1-[1-\overline{G}^2(x;\xi)]^b)^{-(\beta(l+1)+1)} = \sum_{p=0}^{\infty}(-1)^p\binom{-(\beta(l+1)+1)}{p}[1-\overline{G}^2(x;\xi)]^{bp},$$

$$[1-\overline{G}^2(x;\xi)]^{b(\beta l+\beta+p)-1} = \sum_{q=0}^{\infty}(-1)^q\binom{b(\beta l+\beta+p)-1}{q}\overline{G}^{2q}(x;\xi)$$

and

$$\overline{G}^{2q+1}(x;\xi) = \sum_{k=0}^{\infty}(-1)^k\binom{2q+1}{k}G^k(x;\xi)$$

we can write

$$\begin{aligned} f(x)F(x)^{j+i-1} &= \sum_{z,w,l,p,q,k=0}^{\infty}\frac{4b\beta(-1)^{w+l+p+q+k}(z+w+1)^l}{l!}\binom{-(j+i+1)}{z} \\ &\times \binom{j+i-1}{w}\binom{-(\beta(l+1)+1)}{p}\binom{b(\beta l+\beta+p)-1}{q} \\ &\times \binom{2q+1}{k}g(x;\xi)G^k(x;\xi). \end{aligned} \tag{13}$$

Hence, the distribution of the i^{th} order statistic from the HLOW-TL-G family of distributions can be obtained from the equation

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{z,w,l,p,q,k=0}^{\infty}\sum_{j=0}^{n-j}\frac{4b\beta(-1)^{w+l+p+q+k}(z+w+1)^l}{l!}\binom{n-i}{j} \\ &\times \binom{-(j+i+1)}{z}\binom{j+i-1}{w}\binom{-(\beta(l+1)+1)}{p}\binom{b(\beta l+\beta+p)-1}{q} \\ &\times \binom{2q+1}{k}g(x;\xi)G^k(x;\xi) \\ &\times \sum_{k=0}\psi_k^* h_k(x;\xi), \end{aligned} \tag{14}$$

where $h_k(x;\xi) = (k+1)g(x;\xi)G^k(x;\xi)$ is the Exp-G distribution with power parameter k and

$$\begin{aligned} \psi_k^* &= \frac{1}{B(i,n-i+1)}\sum_{z,w,l,p,q=0}^{\infty}\sum_{j=0}^{n-j}\frac{4b\beta(-1)^{w+l+p+q+k}(z+w+1)^l}{l!(k+1)}\binom{n-i}{j} \\ &\times \binom{-(j+i+1)}{z}\binom{j+i-1}{w}\binom{-(\beta(l+1)+1)}{p}\binom{b(\beta l+\beta+p)-1}{q}\binom{2q+1}{k}. \end{aligned} \tag{15}$$

Therefore, the distribution of the i^{th} order statistic from the HLOW-TL-G family of distributions can be obtained directly from the Exp-G distribution.

3.4. Probability Weighted Moments

Probability Weighted Moments (PWMs), say $\phi_{j,i}$ of $X\sim$ HLOW-TL-G (b,β,ξ) distribution is given by

$$\phi_{j,i}=E(X^jF(X)^i)=\int_{-\infty}^{\infty}x^jf(x)F(x)^idx.$$

Using equation (13), we can write

$$\begin{aligned}f(x)F(x)^i &= \sum_{z,w,l,p,q,k=0}^{\infty}\frac{4b\beta(-1)^{w+l+p+q+k}(z+w+1)^l}{l!}\binom{-(i+2)}{z}\\&\times\binom{i}{w}\binom{-(\beta(l+1)+1)}{p}\binom{b(\beta l+\beta+p)-1}{q}\\&\times\binom{2q+1}{k}g(x;\xi)G^k(x;\xi).\end{aligned}$$

Using the properties of the Exp-G distribution, the PWMs of the HLOW-TL-G family of distributions is given by

$$f(x)F(x)^i = \sum_{k=0}^{\infty}\eta_k h_k(x;\xi),$$

where

$$\begin{aligned}\eta_k &= \sum_{z,w,l,p,q=0}^{\infty}\frac{4b\beta(-1)^{w+l+p+q+k}(z+w+1)^l}{l!(k+1)}\binom{-(i+2)}{z}\\&\times\binom{i}{w}\binom{-(\beta(l+1)+1)}{p}\binom{b(\beta l+\beta+p)-1}{q}\\&\times\binom{2q+1}{k}\end{aligned}$$

and $h_k(x;\xi)=(k+1)g(x;\xi)[G(x;\xi)]^k$ is an Exp-G distribution with power parameter k. Therefore, the PWM of the HLOW-TL-G family of distributions is given by

$$\begin{aligned}\phi_{j,i} &= \sum_{k=0}^{\infty}\eta_k\int_{-\infty}^{\infty}x^j h_k(x;\xi)dx\\ &= \sum_{k=0}^{\infty}\eta_k E(H_k^j),\end{aligned}$$

where H_k^j is the j^{th} power of an Exp-G distributed random variable with power parameter k.

3.5. Entropy

Entropy is a measure of variation of uncertainty for a random variable X with *pdf* $f(x)$. There are two famous measures of entropy, namely Shannon entropy by Shannon(1951) and Rényi entropy by Rényi(1960). Rényi entropy is defined by

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\int_0^{\infty}f^{\nu}(x)dx\right],$$

where $\nu>0$ and $\nu\neq 1$. By taking $f(x)$ to be the *pdf* of the HLOW-TL-G family of distributions we get

$$\int_0^{\infty}f^{\nu}(x)dx=\int_0^{\infty}\frac{(4b\beta)^{\nu}g^{\nu}(x;\xi)\overline{G}^{\nu}(x;\xi)[1-\overline{G}^2(x;\xi)]^{(b\beta-1)\nu}e^{\left(-\nu\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)}}{(1-[1-\overline{G}^2(x;\xi)]^b)^{(\beta+1)\nu}\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right)^{2\nu}}dx.$$

By considering the following series expansions

$$\begin{aligned}\left(1+\exp\left(-\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)\right)^{-2\nu} &= \sum_{z=0}^{\infty}\binom{-2\nu}{z}\\ &\times\exp\left(-z\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right),\end{aligned}$$

$$\exp\left(-(z+1)\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta}\right)=\sum_{w=0}^{\infty}\frac{(-1)^w(z+1)^w}{w!}\left[\frac{[1-\overline{G}^2(x;\xi)]^b}{1-[1-\overline{G}^2(x;\xi)]^b}\right]^{\beta w},$$

$$(1-[1-\overline{G}^2(x;\xi)]^b)^{-(\beta(w+\nu)+\nu)}=\sum_{p=0}^{\infty}(-1)^p\binom{-(\beta(w+\nu)+\nu)}{p}[1-\overline{G}^2(x;\xi)]^{bp},$$

$$[1-\overline{G}^2(x;\xi)]^{b(\beta w+\beta\nu+p)-\nu}=\sum_{q=0}^{\infty}(-1)^q\binom{b(\beta w+\beta\nu+p)-\nu}{q}\overline{G}^{2q}(x;\xi)$$

and

$$\overline{G}^{2q+\nu}(x;\xi)=\sum_{j=0}^{\infty}(-1)^j\binom{2q+\nu}{j}G^j(x;\xi)$$

yields

$$\begin{aligned}\int_0^{\infty}f^{\nu}(x)dx &= \sum_{z,w,p,q,j=0}^{\infty}\frac{(-1)^{w+p+q+j}(z+1)^w}{w!}\binom{-2\nu}{z}\binom{-(\beta(w+\nu)+\nu)}{p}\\ &\times\binom{b(\beta w+\beta\nu+p)-\nu}{q}\binom{2q+\nu}{j}[\frac{1}{[j/\nu+1]}]^{\nu}\\ &\times\int_0^{\infty}\left[(j/\nu+1)g(x;\xi)G^{j/\nu}(x;\xi)\right]^{\nu}dx.\end{aligned}$$

Therefore, the Rényi entropy of the HLOW-TL-G family of distributions is given by

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\sum_{j=0}^{\infty}\Phi_j e^{(1-\nu)I_{REG}}\right], \tag{16}$$

where $I_{REG}=\int_0^{\infty}\left[(j/\nu+1)g(x;\xi)G^{j/\nu}(x;\xi)\right]^{\nu}dx$ is the Rényi entropy of Exp-G distribution with power parameter j/ν and

$$\begin{aligned}\Phi_j &= \sum_{z,w,p,q=0}^{\infty}\frac{(-1)^{w+p+q+j}(z+1)^w}{w!}\binom{-2\nu}{z}\binom{-(\beta(w+\nu)+\nu)}{p}\\ &\times\binom{b(\beta w+\beta\nu+p)-\nu}{q}\binom{2q+\nu}{j}[\frac{1}{[j/\nu+1]}]^{\nu}.\end{aligned}$$

4. Maximum Likelihood Estimation

If $X_i\sim HLOW-TL-G(b,\beta;\xi)$ with the parameter vector $\Psi=(b,\beta;\xi)^T$. The total log-likelihood $\ell=\ell(\Psi)$ from a random sample of size n is given by

$$\begin{aligned}\ell &= n\log(4b\beta)+\sum_{i=1}^{n}\log[g(x_i;\xi)]+\sum_{i=1}^{n}\log[\overline{G}(x_i;\xi)]+(b\beta-1)\sum_{i=1}^{n}\log[1-\overline{G}^2(x_i;\xi)]\\ &-\sum_{i=1}^{n}\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}-(\beta+1)\sum_{i=1}^{n}\log[1-(1-\overline{G}^2(x_i;\xi))^b]\\ &-2\sum_{i=1}^{n}\log\left[1+\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\right].\end{aligned}$$

The score vector $U = \left(\frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \xi_k}\right)$ has elements given by:

$$\begin{aligned}
\frac{\partial \ell}{\partial b} &= \frac{n}{b} + \beta \sum_{i=1}^{n} \log[1-\overline{G}^2(x_i;\xi)] - \sum_{i=1}^{n} \left(\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right)^{\beta-1} \\
&\times \frac{\beta(1-\overline{G}^2(x_i;\xi))^b \log(1-\overline{G}^2(x_i;\xi))}{(1-(1-\overline{G}^2(x_i;\xi))^b)^2} \\
&+ (\beta+1)\sum_{i=1}^{n} \frac{(1-\overline{G}^2(x_i;\xi))^b \log(1-\overline{G}^2(x_i;\xi))}{1-(1-\overline{G}^2(x_i;\xi))^b} \\
&+ 2\beta \sum_{i=1}^{n} \frac{\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\log[1-\overline{G}^2(x_i;\xi)]}{\left(1+\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\right)(1-(1-\overline{G}^2(x_i;\xi))^b)^{\beta+1}} \\
&\times \left((1-\overline{G}^2(x_i;\xi))(1-\overline{G}^2(x_i;\xi))^{b\beta-1}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + b\sum_{i=1}^{n} \log[1-\overline{G}^2(x_i;\xi)] - \sum_{i=1}^{n} \log[1-(1-\overline{G}^2(x_i;\xi))^b] \\
&- \sum_{i=1}^{n} \left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta} \log\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right] \\
&+ 2\sum_{i=1}^{n} \frac{\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\log\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right](1-\overline{G}^2(x_i;\xi))^{b\beta}}{\left(1+\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\right)(1-(1-\overline{G}^2(x_i;\xi))^b)^{\beta}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \xi_k} &= \sum_{i=1}^{n} \frac{1}{g(x_i;\xi)} \frac{\partial g(x_i;\xi)}{\partial \xi_k} - 2\sum_{i=1}^{n} \frac{1}{[\overline{G}(x_i;\xi)]} \frac{\partial [\overline{G}(x_i;\xi)]}{\partial \xi_k} \\
&- (b\beta - 1) \sum_{i=1}^{n} \frac{1}{1-\overline{G}^2(x_i;\xi)} \frac{\partial \big[1-\overline{G}^2(x_i;\xi)\big]}{\partial \xi_k} - \sum_{i=1}^{n} \left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b} \right]^{\beta-1} \\
&\times \frac{(1-\overline{G}^2(x;\xi))^b \frac{\partial [1-(1-\overline{G}^2(x_i;\xi))^b]}{\partial \xi_k} - (1-(1-\overline{G}^2(x_i;\xi))^b) \frac{\partial (1-\overline{G}^2(x_i;\xi))^b}{\partial \xi_k}}{(1-(1-\overline{G}^2(x_i;\xi))^b)^2} \\
&- (\beta+1) \sum_{i=1}^{n} \frac{1}{1-(1-\overline{G}^2(x_i;\xi))^b} \frac{\partial \big[1-(1-\overline{G}^2(x_i;\xi))^b\big]}{\partial \xi_k} \\
&- 2\sum_{i=1}^{n} \frac{1}{\left[1+\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\right]} \frac{\partial \left[1+\exp\left(-\left[\frac{(1-\overline{G}^2(x_i;\xi))^b}{1-(1-\overline{G}^2(x_i;\xi))^b}\right]^{\beta}\right)\right]}{\partial \xi_k},
\end{aligned}$$

respectively. These partial derivatives are not in closed form and can be solved using R, MATLAB, and SAS software by the use of iterative methods.
The (2+p) $\times$ (2+p) Fisher information matrix $J(\Psi)$ is given by

$$J(\Psi) = \begin{pmatrix} J_{bb}(\Psi) & J_{b\beta}(\Psi & J_{b\xi}(\Psi) \\ J_{\beta b}(\Psi) & J_{\beta\beta}(\Psi) & J_{\beta\xi}(\Psi) \\ J_{\xi b}(\Psi) & J_{\xi\beta}(\Psi) & J_{\xi\xi}(\Psi) \end{pmatrix}, \tag{17}$$

where $J_{i,j} = \frac{-\partial^2 \ell(\Delta)}{\partial i \partial j}$, for $i,j = b,\beta,\xi$, where ξ is a p component vector, $J_{\xi\xi}(\Psi)$ is a $p \times p$ matrix, $J_{b\xi}(\Psi)$ and $J_{\beta\xi}(\Psi)$ has $p \times 1$ components, respectively. We use the Fisher information matrix to estimate the asymptotic confidence intervals of the parameters $\Psi = (b,\beta,\xi)$. Under the usual regularity conditions $\hat{\Psi}$ is asymptotically normal distributed, that is $\hat{\Psi} \sim N(\underline{0}, I^{-1}(\Psi))$ as $n \longrightarrow \infty$, where $I(\Psi)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\Psi)$ is replaced by $J(\hat{\Psi})$, the information matrix evaluated at $\hat{\Psi}$.

5. Some Special Cases

In this section, we present five special cases of the HLOW-TL-G family of distributions. We considered cases when the baseline distributions are Burr XII, Uniform, Power function, Kumaraswamy, and Weibull distributions.

5.1. *Half Logistic Odd Weibull-Topp-Leone-Burr XII Distribution*

Consider the Burr XII distribution as the baseline distribution with *pdf* and *cdf* given by $g(x;c,k) = ckx^{c-1}(1+x^c)^{-k-1}$ and $G(x;c,k) = 1-(1+x^c)^{-k}$, respectively, for $c,k>0$. The *cdf* and *pdf* of the half logistic odd Weibull-Topp-Leone-Burr XII (HLOW-TL-BXII) distribution are given by

$$F_{HLOW-TL-BXII}(x;b,\beta,c,k) = \frac{1-\exp\left(-\left[\frac{[1-(1+x^c)^{-2k}]^b}{1-[1-(1+x^c)^{-2k}]^b}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-(1+x^c)^{-2k}]^b}{1-[1-(1+x^c)^{-2k}]^b}\right]^{\beta}\right)}$$

and

$$\begin{aligned} f_{HLOW-TL-BXII}(x;b,\beta,c,k) &= \frac{4b\beta ckx^{c-1}(1+x^c)^{-2k-1}[1-(1+x^c)^{-2k}]^{b\beta-1}}{(1-[1-(1+x^c)^{-2k}]^b)^{\beta+1}} \\ &\times \frac{\exp\left(-\left[\frac{[1-(1+x^c)^{-2k}]^b}{1-[1-(1+x^c)^{-2k}]^b}\right]^{\beta}\right)}{\left(1+\exp\left(-\left[\frac{[1-(1+x^c)^{-2k}]^b}{1-[1-(1+x^c)^{-2k}]^b}\right]^{\beta}\right)\right)^2}, \end{aligned}$$

respectively, for $b,\beta,c,k>0$. By setting $k=1$ and $c=1$, we obtain the half logistic odd Weibull-Topp-Leone-log logistic (HLOW-TL-LLoG) and the half logistic odd Weibull-Topp-Leone-Lomax (HLOW-TL-Lx), respectively from the HLOW-TL-BXII distribution.

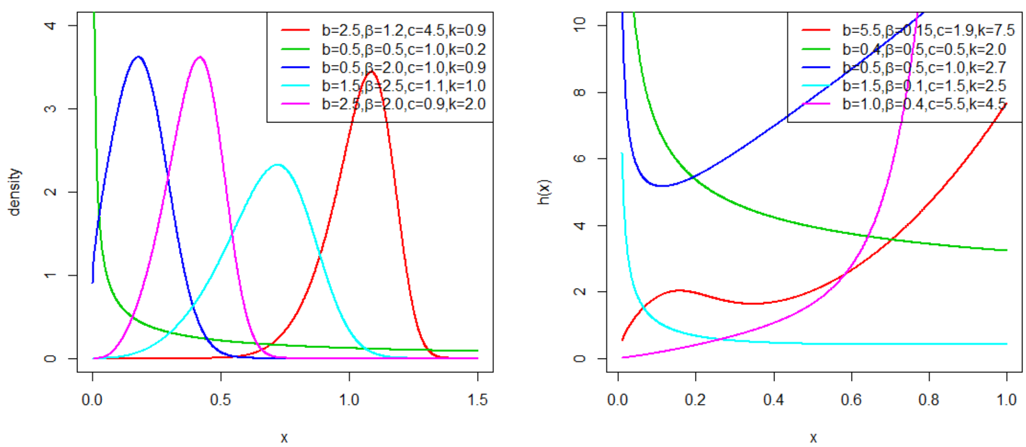

Fig. 1: Plots of the *pdf* and *hrf* for the HLOW-TL-BXII distribution

Figure 1 shows the graphs of the *pdf* and hazard rate function (*hrf*) of the HLOW-TL-BXII distribution. The *pdf* can take various shapes that include almost symmetric, reverse-J, left, or right-skewed. Furthermore, the *hrf* for the HLOW-TL-BXII distribution exhibit increasing, decreasing, reverse-J, bathtub, and upside bathtub followed by bathtub shapes.

5.2. Half Logistic Odd Weibull-Topp-Leone-Uniform Distribution

By taking the baseline distribution to be a uniform distribution with *pdf* and *cdf* given by $g(x) = 1/\theta$ and $G(x, \theta) = x/\theta$, respectively, for $0 < x < \theta$, obtain the half logistic odd Weibull-Topp-Leone-Uniform (HLOW-TL-U) distribution with *cdf* and *pdf* given by

$$F_{HLOW-TL-U}(x; b, \beta, \theta) = \frac{1 - \exp\left(-\left[\frac{[1-(1-x/\theta)^2]^b}{1-[1-(1-x/\theta)^2]^b}\right]^\beta\right)}{1 + \exp\left(-\left[\frac{[1-(1-x/\theta)^2]^b}{1-[1-(1-x/\theta)^2]^b}\right]^\beta\right)}$$

and

$$f_{HLOW-TL-U}(x; b, \beta, \theta) = \frac{4b\beta(1-x/\theta)[1-(1-x/\theta)^2]^{b\beta-1}}{\theta(1-[1-(1-x/\theta)^2]^b)^{\beta+1}} \frac{\exp\left(-\left[\frac{[1-(1-x/\theta)^2]^b}{1-[1-(1-x/\theta)^2]^b}\right]^\beta\right)}{\left(1+\exp\left(-\left[\frac{[1-(1-x/\theta)^2]^b}{1-[1-(1-x/\theta)^2]^b}\right]^\beta\right)\right)^2},$$

respectively, for $b, \beta > 0$ and , $0 < x < \theta$.

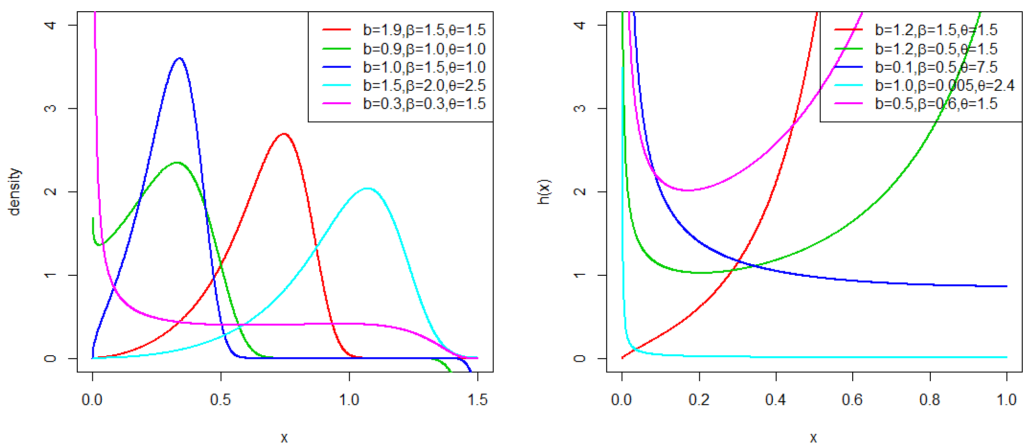

Fig. 2: Plots of the *pdf* and *hrf* for the HLOW-TL-U distribution

Figure 2, shows the graphs of the *pdf*s and *hrf*s of the HLOW-TL-U distribution. The *pdf*s can take various shapes that include left or right-skewed and almost symmetric shapes. The *hrf* exhibit increasing, decreasing, reverse-J, and bathtub shapes.

5.3. Half Logistic Odd Weibull-Topp-Leone-Power Function Distribution

By considering the power distribution as the baseline distribution with *pdf* and *cdf* given by $g(x;\theta,k)=k\theta^k x^{k-1}$ and $G(x;\theta,k)=(\theta x)^k$, for $0<x<1/\theta$, $\theta,k>0$, respectively, we get the half logistic odd Weibull-Topp-Leone-power function (HLOW-TL-PF) distribution with *cdf* and *pdf* given by

$$F_{HLOW-TL-PF}(x;b,\beta,\theta,k)=\frac{1-\exp\left(-\left[\frac{[1-(1-(\theta x)^k)^2]^b}{1-[1-(1-(\theta x)^k)^2]^b}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-(1-(\theta x)^k)^2]^b}{1-[1-(1-(\theta x)^k)^2]^b}\right]^{\beta}\right)}$$

and

$$\begin{aligned} f_{HLOW-TL-PF}(x;b,\beta,\theta,k) &= \frac{4b\beta k\theta^k x^{k-1}(1-(\theta x)^k)[1-(1-(\theta x)^k)^2]^{b\beta-1}}{(1-[1-(1-(\theta x)^k)^2]^b)^{\beta+1}} \\ &\times \frac{\exp\left(-\left[\frac{[1-(1-(\theta x)^k)^2]^b}{1-[1-(1-(\theta x)^k)^2]^b}\right]^{\beta}\right)}{\left(1+\exp\left(-\left[\frac{[1-(1-(\theta x)^k)^2]^b}{1-[1-(1-(\theta x)^k)^2]^b}\right]^{\beta}\right)\right)^2}, \end{aligned}$$

respectively, for $b,\beta,\theta,k>0$.

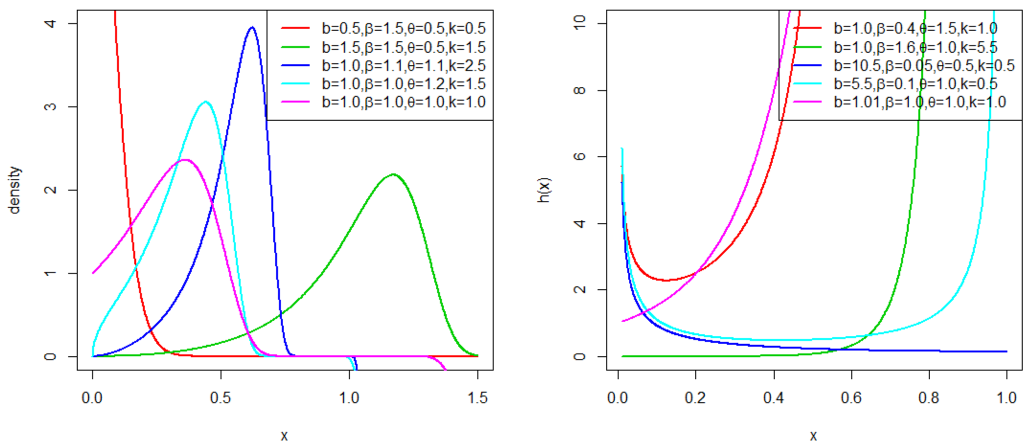

Fig. 3: Plots of the *pdf* and *hrf* for the HLOW-TL-P distribution

Figure 3, shows the graphs of the *pdf*s and *hrf*s for the HLOW-TL-PF distribution. The *pdf* of the HLOW-TL-PF distribution takes various shapes that include left or right skewed and reverse-J. The distribution also addresses variation in kurtosis. The *hrf* exhibit increasing,decreasing, J, reverse-J, and bathtub shapes.

5.4. Half Logistic Odd Weibull-Topp-Leone-Kumaraswamy Distribution

By taking the Kumaraswamy distribution as the baseline distribution with *pdf* and *cdf* given by $g(x;\alpha,\theta)=\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}$ and $G(x;\alpha,\theta)=1-(1-x^{\alpha})^{\theta}$, for $\alpha,\theta>0$, respectively, we get the half logistic odd Weibull-Topp-Leone-Kumaraswamy (HLOW-TL-Kw) distribution with *cdf* and *pdf* given by

$$F_{HLOW-TL-Kw}(x;b,\beta,\alpha,\theta)=\frac{1-\exp\left(-\left[\frac{[1-(1-x^{\alpha})^{2\theta}]^{b}}{1-[1-(1-x^{\alpha})^{2\theta}]^{b}}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-(1-x^{\alpha})^{2\theta}]^{b}}{1-[1-(1-x^{\alpha})^{2\theta}]^{b}}\right]^{\beta}\right)}$$

and

$$\begin{aligned} f_{HLOW-TL-Kw}(x;b,\beta,\alpha,\theta) &= \frac{4b\beta\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{2\theta-1}[1-(1-x^{\alpha})^{2\theta}]^{b\beta-1}}{(1-[1-(1-x^{\alpha})^{2\theta}]^{b})^{\beta+1}} \\ &\times \frac{\exp\left(-\left[\frac{[1-(1-x^{\alpha})^{2\theta}]^{b}}{1-[1-(1-x^{\alpha})^{2\theta}]^{b}}\right]^{\beta}\right)}{\left(1+\exp\left(-\left[\frac{[1-(1-x^{\alpha})^{2\theta}]^{b}}{1-[1-(1-x^{\alpha})^{2\theta}]^{b}}\right]^{\beta}\right)\right)^{2}}, \end{aligned}$$

respectively, for $b,\beta,\alpha,\theta>0$.

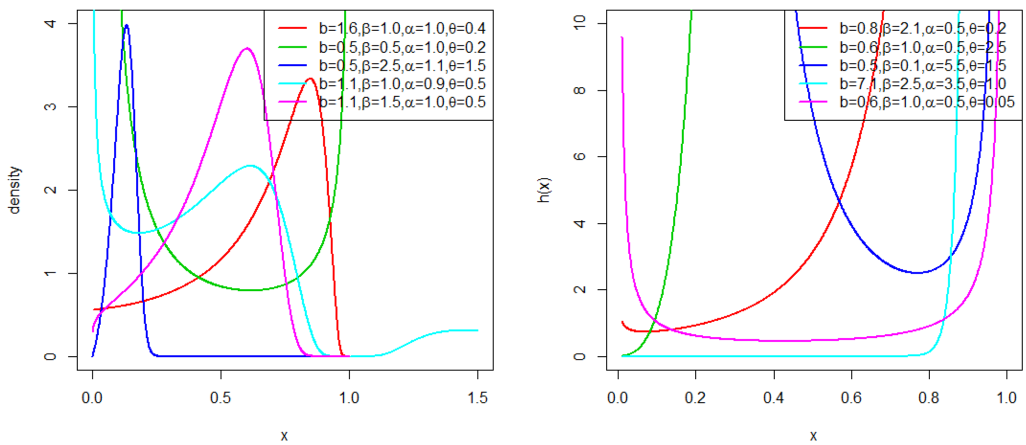

Fig. 4: Plots of the *pdf* and *hrf* for the HLOW-TL-Kw distribution

Figure 4 shows the graphs of the *pdf*s and *hrf*s for the HLOW-TL-Kw distribution. The *pdf* of the HLOW-TL-Kw distribution take various shapes that include U, left, or right skewed. The distribution also addresses variation in kurtosis. The *hrf* exhibit increasing, J, and bathtub shapes.

5.5. *Half Logistic Odd Weibull-Topp-Leone-Weibull Distribution*

Consider the Weibull distribution as the baseline distribution with *pdf* and *cdf* given by $g(x;\lambda,\omega)=\lambda\omega x^{\omega-1}e^{-\lambda x^{\omega}}$ and $G(x;\lambda,\omega)=1-e^{-\lambda x^{\omega}}$, respectively, for $\lambda,\omega>0$. The *cdf* and *pdf* of the half logistic odd Weibull-Topp-Leone-Weibull (HLOW-TL-W) distribution are given by

$$F_{HLOW-TL-W}(x;b,\beta,\lambda,\omega)=\frac{1-\exp\left(-\left[\frac{[1-e^{-2\lambda x^{\omega}}]^{b}}{1-[1-e^{-2\lambda x^{\omega}}]^{b}}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-e^{-2\lambda x^{\omega}}]^{b}}{1-[1-e^{-2\lambda x^{\omega}}]^{b}}\right]^{\beta}\right)}$$

and

$$\begin{aligned}f_{HLOW-TL-W}(x;b,\beta,\lambda,\omega) &= \frac{4b\beta\lambda\omega x^{\omega-1}e^{-3\lambda x^{\omega}}[1-e^{-2\lambda x^{\omega}}]^{b\beta-1}}{(1-[1-e^{-2\lambda x^{\omega}}]^{b})^{\beta+1}}\\ &\times\frac{\exp\left(-\left[\frac{[1-e^{-2\lambda x^{\omega}}]^{b}}{1-[1-e^{-2\lambda x^{\omega}}]^{b}}\right]^{\beta}\right)}{\left(1+\exp\left(-\left[\frac{[1-e^{-2\lambda x^{\omega}}]^{b}}{1-[1-e^{-2\lambda x^{\omega}}]^{b}}\right]^{\beta}\right)\right)^{2}},\end{aligned}$$

respectively, for $b,\beta,\lambda,\omega>0$.

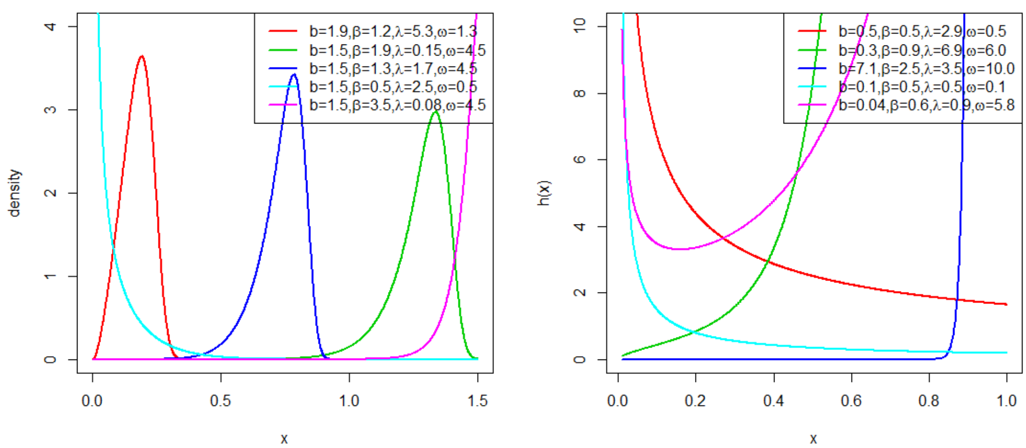

Fig. 5: Plots of the *pdf* and *hrf* for the HLOW-TL-W distribution

Figure 5 shows the graphs of the *pdf*s and *hrf*s of the HLOW-TL-W distribution. The *pdf* can take various shapes that include J, reverse-J, left, or right-skewed. Furthermore, the *hrf* for the HLOW-TL-W distribution exhibit increasing, decreasing, reverse-J, J, and bathtub shapes.

6. Simulation Study

We conducted a simulation study from the HLOW-TL-LLoG distribution to evaluate the performance of the maximum likelihood estimates. We consider sample sizes n= 50, 100, 200, 400, 800 and 1000. We simulated N=1000 times for each sample. We estimate the mean, root mean square error (RMSE), and average bias. We consider simulations for the following sets of initial parameters values ($I: b = 1.0, \beta = 1.0, c = 1.0$), ($II: b = 1.5, \beta = 1.0, c = 1.0$), ($III: b = 1.0, \beta = 1.0, c = 1.5$) and ($IV: b = 1.0, \beta = 1.5, c = 1.0$). If the model performs better, we except the mean to approximate the true parameter values, the RMSE, and bias to decay toward zero for an increase in sample size. From the results in Table 3, the mean values approximate the true parameter values, RMSE and bias decay towards zero for all the parameter values.

7. Applications

We expose the HLOW-TL-LLoG distribution to two data sets of varying skewness to demonstrate the flexibility of the new model compared to well known equi-parameter non-nested models. We also compared the HLOW-TL-LLoG model to its nested models so as to show the flexibility enjoyed by adding the additional parameters to the baseline distribution. Maximum likelihood estimation technique through the use of R software is used to estimate the model parameters. The model parameters estimates are reliable and dependable as supported by the behavior of simulation results. We present model parameters estimates (standard errors in parenthesis) in Tables **??** and 5.

Model performance was assessed by utilizing the following goodness-of-fit statistics; -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling (A^*) (see Chen and Balakrishnan(1995) for details), Kolmogorov-Smirnov (K-S) statistic (and its p-value). The best model has smaller values of these statistics and a bigger p-value for the K-S statistic.

Furthermore, to demonstrate how the HLOW-TL-LLoG model fit the given data sets, we provide fitted densities and histogram of the data, probability plots (as described by Chambers *et al.*(1983)). The plots are shown in Figures 6 and 7.

The HLOW-TL-LLoG model was compared to the following non-nested models: the Topp-Leone-Marshall-Olkin-log-logistic (TL-MO-LLo) by Chipepa *et al.*(2020), exponentiated Weibull by Pal *et al.*(2006), Marshall-Olkin extended inverse Weibull (MOIW) by Pakungwati *et al.*(2018), Marshall-Olkin extended Weibull (MOEW) by Barreto-Souza and Bakouch(2013), Topp-Leone generalized exponential (TL-GE) by Sangsanit and Bodhisuwan(2016), Topp-Leone-Weibull (TL-W) by

		I:	$b=1.0, \beta=1.0, c=1.0$		II:	$b=1.5, \beta=1.0, c=1.0$	
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	1.198742	0.670467	0.198742	1.475756	0.551835	-0.024245
	50	1.194917	0.583723	0.194917	1.546171	0.482082	0.046171
	100	1.154124	0.490620	0.154124	1.569252	0.440803	0.069252
b	200	1.091532	0.381202	0.091532	1.562534	0.393072	0.062534
	400	1.056788	0.254272	0.056788	1.580414	0.346499	0.080414
	800	1.019481	0.155520	0.019481	1.552770	0.285582	0.052770
	1000	1.016985	0.130398	0.016985	1.555542	0.263057	0.055542
	25	2.496573	3.154074	1.496573	2.092720	3.119717	1.092720
	50	2.152205	2.648576	1.152205	2.147152	3.113453	1.147152
	100	1.819582	2.197011	0.819582	2.105784	2.872665	1.105784
β	200	1.443098	1.599337	0.443098	1.921575	2.664450	0.921575
	400	1.166439	0.743856	0.166439	1.786585	2.231715	0.786585
	800	1.042295	0.209920	0.042295	1.506721	1.721239	0.506721
	1000	1.032600	0.162442	0.032600	1.420757	1.436552	0.420757
	25	1.249829	1.212281	0.249829	1.672248	1.798843	0.672248
	50	1.033781	0.768525	0.033781	1.253909	1.112435	0.253909
	100	0.980622	0.552889	-0.019378	1.102464	0.796578	0.102464
c	200	0.996824	0.428958	-0.003176	1.052738	0.641138	0.052738
	400	0.979565	0.290557	-0.020435	0.971645	0.495546	-0.028356
	800	0.997751	0.204238	-0.002249	0.977734	0.400124	-0.022266
	1000	0.994521	0.174264	-0.005479	0.962519	0.362166	-0.037481
		III:	$b=1.0, \beta=1.0, c=1.5$		IV:	$b=1.0, \beta=1.5, c=1.0$	
	25	1.195559	0.675460	0.195559	1.248401	0.761135	0.248401
	50	1.201429	0.591105	0.201429	1.245833	0.678238	0.245833
	100	1.149928	0.492230	0.149928	1.191079	0.586957	0.191079
b	200	1.095099	0.389228	0.095099	1.141021	0.494869	0.141021
	400	1.055518	0.255139	0.055518	1.103634	0.375530	0.103633
	800	1.017853	0.154659	0.017853	1.046501	0.254100	0.046501
	1000	1.015717	0.129858	0.015717	1.040563	0.218538	0.040563
	25	2.565671	3.337909	1.565671	4.279704	4.956300	2.779704
	50	2.230741	2.841546	1.230741	3.698364	4.204864	2.198364
	100	1.854389	2.358496	0.854389	3.124294	3.581765	1.624294
β	200	1.485031	1.748237	0.485031	2.572827	2.760514	1.072827
	400	1.170215	0.790909	0.170215	2.080880	1.831913	0.580880
	800	1.040441	0.209006	0.040441	1.701678	0.829364	0.201678
	1000	1.031279	0.161598	0.031279	1.666435	0.810587	0.166435
	25	1.886307	1.836478	0.386307	1.398391	1.896065	0.398390
	50	1.537226	1.139421	0.037226	1.073266	1.090323	0.073266
	100	1.479773	0.832512	-0.020227	1.003176	0.695425	0.003176
c	200	1.492349	0.648865	-0.007651	0.989917	0.544675	-0.010083
	400	1.472670	0.438289	-0.027330	0.960574	0.391540	-0.039426
	800	1.499363	0.305149	-0.000637	0.987321	0.286160	-0.012679
	1000	1.493585	0.260748	-0.006415	0.982431	0.247544	-0.017569

Table 3: Monte Carlo Simulation Results for HLOW-TL-LLoG Distribution: Mean, RMSE and Average Bias

Rezaei *et al.*(2016), Weibull-Lomax (WLx) by Jamal *et al.*(2019), Marshall-Olkin log-logistc (MO-LLoG) distribution by Wenhao(2013) and alpha power Weibull (APW) by Nassar Nassar *et al.*(2018) distributions. The pdfs of the non-nested distributions are:

$$f_{TL-MO-LLo}(x;b,\delta,c)=\frac{2b\delta^2cx^{c-1}(1+x^c)^{-3}}{[1-\bar{\delta}(1+x^c)^{-1}]^3}\left[1-\frac{\delta^2[1+x^c]^{-2}}{[1-\bar{\delta}(1+x^c)^{-1}]^2}\right]^{b-1},$$

for $b,\delta,c>0$,

$$f_{EW}(x;\alpha,\beta,\delta)=\alpha\beta\delta x^{\beta-1}e^{-\alpha x^{\beta}}(1-e^{-\alpha x^{\beta}})^{\delta},$$

for $\alpha,\beta,\delta>0$,

$$f_{MOEW}(x;\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda^{\gamma}x^{\gamma-1}e^{-\lambda x^{\gamma}}}{(1-\overline{\alpha}e^{-\lambda x^{\omega}})^{2}},$$

for $\alpha,\gamma,\lambda>0$,

$$f_{TL-GE}(x;\alpha,\beta,\lambda)=2\alpha\beta\lambda e^{-\lambda x}(1-(1-e^{-\lambda x})^{\beta}(1-e^{-\lambda x})^{\beta\alpha-1}(2-(1-e^{-\lambda x})^{\beta}))^{\alpha-1},$$

for $\alpha,\beta,\lambda>0$,

$$f_{TL-W}(x;a,b,\alpha)=2\alpha bax^{b-1}e^{-2(ax)^{b}}[1-e^{-2(ax)^{b}}]^{\alpha-1},$$

$$f_{WLx}(x;a,b,\alpha)=\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1}\exp\left(-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right),$$

for $a,b,\alpha>0$,

$$f_{MOIW}(x;\alpha,\theta,\lambda)=\frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}e^{-(\theta x)^{-\lambda}}}{[\alpha-(\alpha-1)e^{-(\theta x)^{-\lambda}}]^{2}},$$

for $\alpha,\theta,\lambda>0$,

$$f_{MO-LLoG}(x;\alpha,\beta,\gamma)=\frac{\alpha^{\beta}\beta\gamma x^{\beta-1}}{(x^{\beta}+\alpha^{\beta}\gamma)^{2}},$$

for $\alpha,\beta,\gamma>0$,

$$f_{TL-W}(x;a,b,\alpha)=2\alpha bax^{b-1}e^{-2(ax)^{b}}[1-e^{-2(ax)^{b}}]^{\alpha-1},$$

for $a,b,\alpha>0$ and

$$f_{APW}(x;\alpha,\beta,\theta)=\frac{\log(\alpha)}{(\alpha-1)}\beta\theta x^{\beta-1}e^{-\theta x^{\beta}}\alpha^{1-e^{-\theta x^{\beta}}},$$

for $\alpha,\beta,\theta>0$.

7.1. 20 mm Fibers Data

The first real data set was originally reported by Badar and Priest(1982), which represents the strength measured in GPa for single carbon fibers and impregnated at gauge lengths of 1, 10, 20, and 50 mm. Impregnated tows of 100 fibers were tested at gauge lengths of 20, 50, 150, and 300 mm. Here, we consider the data set of single fibers of 20 mm in gauge with a sample of size 63. The observations are:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

	Estimates			Statistics							
Model	b	β	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HLOW-TL-G	1.0354×10^{3}	0.4888	3.3650	112.0	118.0	118.4	124.4	0.0356	0.2359	0.0620	0.9687
	(4.0919×10^{-5})	(0.0446)	(0.0789)								
HLOW-TL-G$(1,\beta,c)$	1	0.0535	7.4874	298.3	302.3	302.5	306.6	0.0596	0.3673	0.7088	$< 2.2\times10^{-16}$
	-	(0.0372)	(5.1761)								
HLOW-TL-G$(b,1,c)$	58.1839	1	1.8990	117.5	121.5	121.7	125.8	0.0847	0.5826	0.1021	0.5271
	(18.2743)	-	(0.1362)								
HLOW-TL-G$(b,2,c)$	11.5483	2	1.0346	122.7	126.7	126.9	131.0	0.1276	0.8778	0.0886	0.7054
	(1.9126)	(0.0787)	-								
HLOW-TL-G$(b,1,1)$	9.1144	1	1	166.5	168.5	168.6	170.7	0.0683	0.4540	0.2807	9.7330×10^{-5}
	(0.7827)	-	-								
HLOW-TL-G$(1,\beta,1)$	1	0.3306	1	350.2	352.2	352.3	354.4	0.0599	0.3722	0.7566	$< 2.2\times10^{-16}$
	-	(0.0224)	-								
HLOW-TL-G$(1,1,c)$	1	1	0.2199	550.8	552.8	552.9	554.9	0.0573	0.3370	0.9483	$< 2.2\times10^{-16}$
	-	-	(0.0220)								
	b	δ	c								
TL-MO-LLo	3.8957	102.0401	4.6049	113.7	119.7	120.1	126.2	0.0777	0.4087	0.0873	0.7226
	(4.1429)	(223.4681)	(1.2483)								
	α	β	δ								
EW	2.2849	0.9279	353.02	113.6	119.6	120.0	126.0	0.0731	0.3790	0.0931	0.6452
	(0.1999)	(0.0839)	(4.5861×10^{-4})								
	a	b	α								
TLW	0.5401	1.4541	37.1349	112.6	118.6	119.0	125.1	0.0519	0.2976	0.7989	$< 2.2\times10^{-16}$
	(0.3363)	(0.7596)	(79.2830)								
WLx	0.2653	3.8260	10.3247	121.9	127.9	128.3	134.4	0.1124	0.7839	0.0801	0.8138
	(0.2261)	(9.1950)	(7.0369)								
	α	β	λ								
TLGE	3.5721	17.5657	1.1719	112.8	118.8	119.2	125.2	0.0641	0.3399	0.0819	0.7917
	(18.7661)	(84.6400)	(0.5386)								
	α	λ	θ								
MOIW	10.3319	7.9024	0.4538	113.7	119.7	120.2	126.2	0.0736	0.3935	0.0826	0.7829
	(12.4659)	(1.1449)	(0.0485)								
	α	γ	λ								
MOEW	0.0266	8.3560	0.2158	115.0	121.0	121.4	127.4	0.0770	0.4520	0.0883	0.7093
	(0.0415)	(1.0019)	(0.0328)								
	α	γ	λ								
MO-LLoG	2.4200	8.6457	6.2842	115.7	121.7	122.1	128.1	0.0889	0.5018	0.0907	0.6777
	(0.0815)	(0.8902)	(0.0036)								
	α	β	θ								
APW	0.0153	6.1786	1.8016×10^{-4}	118.3	124.3	124.7	130.8	0.0915	0.6216	0.0896	0.6928
	(0.0314)	(5.3276×10^{-5})	(9.2507×10^{-5})								

Table 4: Parameter estimates and goodness of fit statistics for various models fitted for 20 mm fibers data set

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 1.6743\times10^{-9} & 9.2982\times10^{-7} & -3.2184\times10^{-6} \\ 9.2982\times10^{-7} & 1.9895\times10^{-3} & -1.5271\times10^{-3} \\ -3.2184\times10^{-6} & -1.5271\times10^{-3} & 6.2324\times10^{-3} \end{bmatrix}$$

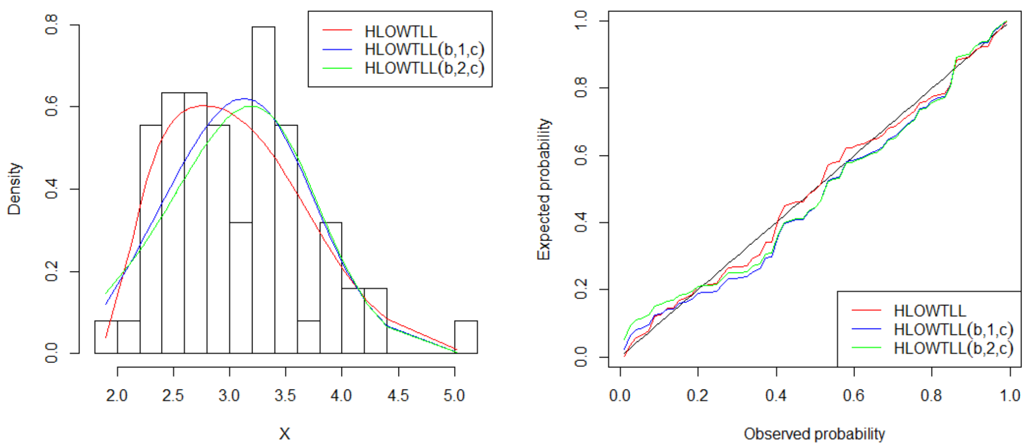

Fig. 6: Fitted *pdf*s and probability plots for 20 mm fibers data set

and the 95% confidence intervals for the model parameters are given by
$b \in [1.0354 \times 10^3 \pm 8.0202 \times 10^{-5}]$, $\beta \in [0.48880 \pm 0.0874]$ and $c \in [3.3650 \pm 0.1547]$.

The HLOW-TL-LLoG model has the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic. We, therefore, conclude that the HOW-TL-LLoG model performs better than the selected equal-parameter non-nested models on 20 mm fibers data set. We also conduct a likelihood ratio test in Section 7.3 to test if the full model performs better than the nested models. From the fitted densities, we can see the flexibility shown by the HLOW-TL-G distribution in addressing skewness in the data set.

7.2. Failure Times Data

The second data set represent failure times (per 1000h) of 50 components. The data is from Murthy *et al.*(2004) and was also analyzed by Oluyede *et al.*(2018). The data are 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.3918 & 0.0114 & -0.0908 \\ 0.0114 & 0.0180 & -0.1123 \\ -0.0908 & -0.1123 & 0.7159 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$b \in [3.0026 \pm 1.2269]$, $\beta \in [0.1995 \pm 0.2631]$ and $c \in [1.4429 \pm 1.6585]$.

	Estimates			Statistics							
Model	b	β	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
HLOW-TL-G	3.0026	0.1995	1.4429	200.2	206.2	206.7	211.9	0.1095	0.7121	0.1029	0.6279
	(0.6260)	(0.1342)	(0.8462)								
HLOW-TL-G$(1,\beta,c)$	1	0.1295	2.1908	219.0	223.0	223.3	226.9	0.1998	1.2658	0.2192	0.0137
	-	(0.0774)	(1.3009)								
HLOW-TL-G$(b,1,c)$	2.7245	1	0.3433	204.4	208.4	208.7	212.3	0.1495	0.9398	0.1448	0.2225
	(0.2960)	-	(0.0337)								
HLOW-TL-G$(b,2,c)$	2.5561	2	0.1737	205.5	209.5	209.7	213.3	0.1564	0.9838	0.1414	0.2461
	(0.1486)	(0.0190)	-								
HLOW-TL-G$(b,1,1)$	7.3622	1	1	724.9	726.9	727.0	728.8	0.1072	0.7628	0.4427	1.9130×10^{-9}
	(0.3414)	-	-								
HLOW-TL-G$(1,\beta,1)$	1	0.2734	1	222.1	224.1	224.1	226.0	0.1713	1.0903	0.2526	0.0027
	-	(0.0220)	-								
HLOW-TL-G$(1,1,c)$	1	1	0.2099	313.7	315.7	315.8	317.6	0.1639	1.0195	0.5853	8.8820×10^{-16}
	-	-	(0.0205)								
	b	δ	c								
TL-MO-LLo	0.1376	2984.40	3.1222	206.0	212.0	212.5	217.7	0.1485	1.0015	0.1569	0.1529
	(0.0195)	(2.2197×10^{-5})	(0.1872)								
	α	β	δ								
EW	5.7282	0.1099	201.02	208.7	214.7	215.2	220.4	0.2095	1.2940	0.1596	0.1404
	(0.1672)	(0.0111)	(8.4065×10^{-4})								
	a	b	α								
TLW	0.1180	0.7706	0.7845	204.7	210.7	211.2	216.4	0.1553	0.9870	0.2944	0.0002
	(0.1421)	(0.9927)	(1.5466)								
WLx	5.9651	0.0379	0.5193	203.2	209.2	209.8	215.0	0.1317	0.8815	0.1441	0.2274
	(15.5413)	(0.1127)	(0.0975)								
	α	β	λ								
TLGE	1.4914×10^{3}	7.2545×10^{-3}	0.0414	202.7	208.7	209.3	214.5	0.1461	0.9050	0.1257	0.3770
	(7.1220×10^{-9})	(7.9852×10^{-4})	(0.0145)								
	α	λ	θ								
MOIW	6.6220	0.8209	10.4626	208.5	214.5	215.0	220.2	0.2109	1.2973	0.1410	0.2492
	(6.9473)	(0.1264)	(8.7858)								
	α	γ	λ								
MOEW	0.6042	8.7273	0.2650	204.4	210.4	211.0	216.2	0.1528	0.9525	0.1147	0.4912
	(0.5532)	(0.1396)	(0.1934)								
	α	γ	λ								
MO-LLoG	0.3938	0.9108	2.6172	211.0	217.0	217.5	222.7	0.2138	1.3281	0.1355	0.2910
	(0.1182)	(0.1033)	(0.0195)								
	α	β	θ								
APW	0.6459	0.6886	0.4664	206.6	210.6	211.1	216.3	0.1525	0.9535	0.1219	0.4138
	(0.7955)	(0.1037)	(0.2147)								

Table 5: Parameter estimates and goodness of fit statistics for various models fitted for failure times data set

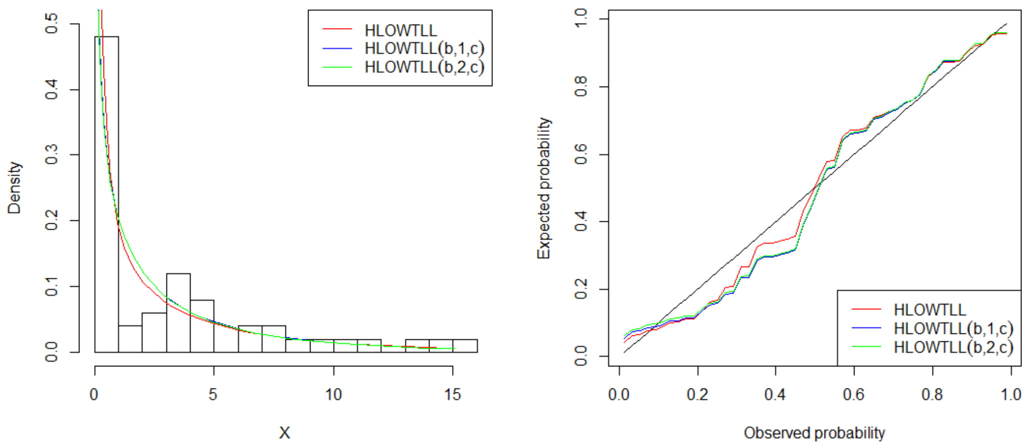

Fig. 7: Fitted *pdf*s and probability plots for failure times data set

Also, from the second example, we can conclude from the HLOW-TL-LLoG model performs than the selected non-nested models since the HLOW-TL-LLoG distribution has lower values for the goodness-of-fit statistics. We, therefore, recommend researchers to use the HLOW-TL-LLoG distribution in case of skewed data other than classical distributions or other generalizations of this classical distribution.

7.3. *Likelihood Ratio Test*

We present the results of the Likelihood ratio test for testing if the HLOW-TL-LLoG model performs better than its nested models. We conclude from the values of the $\chi 2$ shown in Table 6 at 5% level of significance that the HLOW-TL-LLoG model performs better than its nested models on the two data sets considered in this paper.

	20 mm Fibers Data	Failure Times Data
Model	$\chi 2$ (p-value)	$\chi 2$ (p-value)
HLOW-TL-G$(1,\beta,c)$	186.3 (<0.00001)	18.8 (0.00002)
HLOW-TL-G$(b,1,c)$	5.5 (0.19016)	4.2 (0.04042)
HLOW-TL-G$(b,2,c)$	10.7 (0.00107)	5.3 (0.02133)
HLOW-TL-G$(b,1,1)$	54.5 (<0.00001)	524.7 (<0.00001)
HLOW-TL-G$(1,\beta,1)$	238.2 (<0.00001)	21.9 (0.00002)
HLOW-TL-G$(1,1,c)$	438.8 (<0.00001)	113.5 (<0.00001)

Table 6: Likelihood Ratio Test Results

8. Concluding Remarks

A new family of distributions referred to as the half logistic odd Weibull-Topp-Leone-G (HLOW-TL-G) family of distributions is developed. We derive the statistical properties of the proposed family of distributions. The new family of distributions can be expressed as an infinite linear combination of Exp-G distribution. We derive maximum likelihood estimates of the HLPW-TL-G family of distributions. A simulation study on the HLOW-TL-LLoG distribution was also conducted to assess the consistency of the maximum likelihood estimates. The HLOW-TL-LLoG distribution was applied to two real data examples and compared to several equal-parameter non-nested models. The HLOW-TL-LLoG distribution performed better than the selected non-nested models on the two data sets considered in this paper.

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