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Full Length Research Paper

Chebyshev collocation approach to stability of blood flows in a large artery

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In this study, treating the large artery as a rigid channel with uniform width and the blood as an incompressible Newtonian fluid with variable viscosity due to transverse variation in hematocrit ratio, the basic flow structure and its temporal stability to small disturbances were studied. A fourth-order Eigenvalues problem which reduces to the well known Orr–Sommerfeld equation in some limiting cases was obtained and solved numerically by a spectral collocation technique with expansions in Chebyshev polynomials implemented in MATLAB. Graphical results for the basic flow axial velocity, disturbance growth rate and marginal stability curve are presented and discussed. It is worth pointing out that a transverse increase in the blood hematocrit ratio towards the central region of the artery had a stabilizing effect on the flow.

Key words: Arterial blood flow, hematocrit ratio, variable viscosity, temporal stability, Chebyshev spectral collocation technique.

INTRODUCTION

The arterial blood flow provides a way for glucose, oxygen and hormones to reach various organs around the body. The first set of blood leaves the heart from the left ventricle into the biggest artery called the aorta (Pedley, 1980). It is important that fresh blood from aorta goes directly to the brain because the brains need oxygen constantly to avoid irreversible damage to it. Blood is a suspension of cells in plasma and can be separated into microscopically visible element and liquid plasma (Fung, 1984). The elements are red cells or erythrocytes, the white cells or leukocyte, and the platelets or thrombocytes (Figure 1). The concentration of erythrocytes in the blood has a strong influence on blood viscosity. At a hematocrit of 40 to 45%, blood viscosity is approximately 3 times the value for plasma and approximately 5 times that of water (Matral et al., 1987). Blood viscosity shows a curvilinear relation with the hematocrit and it increases sharply when the hematocrit is raised much beyond the normal range (Burton, 1966).

Hematocrit (hemato from the Greek haima = blood; crit from the Greek krinein = to separate) is the ratio of the volume of packed red blood cells to the total blood volume and is therefore also known as the packed cell volume, or PCV. In healthy adult individuals the red blood cells constitute approximately 40 to 48%, whereas newborns may have hematocrits of up to 60% (Fuchs et al., 1987). In the course of blood flow in the large arteries, the red blood cells in the vicinity of arterial wall move to the central region of the artery so that the hematocrit ratio becomes quite low near the arterial wall, which results in lower viscosity in this region (Ditzel and Kampmann, 1971; Pedley, 1980). Moreover, due to high shear rate near the arterial wall, the viscosity of blood is further reduced. Therefore, for flow problems in large blood vessels, the blood may be treated as Newtonian fluid with transverse variation in viscosity, which takes its maximum value at the central region of the artery and minimum value near the arterial wall.

Furthermore, blood viscosity and its major determinants such as hematocrit ratio variation, may be important risk factor for the early development atherosclerosis (Koenig and Ernst, 1992). Atherosclerosis refers to the occlusion of the arterial lumen. Therefore, from the mathematical standpoint, the study of variable viscosity arterial blood flow and its temporal stability to small disturbances is paramount in understanding pathological situations in the cardiovascular systems (Orszag, 1971; Makinde, 2005). Meanwhile, the stability of a constant viscosity Newtonian

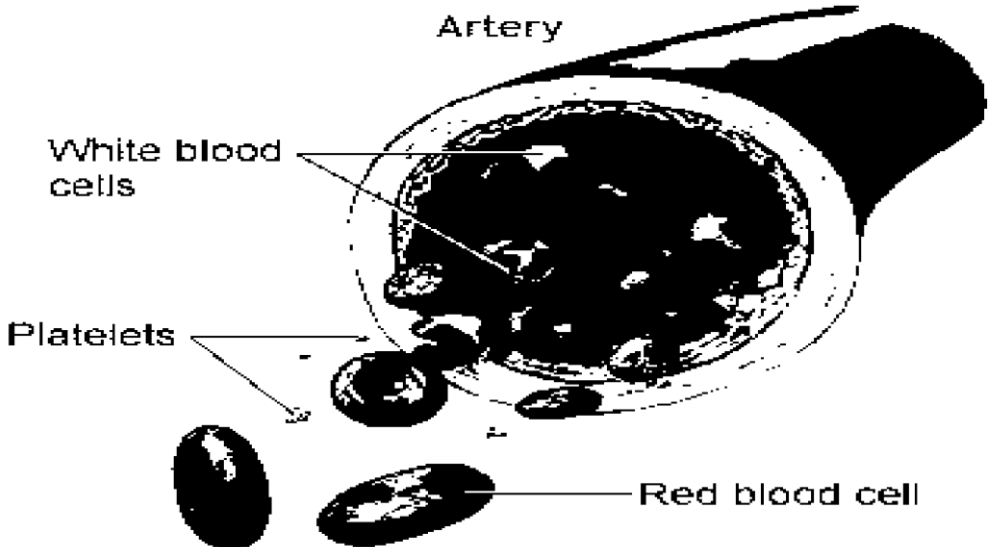

Figure 1. Content of blood (Fung, 1984).

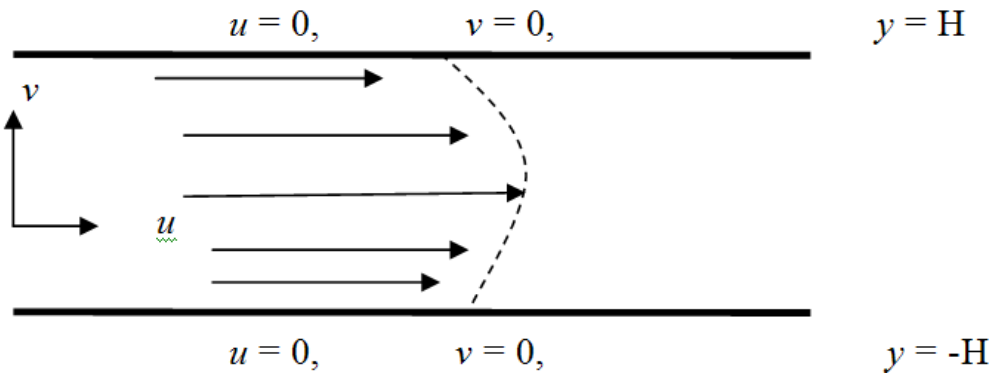

Figure 2. Geometry of the problem.

fluid flowing through channels of varying width to small disturbances has been investigated both analytically and numerically by several authors (Squire, 1933; Ho and Denn, 1977; Drazin, 1999; Turkyilmazoglu et al., 2000; Makinde and Mhone, 2007; Turkyilmazoglu, 2007; Makinde, 2009). Recently, Prakash and Makinde (2011) investigated the combined effects of magnetic field and thermal radiation on arterial blood flow in the presence of hematocrit variation due to blood erythrocytes concentration. To the best of our knowledge, there has not been any reported study concerning stability of arterial blood flow with varying hematocrit ratio.

Motivated by the scarcity of such investigation in the literature, the temporal development of small disturbances in a variable viscosity arterial blood flow due to transverse variation in hematocrit ratio was investigated. Treating artery as a rigid channel with uniform width and the blood as an incompressible Newtonian fluid with variable viscosity, the linear stability analysis was performed. The followings were accomplished: problem formulation and solution for the steady basic flow; derivation of the Eigenvalues problem for temporal development of small disturbances; solving the resultant Eigenvalues problem by employing the Chebyshev spectral collocation numerical technique (the results obtained were discussed quantitatively).

MATHEMATICAL MODEL

For the development of mathematical model (Figure 2), the following assumptions were made:

(i) In the large artery, blood is assumed to be an incompressible Newtonian fluid
(ii) Viscosity of blood varies transversely with maximum value in the central region of the artery due to hematocrit ratio variation.
(iii) A two-dimensional flow problem is considered.

The governing equations of continuity and momentum for symmetric flow of blood through an artery in dimensionless form under the above mentioned assumption are (Mekheimer, 2004):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{2}{\mathrm{Re}}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{2}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \tag{3}$$

Where, t is the time; P is the pressure; *Re* is the Reynolds number; *x* is the coordinate in the stream-wise direction; *y* is the normal coordinate and (u, v) are the velocity components in the *x* and *y* directions, respectively. Based on the transverse variation in the hematocrit ratio within the blood vessel, the variable viscosity function μ is modeled as:

$$\mu(y)=\sec h(\beta y), \tag{4}$$

Where, β is the viscosity variation parameter. The shape of the profile given by Equation 4 is valid only for very dilute suspension of red cells (Matral et al., 1987). The governing Equations (1 - 3) have been non-dimensionalized using the following dimensionless variables:

$$\begin{gathered}u=\frac{\bar{u}}{U_0},\ v=\frac{\bar{v}}{U_0},\ t=\frac{U_0\bar{t}}{H},\ x=\frac{\bar{x}}{H},\ y=\frac{\bar{y}}{H},\ P=\frac{\bar{P}}{\rho U_0^2},\\ \mu=\frac{\bar{\mu}}{\mu_0},\ \nu=\frac{\mu_0}{\rho},\ \mathrm{Re}=\frac{U_0H}{\nu},G=-\frac{\partial P}{\partial x},\ \beta=H\gamma.\end{gathered} \tag{5}$$

Where, *H* is the channel characteristic half width; ρ is the fluid density; U_0 is the velocity scale; ν is the kinematic viscosity coefficient and *G* is the constant axial pressure gradient parameter. The basic steady state of the arterial blood flow system corresponds to a parallel flow with velocities $u = U(y)$ and $v = 0$. The equation and the boundary conditions describing the basic steady state flow are:

$$\frac{d}{dy}\left(\mu\frac{dU}{dy}\right)=-G,\ \frac{dU}{dy}(0)=0,\ \ U(1)=0, \tag{6}$$

and the solution is given by:

$$U(y;\beta>0)=\frac{G}{\beta^2}\left[\beta\left(\sinh(\beta)-y\sinh(\beta y)\right)+\cosh(\beta y)-\cosh(\beta)\right] \tag{7a}$$

$$U(y;\beta\to 0)\approx\frac{G}{2}\left(1-y^2\right)+\frac{G}{8}\left(1-y^4\right)\beta^2+O(\beta^4) \tag{7b}$$

Stability analysis

Herein, the temporal stability analysis of two-dimensional small disturbances imposed on the basic flow is considered (Squire, 1933). Let

$$u(x,y,t)=U(y)+\hat{u}(x,y,t),\ v(x,y,t)=\hat{v}(x,y,t),\ p(x,y,t)=P(x)+\hat{p}(x,y,t), \tag{8}$$

Where $\hat{u}$, $\hat{v}$ and $\hat{p}$ are very small so that products and higher powers can be neglected. Substituting Equation 8 into Equations 1 to 3 and neglecting the nonlinear terms in the disturbance quantities, we obtained:

$$\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}=0, \tag{9}$$

$$\frac{\partial\hat{u}}{\partial t}+U\frac{\partial\hat{u}}{\partial x}+\hat{v}\frac{dU}{dy}=-\frac{\partial\hat{P}}{\partial x}+\frac{2}{\mathrm{Re}}\frac{\partial}{\partial x}\left(\mu\frac{\partial\hat{u}}{\partial x}\right)+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\hat{u}}{\partial y}+\frac{\partial\hat{v}}{\partial x}\right)\right] \tag{10}$$

$$\frac{\partial\hat{v}}{\partial t}+U\frac{\partial\hat{v}}{\partial x}=-\frac{\partial\hat{P}}{\partial y}+\frac{2}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\mu\frac{\partial\hat{v}}{\partial y}\right)+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\hat{u}}{\partial y}+\frac{\partial\hat{v}}{\partial x}\right)\right], \tag{11}$$

Following Orszag (1971), we sought a normal mode solution for Equations 9 to 11 defined in terms of a stream-function as:

$$\psi(x,y,t)=\phi(y)e^{i\alpha(x-ct)}, \tag{12}$$

Where, $\phi(y)$ is the amplitude function and c,α are the disturbances wave speed and wave number respectively. The disturbance velocity components can be expressed as follows:

$$\hat{u}=\frac{\partial\psi}{\partial y}=\phi'(y)e^{i\alpha(x-ct)}, \tag{13}$$

$$\hat{v}=-\frac{\partial\psi}{\partial x}=-i\alpha\phi(y)e^{i\alpha(x-ct)}, \tag{14}$$

Where the prime symbol denotes differentiation with respect to *y*. Substituting Equations 12 to 14 into Equations 9 to 11 and eliminating the pressure terms yields;

$$\begin{aligned}(U-c)(\phi''-\alpha^2\phi)-U''\phi=&\frac{\mu}{i\alpha\,\mathrm{Re}}\left(\phi^{iv}-2\alpha^2\phi''+\alpha^4\phi\right)\\&+\frac{2\mu'}{i\alpha\,\mathrm{Re}}\left(\phi'''-\alpha^2\phi'\right)+\frac{\mu''}{i\alpha\,\mathrm{Re}}\left(\phi''+\alpha^2\phi\right),\end{aligned} \tag{15}$$

with the boundary conditions;

$$\begin{aligned}&\phi(-1)=\phi'(-1)=0,\\&\phi(1)=\phi'(1)=0.\end{aligned}$$

It is noteworthy that Equation 15 reduces to the classical Orr-Sommerfield equation (Drazin, 1980) when μ =1, which correspond to constant blood viscosity situation with β =0. In order to find a non-trivial function ϕ satisfying Equation 15 with boundary conditions 16, the parameters $\alpha, \mathrm{Re}, \beta$ and c must satisfy a certain complex Eigenvalue relation, say:

$$F(\alpha,c,\beta,\mathrm{Re})=0. \tag{17}$$

For temporal development of the disturbances, α is real and c is

complex which can be expressed as:

$$c = c_r(\alpha, \beta, \mathrm{Re}) + ic_i(\alpha, \beta, \mathrm{Re}) \quad (18)$$

The imaginary part of Equation 18 determines whether the disturbances grow or decay. When $\alpha c_i > 0$ the disturbances grow, while when $\alpha c_i = 0$ they neither grow nor decay, in this case the disturbance modes are said to be neutrally stable.

Computational approach

The Eigenvalue problem in Equation 15 with the condition (16) is solved using the Chebyshev spectral collocation method (Turkyilmazoglu and Gajjar, 1999; Makinde and Mhone, 2007). We expand the solution of the differential equation and its boundary conditions as a finite series in Chebyshev polynomials of the form:

$$\phi(y) \approx \phi_{\mathrm{N}}(y_j) = \sum_{k=o}^{\mathrm{N}} \tilde{\phi}_k \mathrm{T}_k(y_j), \quad j = 0,1,...,\mathrm{N} \quad (19)$$

Where T_k is the k[th] Chebyshev polynomial defined by:

$$\mathrm{T}_0(y) = 1,\ \mathrm{T}_1(y) = y,\ \mathrm{T}_{k+1}(y) - 2y\mathrm{T}_k(y) + \mathrm{T}_{k-1}(y) = 0, \quad (-1 \le y \le 1), \quad (20)$$

$\tilde{\phi}_k$ represents the unknown coefficients and y_j are the Gauss-Lobatto collocation points (Canuto et al. 1988) on the interval [-1,1] defined by:

$$y_j = \cos\frac{\pi j}{\mathrm{N}}, \quad j = 0,1,...,\mathrm{N}. \quad (21)$$

Substituting Equation 21 into Equation 19 and requiring that the differential Equation 15 be satisfied at the *N* collocation points, we obtain (*N*+1) x (*N*+1) algebraic equations which form the Eigenvalue problem;

$$\mathrm{E}\phi^T = c\mathrm{B}\phi^T, \quad (22)$$

where $$\phi^{\mathrm{T}} = (\tilde{\phi}_0, \tilde{\phi}_1, ..., \tilde{\phi}_{\mathrm{N}}), \quad (23)$$

is the transpose of the column vector ϕ. The clamped boundary conditions are incorporated explicitly in the first two and last rows of the matrices *E* and *B* by setting:

$$\mathrm{E}(m,n) = \begin{cases} 1 & m = n = 0; \\ 0 & m = 0,\ n = 1, ..., N; \\ \sum_{n=0}^{\mathrm{N}} D_{0n} & m = 1,\ n = 0, ..., N; \\ \tilde{\mathrm{E}}(m,n) & m = 1,...,N\text{-}2,\ n = 0, ..., N; \\ \sum_{n=0}^{\mathrm{N}} D_{\mathrm{N}n} & m = N\text{-}1,\ n = 0, ..., N; \\ 0 & m = N,\ n = 1, ..., N\text{-}1; \\ 1 & m = N,\ n = N; \end{cases} \quad (24)$$

$$\mathrm{B}(m,n) = \begin{cases} 0 & m = 0,1,\mathrm{N}-1,\mathrm{N}\ \ n = 0,...,\mathrm{N}; \\ \tilde{\mathrm{B}}(m,n) & m = 2,...,\mathrm{N}-2,\ \ n = 0,...,\mathrm{N}; \end{cases} \quad (25)$$

Where:

$$\tilde{\mathrm{E}} = U(D^2 - \alpha^2 I) - U'' + \frac{i\mu}{\alpha\,\mathrm{Re}}(D^4 - 2\alpha^2 D^2 + \alpha^4 I) + \frac{2i\mu'}{\alpha\,\mathrm{Re}}(D^3 - \alpha^2 D) + \frac{i\mu''}{\alpha\,\mathrm{Re}}(D^2 + \alpha^2 I) \quad (26)$$

$$\tilde{\mathrm{B}} = (D^2 - \alpha^2 I) \quad (27)$$

U= diag [$U(y_j)$]; *I* is the (*N*+1)x(*N*+1) identity matrix and *D* is the usual differential matrix (cf Canuto et al., 1988). Here diag [ ] means that the entries are placed on the main diagonal of an (*N*+1)x(*N*+1) matrix with the rest of the entries being zero, which usually results to matrix *B* becoming singular. The problem is avoided by employing the idea of Weidmann and Reddy (2000) by using Hermite interpolating polynomials that satisfy the boundary conditions. Thus, we obtained:

$$\tilde{\phi}_0 = 0,\ \sum_{n=0}^{\mathrm{N}} D_{0n}\tilde{\phi}_n = 0 \text{ on } y = 1, \quad (28)$$

$$\tilde{\phi}_{\mathrm{N}} = 0, \qquad \sum_{n=0}^{\mathrm{N}} D_{\mathrm{N}n}\tilde{\phi}_n = 0 \text{ on } y = -1. \quad (29)$$

RESULTS AND DISCUSSION

The Eigen solutions of the generalized Eigenvalue problem (Equations 22 to 29) obtained numerically are presented herein. The numerical solutions have been verified for correctness by comparing with the results obtained by Orszag (1971) for β = 0. The blood viscosity variation model and the flow basic velocity are computed from Equations 4 and 6 and the Chebyshev spectral collocation method is implemented in MATLAB 5.1 to compute the fastest growing mode, although there is no reason to believe that more than one mode of the present problem grows for given fixed values of β, *G*, α and *Re*. Figure 3 shows the transverse variation of arterial blood viscosity with maximum value in the central region and minimum value at the arterial wall. It is noteworthy that increasing values of viscosity variation parameter β leads to a transverse increase in the hematocrit ratio towards the central region of the artery.

Figure 4 illustrates the arterial blood velocity profile. Generally, a parabolic plane Poiseuille profile is observed with maximum value along the centerline and minimum at the wall. However, an increase in the blood viscosity variation parameter causes a further increase in the blood velocity. Table 1 shows the numerical results for the Eigenvalues of the most unstable mode for increasing values of β at fixed values of *G*, α and *Re*. It is interesting

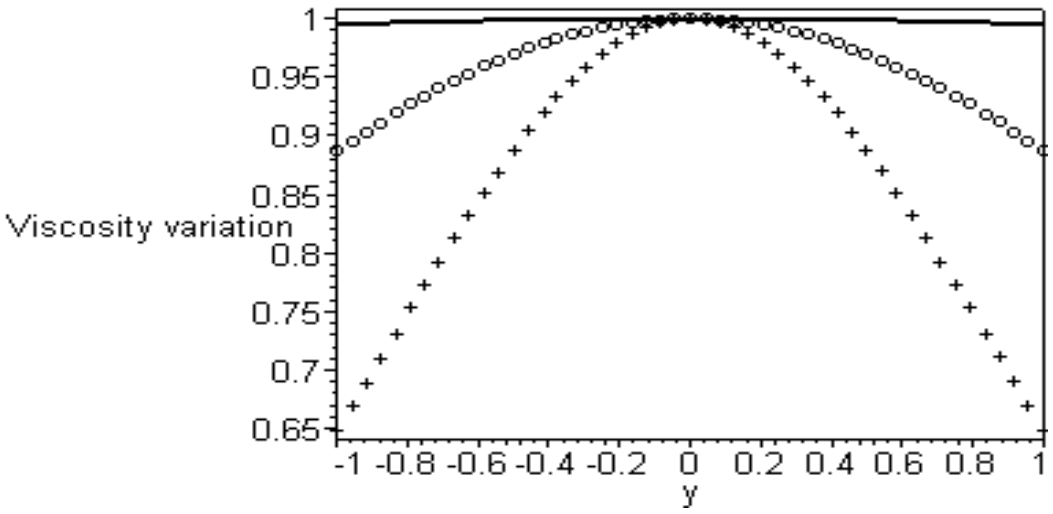

Figure 3. Arterial blood viscosity variation. ——— β = 0.1; **ooooo** β =0.5; **×××××** β = 1.0.

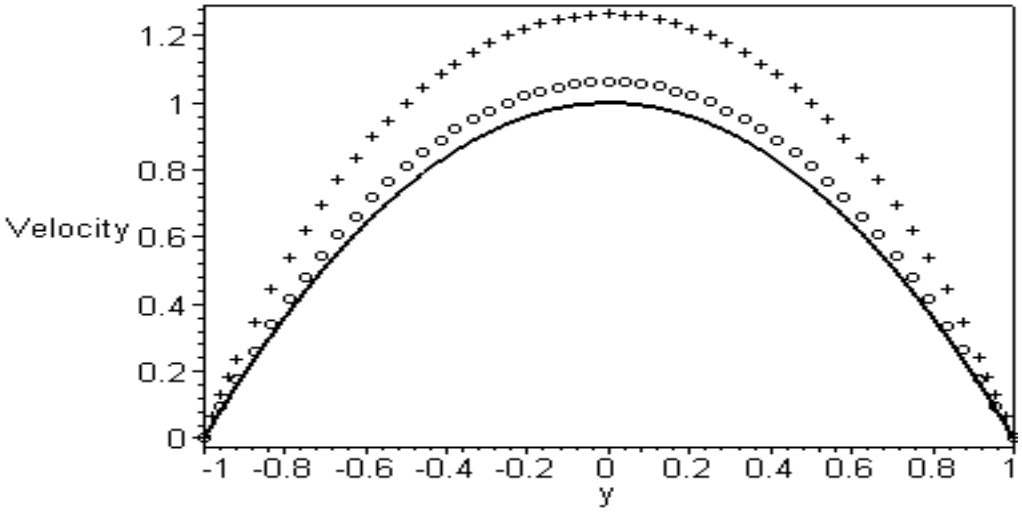

Figure 4. Velocity profile, G = 2; ——— β = 0.1; **ooooo** β =0.5; **×××××** β = 1.0.

Table 1. Computation showing the Eigenvalues of the most unstable mode (G=2, Re = 20000, α = 1).

Variable viscosity parameter (β)	Wave speed (c)
0.0	0.23752648883586 + 0.00373967069853i
0.1	0.23783691590891 + 0.00364707995371i
0.2	0.23876667955404 + 0.00337568662317i
0.3	0.24031185079476 + 0.00294413960238i
0.4	0.24246778013888 + 0.00238189266321i
0.5	0.24523135670077 + 0.00172677600741i
0.6	0.24860316803977 + 0.00102176825239i
0.7	0.25258895099082 + 0.00031134842668i
0.8	0.25719998746173 - 0.00036202436167i
0.9	0.26245247457356 - 0.00096074508379i
1.0	0.26836621542410 - 0.00145385714220i

Table 2. Computations showing the critical values at which unstable modes begin to exist (G = 2).

β	α_c	Re_c
0.0	1.02052	5772.2283
0.2	1.01260	6053.6434
0.4	0.99225	6886.2840
0.6	0.96705	8176.7969
0.8	0.94512	9649.2259
1.0	0.93110	10878.4282

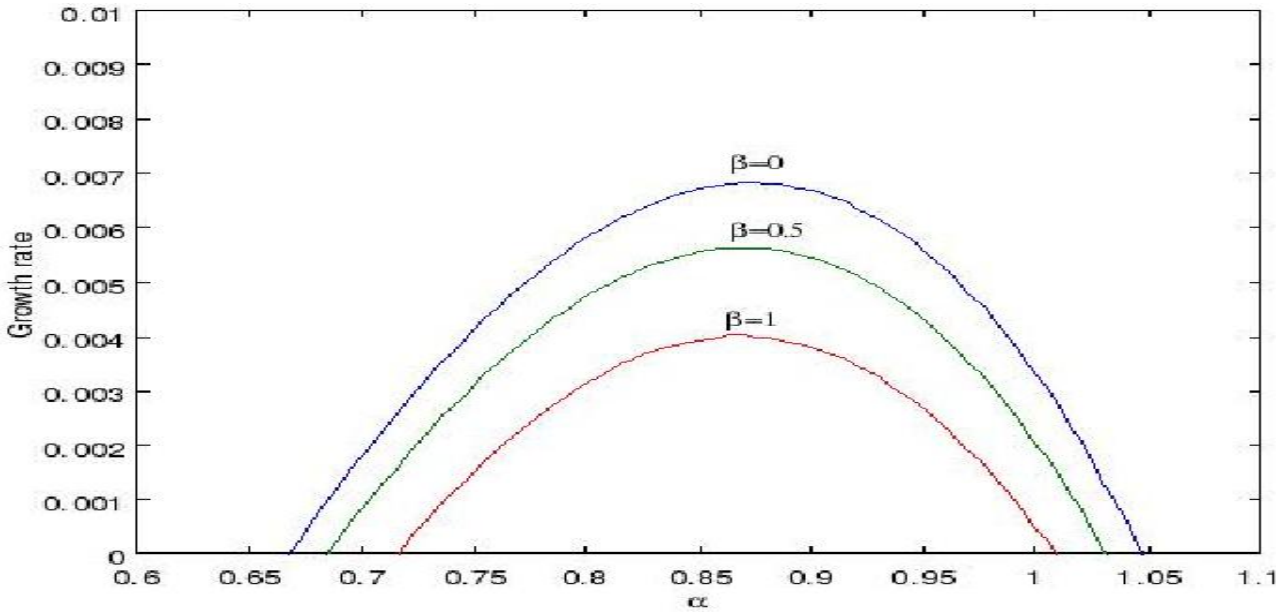

Figure 5. Growth rate αc_i for Re = 20000.

to note that a slight increase in the values of β due to increasing transverse hematocrit ratio variation has the effect of decreasing the imaginary parts of the wave speed. This shows that an increase in blood viscosity transverse variation has a stabilizing effect on the flow. Table 2 shows the critical Reynolds number Re_c and the critical wave number α_c at the instability threshold for varying values of β. For β =0, the result obtained is in perfect agreement with the one reported in Orszag (1971). We observe that an increase in β leads to an increase in the critical Reynolds number and a slight decrease in the critical wave speed. This means that the stable region in (Re, α)-plane increases as the hematocrit ratio transverse variation increases (Figure 6).

In addition, Figure 5 shows the variation in the growth rate of the most unstable mode against the wave number. It is interesting to note that increasing values of β have the effect of damping the disturbances. This means that a transverse increase in the hematocrit ratio towards the arterial central region acts like a control parameter that eliminates the growth of small disturbances in the flow field.

Conclusion

The Chebyshev spectral collocation method implemented in MATLAB is employed to investigate the temporal development of small disturbances in a variable viscosity arterial blood flow due to transverse variation in hematocrit ratio. We obtained accurately the critical Reynolds number (Re_c) and the critical wave number (α_c) for increasing values of viscosity variation parameter. It is observed that a transverse increase in the hematocrit ratio towards the arterial central region has a stabilizing effect on the blood flow.

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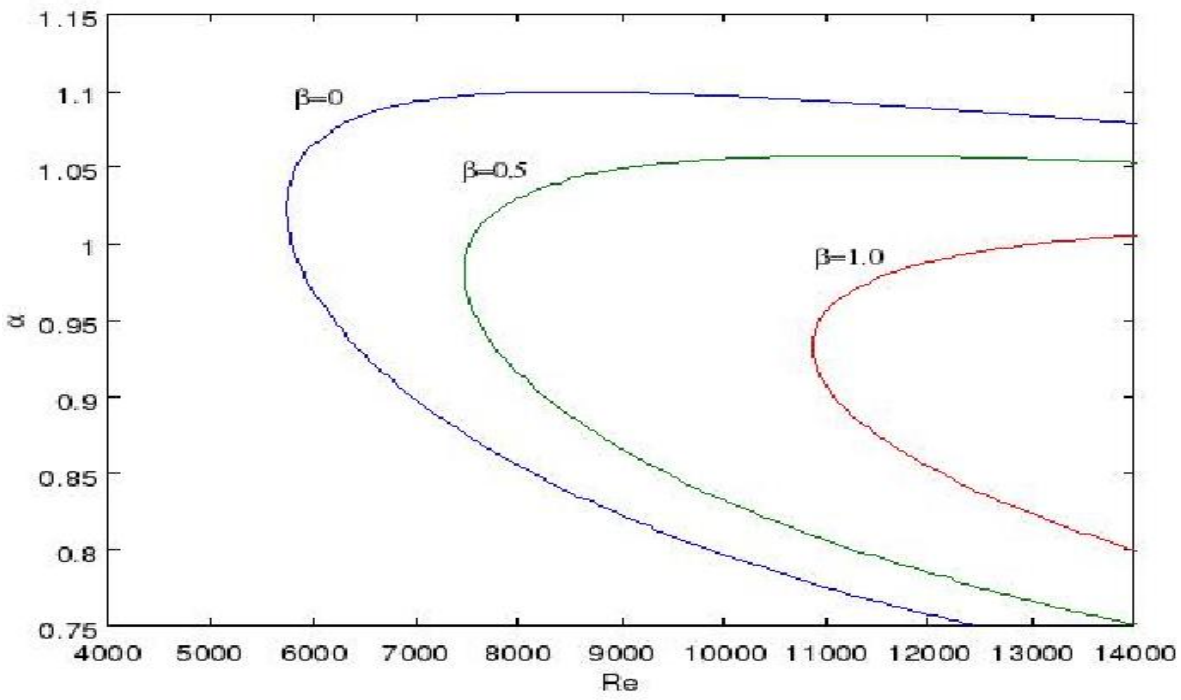

Figure 6. Marginal stability curve for G = 2.

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