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On error probability exponents of many hypotheses optimal testing illustrations

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Abstract. In this paper we study a model of hypotheses testing consisting of with to simple homogeneous stationary Markov chains ith finite number of states such that having different distributions from four possible transmission probabilities.For solving this problem we apply the method of type and large deviation techniques (LTD). The case of two objects having different distributions from to given probability distribution as examined by Ahlswedeh and Haroutunian.

Résumé. Dans cet article nous étudions un modèle de tests d'hypothèses composé de deux chaines de Markov stationnaires homogènes et simples avec un nombre fini d'états ayant différentes distributions parmi quatre probabilités de transition possibles. Pour résoudre ce problème, nous appliquons la méthode des types et des techniques de grandes deviations. Le cas de deux objets ayant différentes distributions issues d'une distribution de probabilité donnée, a été examiné par Ahlswedeh et Haroutunian.

Key words: Markov chains; Error probabilities; Different distributions; Transition probabilities; Reliabilities.
AMS 2010 Mathematics Subject Classification : 62P30; 62M02.

1. Introduction

Applications of information-theoretical methods in mathematical statistics are reflected in the monographs by Kullback [10], Csiszár and Körner [4], Blahut [2], Csiszár and Shields [5], Zeitouni and Gutman [14]. In the book of Csiszár and Shields [5] different asymptotic aspects of two hypotheses testing for independent identically distributed observations are considered via theory of large deviations. Similar problems for Markov dependence of experiments were investigated by Natarajan [13], Haroutunian [7], [8], Haroutunian and Navaei [9] and others.

Ahlswede and Haroutunian in [1] formulated an ensemble of problems on multiple hypotheses testing for many objects and on identification of hypotheses under reliability requirement.

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The problem of many ($L > 2$) hypotheses testing on distributions of independent observations is studied in [13], [11] via large deviations techniques.

In this paper we investigate a model with two simple homogeneous stationary Markov chains with finite number of states such that having different distributions from four possible transition probabilities. In Section 2 we introduce the concept of Markov chain and the method of type [3] and in Section 3, we apply the result Section 2 for hypotheses testing.

2. Preliminaries

Let $\mathbf{y} = (y_0, y_1, y_2, \ldots, y_N)$, $y_n \in \mathcal{Y} = \{1, 2, \ldots, I\}$, $\mathbf{y} \in \mathcal{Y}^{N+1}$, $N = 0, 1, 2, ...$, be a vectors of observations of a simple homogeneous stationary Markov chain with finite number I of states. The $l = \overline{1, L}$ hypotheses concern the irreducible matrices of the transition probabilities

$$P_l = \{P_l(j|i),\, i = \overline{1, I},\, j = \overline{1, I}\},\ \ l = \overline{1, L}.$$

The stationarity of the chain provides existence for each $l = \overline{1, L}$ of the unique stationary distribution $Q_l = \{Q_l(i),\ i = \overline{1, I}\}$, such that

$$\sum_i Q_l(i)P_l(j|i) = Q_l(j),\quad \sum_i Q_l(i) = 1,\ \ i = \overline{1, I},\ \ j = \overline{1, I}.$$

We define the joint distributions

$$Q_l \circ P_l = \{Q_l(i)P_l(j|i), i = \overline{1, I}, j = \overline{1, I}\},\ \ l = \overline{1, L}.$$

Let us denote $D(Q \circ P \| Q_l \circ P_l)$ Kullback-Leibler divergence

$$D(Q \circ P \| Q_l \circ P_l) = \sum_{i,j} Q(i)P(j|i)[\log Q(i)P(j|i) - \log Q_l(i)P_l(j|i)]$$

$$= D(Q \| Q_l) + D(Q \circ P \| Q \circ P_l),$$

of the distribution

$$Q \circ P = \{Q(i)P(j|i), i = \overline{1, I}, j = \overline{1, I}\},$$

with respect to distribution $Q_l \circ P_l$ where

$$D(Q \| Q_l) = \sum_i Q(i)[\log Q(i) - \log Q_l(i)],\ \ l = \overline{1, L}.$$

Let us name the second order type of vector $\mathbf{y}$ the square matrix of I^2 relative frequencies $\{N(i,j)N^{-1}, i = \overline{1, I},\ j = \overline{1, I}\}$ of the simultaneous appearance in $\mathbf{y}$ of the states i and j on the pairs of neighbor places. It is clear that $\sum_{ij} N(i,j) = N$. Denote by $\mathcal{T}^N_{Q \circ P}$ the set of vectors from $\mathcal{Y}^{N+1}$ which have the second order type such that for some joint PD $Q \circ P$

$$N(i,j) = NQ(i)P(j|i),\quad i = \overline{1, I},\quad j = \overline{1, I}.$$

The set of all joint PD $Q \circ P$ on $\mathcal{Y}$ is denoted by $\mathcal{Q} \circ \mathcal{P}(\mathcal{Y})$ and the set of all possible the second order types for joint PD $Q \circ P$ is denoted by $\mathcal{Q} \circ \mathcal{P}^N(\mathcal{Y})$. Note that if vector $\mathbf{y} \in \mathcal{T}^N_{Q \circ P}$, then

$$\sum_j N(i,j) = NQ(i),\ i = \overline{1,I}, \quad \sum_i N(i,j) = NQ'(j),\ j = \overline{1,I},$$

for somewhat different from PD Q', but in accordance with the definition of $N(i,j)$ we have

$$|NQ(i) - NQ'(i)| \leq 1,\ i = \overline{1,I},$$

and then in the limit, when $N \to \infty$, the distribution Q coincides with Q' and may be taken as stationary for conditional PD P:

$$\sum_i Q(i)P(j|i) = Q(j),\ j \in \mathcal{Y}.$$

The probability of vector $\mathbf{y} \in \mathcal{Y}^{N+1}$ of the Markov chain with transition probabilities P_l and stationary distribution Q_l, is the following

$$Q_l \circ P_l^N(\mathbf{y}) \triangleq Q_l(y_0) \prod_{n=1}^{N} P_l(y_n|y_{n-1}),\ l = \overline{1,I},$$

$$Q_l \circ P_l^N(\mathcal{A}) \triangleq \sum_{\mathbf{y} \in \mathcal{A}} Q_l \circ P_l^N(\mathbf{y}),\ \mathcal{A} \subset \mathcal{Y}^{N+1}.$$

Note also that if $Q \circ P$ is absolutely continuous relative to $Q_l \circ P_l$, then from [3],[7] we have

$$Q_l \circ P_l^N(\mathcal{T}_{Q \circ P}^N) = \exp\{-N(D(Q \circ P \| Q \circ P_l)) + o(1)\},$$

where

$$o(1) = \max(\max_i |N^{-1} \log Q_l(i)| : Q_l(i) > 0),$$

$$(\max_i |N^{-1} \log Q_l(i)| : Q_l(i) > 0) \to 0, \quad \text{when } N \to \infty.$$

and also according [5],[8] this is not difficult to verify taking into account that the number $|\mathcal{T}_{Q \circ P}^N|$ of vectors in $\mathcal{T}_{Q \circ P}^N$ is equal to

$$\exp\{-N(\sum_{i,j} Q(i)P(j|i) \log P(j|i)) + o(1)\}.$$

In the next section we use the results of this section for the case of $L = 12$ Hypotheses LAO testing.

3. Problem Statement and Formulation of Results

Let Y_1 and Y_2 be random variables (RV) taking values in the same finite set $\mathcal{Y}$ with one of $L = 4$ PDs.

Let $(\mathbf{y_1}, \mathbf{y_2}) = ((y_0^1, y_0^2), \ldots, (y_n^1, y_n^2), \ldots, (y_N^1, y_N^2)), y^i \in \mathcal{Y}, \quad i = 1, 2, \quad n = \overline{0, N}$, be a sequence of results of $N+1$ **independent** observations of a simple homogeneses stationary Markov chain with finite number I of states. The goal of the statistician is to define which joint of distributions corresponds to observed sample $(\mathbf{y_1}, \mathbf{y_2})$, which we denote by ϕ_N. For this model the vector (Y_1, Y_2) can have one of six joint probability distributions $Q^{'}_{l_1,l_2} \circ P^{'}_{l_1,l_2}(\mathbf{y_1}, \mathbf{y_2})$, $l_1 \neq l_2$, $l_1, l_2 = \overline{1,4}$ where

$$Q^{'}_{l_1,l_2} \circ P^{'}_{l_1,l_2}(\mathbf{y_1}, \mathbf{y_2}) = Q^{'}_{l_1} \circ P^{'}_{l_1}(\mathbf{y_1}) Q^{'}_{l_2} \circ P^{'}_{l_2}(\mathbf{y_2}).$$

We can take $(Y_1, Y_2) = X$, $\mathcal{Y} \times \mathcal{Y} = \mathcal{X}$ and $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_N)$, $x_n \in \mathcal{X}$, $\mathbf{x} \in \mathcal{X}^{N+1}$, where $x_n = (y_n^1, y_n^2)$, $n = \overline{0, N}$, then we will have six new hypotheses for one object.

$$\begin{array}{ll}
Q^{'}_{1,2} \circ P^{'}_{1,2}(\mathbf{y_1}, \mathbf{y_2}) = Q_1 \circ P_1(\mathbf{x}), & Q^{'}_{2,1} \circ P^{'}_{2,1}(\mathbf{y_1}, \mathbf{y_2}) = Q_4 \circ P_4(\mathbf{x}), \\
Q^{'}_{1,3} \circ P^{'}_{1,3}(\mathbf{y_1}, \mathbf{y_2}) = Q_2 \circ P_2(\mathbf{x}), & Q^{'}_{2,3} \circ P^{'}_{2,3}(\mathbf{y_1}, \mathbf{y_2}) = Q_5 \circ P_5(\mathbf{x}), \\
Q^{'}_{1,4} \circ P^{'}_{1,4}(\mathbf{y_1}, \mathbf{y_2}) = Q_3 \circ P_3(\mathbf{x}), & Q^{'}_{2,4} \circ P^{'}_{2,4}(\mathbf{y_1}, \mathbf{y_2}) = Q_6 \circ P_6(\mathbf{x}),
\end{array}$$

$$\begin{array}{ll}
Q^{'}_{3,1} \circ P^{'}_{3,1}(\mathbf{y_1}, \mathbf{y_2}) = Q_7 \circ P_7(\mathbf{x}), & Q^{'}_{4,1} \circ P^{'}_{4,1}(\mathbf{y_1}, \mathbf{y_2}) = Q_{10} \circ P_{10}(\mathbf{x}), \\
Q^{'}_{3,2} \circ P^{'}_{3,2}(\mathbf{y_1}, \mathbf{y_2}) = Q_8 \circ P_8(\mathbf{x}), & Q^{'}_{4,2} \circ P^{'}_{4,2}(\mathbf{y_1}, \mathbf{y_2}) = Q_{11} \circ P_{11}(\mathbf{x}), \\
Q^{'}_{3,4} \circ P^{'}_{3,4}(\mathbf{y_1}, \mathbf{y_2}) = Q_9 \circ P_9(\mathbf{x}), & Q^{'}_{4,3} \circ P^{'}_{4,3}(\mathbf{y_1}, \mathbf{y_2}) = Q_{12} \circ P_{12}(\mathbf{x}),
\end{array}$$

and thus we have brought the original problem to the identification problem for one object of observation of Markov chain with finite number of states with $L = 12$ hypotheses.

Now, according non-randomized test $\phi_N(\mathbf{x})$ accepts one of the hypotheses H_l, $l = \overline{1,12}$ on the basis of the trajectory $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ of the $N+1$ observations. Let us denote $\alpha^{(N)}_{l|m}(\phi_N)$ the probability to accept the hypothesis H_l under the condition that H_m, $m \neq l$, is true. For $l = m$ we denote $\alpha^{(N)}_{m|m}(\phi_N)$ the probability to reject the hypothesis H_m. It is clear that

$$\alpha^{(N)}_{m|m}(\phi_N) = \sum_{l \neq m} \alpha^{(N)}_{l|m}(\phi_N), \ m = \overline{1,12}. \tag{1}$$

This probability is called the error probability of the m-th kind of the test ϕ_N. The quadratic matrix of 144 error probabilities $\{\alpha^{(N)}_{l|m}(\phi), \ m = \overline{1,12}, \ l = \overline{1,12}\}$ sometimes is called the power of the tests. To every trajectory $\mathbf{x}$ the test ϕ_N puts in correspondence one from 6 hypotheses. So the space $\mathcal{X}^{N+1}$ will be divided into 12 parts, $\mathcal{G}_l^N = \{\mathbf{x}, \ \phi_N(\mathbf{x}) = l\}$, $l = \overline{1,12}$, and $\alpha^{N}_{l|m}(\phi_N) = Q_m \circ P_m(\mathcal{G}_l^N)$, $\quad m, l = \overline{1,12}$.

We study the matrix of "reliabilities",

$$E_{l|m}(\phi) = \lim_{N \to \infty} -\frac{1}{N} \log \alpha_{l|m}(\phi_N), \ m, l = \overline{1,12}. \tag{2}$$

Note that from definitions (1) and (2) it follows that

$$E_{m|m} = \min_{l \neq m} E_{l|m}. \tag{3}$$

$$\mathbf{E}(\phi) = \begin{bmatrix} E_{1|1} & \dots & E_{1|m} & \dots & E_{1|12} \\ \vdots & & \vdots & & \vdots \\ E_{l|1} & \dots & E_{l|m} & \dots & E_{l|12} \\ \vdots & & \vdots & & \vdots \\ E_{12|1} & \dots & E_{12|m} & \dots & E_{12|12} \end{bmatrix}.$$

Definition 1. The test sequence $\Phi^* = (\phi_1, \phi_2, \dots)$ is called LAO if for given family of positive numbers $E_{1|1}, E_{2|2}, \dots, E_{11|11}$, the reliability matrix contains in the diagonal these numbers and the remained 133 its components take the maximal possible values.

Let $P = \{P(j|i)\}$ be a irreducible matrix of transition probabilities of some stationary Markov chain with the same set $\mathcal{X}$ of states, and $Q = \{Q(i),\, i = \overline{1,I}\}$ be the corresponding stationary PD.

For given family of positive numbers $E_{1|1}, E_{2|2}, \dots, E_{11|11}$, let us define the decision rule ϕ^* by the sets

$$\mathcal{R}_l \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_l) \le E_{l|l},\ \ D(Q \| Q_l) < \infty\},\ l = \overline{1,11}, \tag{4}$$

$$\mathcal{R}_6 \triangleq \{Q \circ P : D(Q \circ P \| Q \circ P_l) > E_{l|l},\ l = \overline{1,11}\},$$

$$\mathcal{R}_l^N \triangleq \mathcal{R}_l \cap \mathcal{Q} \circ \mathcal{P}^N(\mathcal{X}), \qquad l = \overline{1,12}.$$

and introduce the functions:

$$E^*_{l|l}(E_{l|l}) \triangleq E_{l|l},\ l = \overline{1,11},$$

$$E^*_{l|m}(E_{l|l}) = \inf_{Q \circ P \in \mathcal{R}_l} D(Q \circ P \| Q \circ P_m),\ m = \overline{1,12},\ l \neq m,\ l = \overline{1,11}, \tag{5}$$

$$E^*_{12|m}(E_{1|1}, \dots, E_{11|11}) \triangleq \inf_{Q \circ P \in \mathcal{R}_12} D(Q \circ P \| Q \circ P_m),\ m = \overline{1,11},$$

and

$$E^*_{12|12}(E_{1|1}, \dots, E_{11|11}) \triangleq \min_{l = \overline{1,11}} E^*_{l|12}.$$

We cite the statement of the general case of large deviation result for types by Natarajan [13].

Theorem 1. *: Let $\mathcal{X} = \{1, 2, \dots, I\}$ be a discrete topological space of finite set of the states of the stationary Markov chain possessing an irreducible transition matrix P and $(\mathcal{X}, \mathcal{A})$ be a measurable space such that $\mathcal{A}$ be a nonempty and open subset or convex subset of joint distributions $Q \circ P$ and Q_m is stationary distribution for P_m, then for the type $Q \circ P(\mathbf{x})$ of a vector $\mathbf{x}$ from $Q_m \circ P_m$ on $\mathcal{X}$:*

$$\lim_{N \to \infty} -\frac{1}{N} \log Q_m \circ P_m^N \{\mathbf{x} : Q \circ P(\mathbf{x}) \in \mathcal{A}\} = \inf_{Q \circ P \in \mathcal{A}} D(Q \circ P \| Q \circ P_m).$$

In this section we use the following lemma.

Lemma 1. *:If elements $E_{m|l}(\varphi^i)$, $\quad m,l = \overline{1,12}, i = 1,2$, are strictly positive, then the following equalities hold for $\Phi = (\varphi^1, \varphi^2)$:*

$$E_{m_1,m_2|l_1,l_2}(\Phi) = E_{m_1|l_1}(\varphi^1) + E_{m_2|l_2}(\varphi^2), \quad \textit{if} \quad m_1 \neq l_1, \quad m_2 \neq l_2, \tag{a}$$

$$E_{m_1,m_2|l_1,l_2}(\Phi) = E_{m_i|l_i}(\varphi^i), \quad \textit{if} \quad m_{3-i} = l_{3-i} \quad m_i \neq l_i, \quad i = 1,2. \tag{b}$$

Proof : From the independence of the objects we can write:

$$\alpha^N_{m_1,m_2|l_1,l_2}(\Phi_N) = \alpha_{m_1|l_1}(\varphi^1)\alpha_{m_2|l_2}(\varphi^2), \quad \text{if} \quad m_1 \neq l_1, \quad m_2 \neq l_2, \tag{c}$$

$$\alpha^N_{m_1,m_2|l_1,l_2}(\Phi_N) = \alpha_{m_i|l_i}(\varphi^i_N)[1-\alpha_{m_3-i|l_3-i}](\varphi^{3-i}_N), \quad \text{if} \quad m_3 - i \neq l_3 - i, \quad m_i \neq l_i \tag{d}$$

According to the definitions (1) and (2) we obtain (a) and (b) from equalities (c) and (d). □

Notice that using Lemma 1, for joint probability distributions $D(Q^{'}_{l_1,l_2} \circ P^{'}_{l_1,l_2} \| Q^{'}_{m_1,m_2})$ and definition of $\alpha^N_{l|m}(\phi_N) = Q_m \circ P_m(\mathcal{G}^N_l)$, $\quad m,l = \overline{1,12}$, it is clear that: When $m_i, l_i = \overline{1,4}$, $i = 1,2$, $m_1 \neq m_2$, $l_1 \neq l_2$, we have

$$D(Q^{'}_{l_1,l_2} \circ P^{'}_{l_1,l_2} \| Q^{'}_{m_1,m_2} \circ P^{'}_{m_1,m_2}) =$$
$$D(Q^{'(1)}_{l_1} \circ P^{'(1)}_{l_1} \| Q^{'(1)}_{m_1} \circ P^{'(1)}_{m_1}) + D(Q^{'(2)}_{l_2} \circ P^{'(2)}_{l_2} \| Q^{'(2)}_{m_2} \circ P^{'(2)}_{m_2}),$$

and for $m_i \neq l_i$, $m_{3-i} = l_{3-i}$, $i = 1,2$,

$$D(Q^{'}_{l_1,l_2} \circ P^{'}_{l_1,l_2} \| Q^{'}_{m_1,m_2} \circ P^{'}_{m_1,m_2}) = D(Q^{'(i)}_{l_i} \circ P^{'(i)}_{l_i} \| Q^{'(i)}_{m_i} \circ P^{'(i)}_{m_i}).$$

For example

$$D(Q^{'}_{1,2} \circ P^{'}_{1,2} \| Q^{'}_{4,2} \circ P^{'}_{4,2}) = D(Q^{'(1)}_{1} \circ P^{'(1)}_{1} \| Q^{'(1)}_{4} \circ P^{'(1)}_{4}).$$

□

Now we formulate the theorem from [9], which we prove by application of Theorem 1.

Theorem 2. *Let $\mathcal{X}$ be a fixed finite set, and $P_1, \cdots, P_{12}$ be a family of distinct distributions of a Markov chain. Consider the following conditions for positive finite numbers $E_{1|1}, \cdots, E_{11|11}$:*

$$0 < E_{1|1} < \min[D(Q_m \circ P_m \| Q_m \circ P_1), \; m = \overline{2,12}], \tag{6}$$

$$0 < E_{l|l} < \min[\min E^*_{l|m}(E_{m|m})_{m=\overline{1,l-1}}, \min D(Q_m \circ P_m \| Q_m \circ P_l)_{m=\overline{l+1,12}}],$$
$$l = \overline{2,11}.$$

Two following statements hold:

a). if conditions (6) are verified, then here exists a LAO sequence of tests ϕ^*, the reliability matrix of which $E^* = \left\{E^*_{l|m}(\phi^*)\right\}$ is defined in (5), and all elements of it are positive,

b). even if one of conditions (6) is violated, then the reliability matrix of an arbitrary test having in diagonal numbers $E_{1|1}, \cdots, E_{11|11}$ necessarily has an element equal to zero (the corresponding error probability does not tend exponentially to zero).

Proof: First we remark that $D(Q \circ P_l \| Q \circ P_m) > 0$, for $l \neq m$, because all measures $P_l, l = \overline{1,12}$, are distinct. Let us prove the statement a) of the Theorem 2 about the existence of the sequence corresponding to a given $E_{1|1}, \cdots, E_{11|11}$ satisfying condition (6). Consider the following sequence of tests ϕ^* given by the sets

$$\mathcal{B}_l^N = \bigcup_{Q \circ P \in \mathcal{R}_l^N} \mathcal{T}^N_{Q \circ P}(\mathbf{x}), \quad l = \overline{1,12}. \tag{7}$$

Notice that on account of condition (6) and the continuity of divergence D for N large enough the sets $\mathcal{R}_l^N, l = \overline{1,12}$ from (4) are not empty. The sets $\mathcal{B}_l^N, l = \overline{1,12}$, satisfy conditions :

$$\mathcal{B}_l^N \bigcap \mathcal{B}_m^N = \emptyset, l \neq m, \qquad \bigcup_{l=1}^{12} \mathcal{B}_l^N = \mathcal{X}^N.$$

Now let us show that, exponent $E_{l|m}(\phi^*)$ for sequence of tests ϕ^* defined in (7) is equal to $E^*_{l|m}$. We know from (4) that $\mathcal{R}_l, l = \overline{1,11}$, are convex subset and $\mathcal{R}_{12}$ is open subset of the decision rule of ϕ^*, therefore $\mathcal{R}_l, l = \overline{1,12}$, satisfy in condition of Theorem 1. With relations (4), (5), by Theorem 1 we have

$$\lim_{N\to\infty} -\frac{1}{N} \log \alpha^N_{l|m}(\phi^*) = \lim_{N\to\infty} -\frac{1}{N} \log Q_m \circ P^N_m(\mathcal{R}_l) = \inf_{Q\circ P \in \mathcal{R}_l} D(Q \circ P \| Q \circ P_m). \tag{8}$$

Now using (2) and (8) we can write

$$E_{l|m}(\phi^*) = \inf_{Q\circ P \in \mathcal{R}_l} D(Q \circ P \| Q \circ P_m) \quad m, l = \overline{1,12}. \tag{9}$$

Using (8), (4) and (5) we can see that all $E^*_{l|m}$ are strictly positive. The proof of part (a) will be finished if one demonstrates that the sequence of the tests ϕ^* is LAO, that is for given finite $E_{1|1}, \cdots, E_{11|11}$ for any other sequence of tests ϕ^{**}

$$E^*_{l|m}(\phi^{**}) \leq E^*_{l|m}(\phi^*), \quad m, l = \overline{1,12}.$$

Let us consider another sequence of tests ϕ^{**}, which is defined by the sets $\mathcal{G}_1^N, \cdots, \mathcal{G}_{12}^N$ such that

$$E^*_{l|m}(\phi^{**}) \geq E^*_{l|m}(\phi^*), \quad m, l = \overline{1,12}.$$

This condition is equivalent to the inequality

$$\alpha^*_{l|m}(\phi^{**}) \leq \alpha^*_{l|m}(\phi^*). \tag{10}$$

We examine the sets $\mathcal{G}_l^N \bigcap \mathcal{B}_l^N, l = \overline{1, 11}$. This intersection can not be empty, because in that case

$$\alpha_{l|l}^{(N)}(\phi^{**}) = Q_l \circ P_l^N(\overline{\mathcal{G}_l^N}) \geq Q_l \circ P_l^N(\mathcal{B}_l^N) \geq$$

$$\geq \max_{Q \circ P: D(Q \circ P \| Q_l \circ P_l) \leq E_{l|l}} Q_l \circ P_l^{(N)}(\mathcal{T}_{Q \circ P}^N(\mathbf{x})) \geq \exp\{-N(E_{l|l} + o(1))\}$$

Let us show that $\mathcal{G}_l^N \bigcap \mathcal{B}_m^N = \emptyset, l = \overline{1, 11}$. If there exists $Q \circ P$ such that $D(Q \circ P \| Q_l \circ P_l) \leq E_{l|l}$ and $\mathcal{T}_{Q \circ P}^N(\mathbf{x}) \in \mathcal{G}_l^N$, then

$$\alpha_{l|m}^{(N)}(\phi^{**}) = Q_m \circ P_m^N(\overline{\mathcal{G}_l^N}) > Q_m \circ P_m^N(\mathcal{T}_{Q \circ P}^N(\mathbf{x})) \geq \exp\{-N(E_{m|m} + o(1))\}$$

When $0 \neq \mathcal{G}_l^N \bigcap \mathcal{T}_{Q \circ P}^N(\mathbf{x}) \neq \mathcal{T}_{Q \circ P}^N(\mathbf{x})$, we also obtain that

$$\alpha_{l|m}^{(N)}(\phi^{**}) = Q_m \circ P_m^N(\mathcal{G}_l^N) > Q_m \circ P_m^N(\mathcal{G}_l^N \bigcap \mathcal{T}_{Q \circ P}^N(\mathbf{x})) \geq \exp\{-N(E_{m|m} + o(1))\}$$

Thus it follows if

a). $l < m$ from (6)we obtain that $E_{l|m}(\phi^{**}) \leq E_{m|m} < E_{l|m}^*(\phi^*)$.

b). $l > m$ then $E_{l|m}(\phi^{**}) \leq E_{m|m} < E_{l|m}^*(\phi^*)$, which contradicts our assumption. Hence we obtain that $\mathcal{G}_l^N \bigcap \mathcal{B}_l^N = \mathcal{B}_l^N, l = \overline{1, 11}$. The following intersection $\mathcal{G}_{12}^N \bigcap \mathcal{B}_{12}^N = \mathcal{B}_{12}^N$ is empty too, because otherwise

$$\alpha_{12|l}^*(\phi^{**}) \geq \alpha_{12|l}^*(\phi^*),$$

which contradicts to (10), in this case $\mathcal{G}_l^N = \mathcal{B}_l^N, l = \overline{1, 12}$. □

According the previous explaining the statement of part b) of theorem is evident, since the violation of one of the conditions (8) reduces to the equality to zero of a least one of the elements $E_{l|m}^*$ defined in (5).

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