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Caputo Derivative-Based FTC Design and Sensor Fault Estimation for Lipschitz Fractional Nonlinear Systems

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Abstract:
Sensor fault estimation and active fault-tolerant control (FTC) are still important problems for guaranteeing the fractional-order nonlinear systems' (FONS) stability under sensor faults. To deal with this problem, this work uses the Caputo fractional derivative framework to construct a unified strategy for simultaneous sensor fault estimation and active (FTC). Sufficient conditions obtained are in the form of linear matrix inequalities (LMIs) from a stability study based on Lyapunov theory. The proposed approach is applied to a two-state system with a biased sensor fault to demonstrate its performance. The simulation results indicate that the adaptive observer can accurately estimate the fault magnitude and has robust tracking ability with large initial estimation errors. Moreover, the fault tolerant controller is capable of stabilizing the system successfully with zero steady state error and without any overshoot.

Keywords: Caputo fractional derivative (CFD), fault-tolerant control (FTC), sensor fault estimation, adaptive observer, fractional nonlinear systems, linear matrix inequalities (LMIs).

تصميم التحكّم المتسامح مع الأخطاء القائم على مشتق كابوتو وتقدير أخطاء المستشعرات للأنظمة ليبشيتز الكسرية الغير خطية

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الملخص
لا يزال تقدير خطأ المستشعر والتحكم المتسامح مع الخطأ من المشكلات المهمة لضمان استقرار الأنظمة غير الخطية ذات الترتيب الكسري في وجود أخطاء المستشعر. للتعامل مع هذه المشكلة، تظهر هذه الورقة نهج مشتق كابوتو الكسري لبناء استراتيجية موحدة لتقدير خطأ الاستشعار في وقت واحد والتحكم المتسامح مع الخطأ. حيث أخذ في الاعتبار شرط متباينة المصفوفة الخطية linear matrix inequalities لدراسة الاستقرار على أساس نظرية ليابونوف Lyapunov theory. يتم تطبيق النهج المقترح على نظام فضاء الحالة من الدرجة الثانية مع خطأ استشعار متحيز وذلك لإثبات أدائه. كما أثبتت نتائج المحاكاة أن المراقب التكيفي يمكنه تقدير حجم الخطأ بدقة ولديه قدرة تتبع قوية مع أخطاء تقدير أولية كبيرة. علاوة

على ذلك، فإن وحدة التحكم المتسامحة مع الأخطاء قادرة على ايصال النظام لحالة الاستقرار بنجاح مع عدم وجود خطأ في حالة الاستقرار وبدون أي تجاوز **.519**

الكلمات المفتاحية: مشتق كابوتو الكسري، التحكم المتسامح مع الأخطاء، تقدير خطأ الاستشعار، المراقب التكيفي، الانظمة الكسرية والغير الخطية، متباينة المصفوفة الخطية.

Introduction

Fractional calculus has been proved to be a powerful mathematical tool to describe complex dynamical systems with memory effects, hereditary properties and nonlocal interactions [1], [2]. Fractional-order systems (FOS) are able to capture the long-range dependencies and memory effects that are naturally present in physical, biological and engineered processes [3]. as opposed to traditional integer-order models, which presume that future behavior is solely based on current circumstances. The Caputo derivative is considered, among the numerous fractional derivative definitions, to be the most widely used in engineering areas, since it permits the usage of beginning conditions in terms of integer-order derivatives which are physically relevant [4]. This property makes the Caputo derivative particularly convenient for practical control design where measurable initial conditions are required [5].

Many engineering areas have demonstrated the advantages of fractional-order models over their integer-order counterparts. Fractional models are able to describe stress-strain relationships which could never be predicted by models based on integer derivatives [6]. FOPID (fractional-order PID) controllers can be more flexible and robust than the conventional PID controllers for control systems [7]. Recently it is reviewed that fractional order controllers are successfully applied to optimize the industrial processes and improve the system stability in various applications [8].

The inverted pendulum is one of the simplest benchmark problems in nonlinear control engineering. Underactuated nature, strong nonlinearities such as sine, cosine and the square of the angular velocity, and its inherent instability make it an ideal testbed for the evaluation of advanced control strategies [9]. The practical effectiveness of fractional-order control has been demonstrated in recent studies for inverted pendulum-cart systems. Sanchez-Rivero et al. presented a model reference adaptive control framework based on fractional gradient which obtained better tracking performance than the conventional controllers in real time experiments on a physical pendulum-cart setup [10].

(FTC) is essential for ensuring system reliability and safety in the presence of component faults [11]. Faults in actuators and sensors can lead to probable manifestations of bias, drift and abrupt signal losses which, if not correctly addressed, can substantially compromise control performance and possibly cause system instability [12]. In (FOS) these faults are particularly challenging with time-varying delays as fault effects can be amplified by delayed state propagation [13]. Recently, Zhao et al. proposed a memoryless observer-based fault detection approach by integrating the Razumikhin theorem for (FONS) with time-varying delays and Caputo derivative analysis [14].

Recent research [15] showed the asymptotic stability robustness by Lyapunov-Krasovskii functions and (LMI) techniques, and presented a fault-tolerant fuzzy resilient control strategy for fractional-order stochastic underactuated systems with actuator saturation. For joint state and sensor defect estimation of continuous-time switching systems, a robust descriptor observer is proposed by Ben Halima et al. [16], presented at the 20th IEEE International Multi-Conference on Systems, Signals and Devices, 2023. A paper on a new exponential approach to dynamic event-triggered leaderless consensus of nonlinear multi-agent systems, which is related to distributed control under communication constraints, will also be presented at the 2023 IEEE Region 10 Conference. [17].

These developments haven't resolved every issue. The simultaneous estimate of system states and sensor defects in Caputo fractional nonlinear systems is rarely researched. Moreover, the coupling of adaptive observers with (LMI) based (FTC) design for (FOS) is still an ongoing research subject. The command-filtered backstepping control using fractional-order command filters is promising to cope with these challenges [18].

To the best of the authors' knowledge, this paper's primary contributions are: a new fractional-order adaptive observer for simultaneous state and sensor fault estimation of Lipschitz nonlinear systems; an active (FTC) law according to the estimated fault; Lyapunov-based (LMI) conditions; and numerical validation on a two-state system with bias fault that hasn't been covered in the literature.

This paper is structured as follows for the remainder. The prerequisites are covered in Section 2. The (FTC) statute, its stability analysis, the observer design, and the problem formulation are all presented in Section 3. In section 4, the simulation results are displayed. Section 5 brings the paper to a close.

Preliminaries

This section contains useful results and terminology of fractional order calculus.

Definition 1:[19] Assume that a represent the integer portion of $\alpha + 1$. (CFD) refers to.

$$ {}^{C}D_{t_0,t}^{\alpha}x(t) = \frac{1}{\Gamma(a-\alpha)}\int_0^t (t - \tau)^{a-\alpha-1}\frac{d^a}{d\tau^a}x(\tau)d\tau\,, \quad (a-1 < \alpha < a) $$

For $0 < \alpha < 1$, this derivative is reduced to:

$$ {}^{C}D_{t_0,t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{d}{d\tau}x(\tau)d\tau $$

$\Gamma(a-\alpha) = \int_0^{+\infty} e^{-t}t^{a-\alpha-1}dt$, is the Gamma function generalizing factorial for non-integer arguments. The Mittag-Leffler (ML) function is most frequently defined in order to solve fractional order difficulties. The function can be thought of as the exponential function's generalization.

Definition 2: [19] The following describes (ML) function:

$$ E_{\alpha,\mu}(\xi) = \sum_{k=0}^{+\infty}\frac{\xi^k}{\Gamma(k\alpha+\mu)} $$

where $\alpha > 0$, $\mu > 0$, $\xi \in \mathbb{C}$. For $\mu = 1$, one has $E_\alpha(\xi) = E_{\alpha,1}(\xi)$, furthermore $E_{1,1}(\xi) = e^{\xi}$.

In this work, we consider the Caputo (FONS) defined by any $t \geq 0$ as follows:

$$ \begin{aligned} {}^{C}D_{t_0,t}^{\alpha}x(t) &= Ax(t) + Bu(t) + f(x(t)) \\ y(t) &= Cx(t) + Df_s(t) \end{aligned} \qquad (1) $$

$x(t) \in \mathbb{R}^n$ represents the system states, $u(t) \in \mathbb{R}^m$ denotes the control input, $f\big(x(t)\big)$ is a nonlinear function satisfies the Lipschitz condition, $y(t) \in \mathbb{R}^p$ signifies the system output and $f_s(t) \in \mathbb{R}^q$ indicates the sensor fault vector. The constant matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, and $D \in \mathbb{R}^{q\times l}$ are well-known.

Definition 3: [19] If there are positive constants m, λ, and $\alpha \in (0,1)$ such that the trajectory satisfies the following for every starting state $x(t_0)$ at any initial time $t_0 \geq 0$, then system (1) is said to be universally (ML) stable.

$$\|x(t)\| \leq m\|x(t_0)\| \, E_{\alpha,1}(-\lambda(t-t_0))^{\alpha}), \forall t \geq t_0,$$

For $t_0 = 0$, this simplifies to:

$$\|x(t)\| \leq m\|x(0)\| E_{\alpha}(-\lambda t^{\alpha}),$$

This stability concept generalizes exponential stability, which is recovered when $\alpha = 1$ since $E_{1,1}(-\lambda t) \; = e^{-\lambda t}$.

Theorem 2.1: [20] Assume that the system's equilibrium point is $x = 0$. Assume that $V: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz in x function that is locally continuously differentiable.

$$\mu_1\|x\|^c \leq V(t, x(t)) \leq \mu_2\|x\|^{cd}$$

$${}^{C}D^{\alpha}_{t_0,t}V(t, x(t)) \leq -\mu_3\|x\|^{cd}$$

where $t \geq 0,\ x \in \mathbb{R}^n,\ \alpha \in (0,1),\ \mu_1, \mu_2, \mu_3, c$ and d are positive, arbitrary constants. In such case, $x = 0$ is (ML) stable worldwide.

Definition 4: [21] If there is a constant $L_f > 0$ such that: such that a function $f: R^n \to R^n$ is Lipschitz continuous, then

$$\| f(x) - f(y) \| \leq L_f \| x - y \|, \qquad \forall x, y \in R^n$$

Lemma 1: [22] For any vectors $x, y \in R^n$ and any scalar $\varepsilon > 0$:

$$2x^T y \leq \frac{1}{\varepsilon} x^T x + \varepsilon y^T y$$

Lemma 2: [22] For any matrix P**:**

$$2x^T P y \leq \frac{1}{\varepsilon} x^T P P^T x + \varepsilon y^T y$$

This inequality is essential for bounding cross-terms that appear in Lyapunov stability analysis.

Lemma 3: [23] Assume that $\alpha \in (0,1)$ and $P \in \mathbb{R}^{n\times n}$ are constant, symmetric, and positive definite matrices. In that case, the following relationship is true:

$$\frac{1}{2}\, {}^{C}D^{\alpha}_{t_0,t}\big(x^T(t)Px(t)\big) \leq x^T(t)P \, {}^{C}D^{\alpha}_{t_0,t}x(t),$$

Lemma 4: [22] Consider R, S, and T, three acceptable dimension matrices, with R and Q being symmetric. Thus

$$\begin{cases} T > 0 \\ R + S^T T^{-1} S < 0 \end{cases} \Leftrightarrow \begin{bmatrix} R & S^T \\ S & -T \end{bmatrix} < 0$$

Observer-Based FTC Scheme

The goal is to design an (FTC) for system (1) to stabilize the states of the closed loop system in the presence of bias sensor faults. Since the suggested method will need these expected signals, we shall estimate the system states and any problems. To begin, let's assume the following.

Assumption 1: There exists a matrix K such that $\big(A - BK - LC - BK_f C\big)$ is Hurwitz.

Assuming that the estimated state equation is supplied by: The adaptive observer is designed to concurrently estimate the system states and sensor failures in specific systems where some of the system states may not be measured.

$$ {}^{C}D_{t_0,t}^{\alpha}\hat{x}(t) = A\hat{x}(t) + Bu(t) + f(\hat{x}(t)) + L(y(t) - \hat{y}(t)) \tag{2} $$

Where, the estimated state vector is denoted by $\hat{x}(t)$, the observer gain matrix, L that will be generated later, and the estimated output:

$$ \hat{y}(t) = C\hat{x}(t) + D\hat{f}_s(t) \tag{3} $$

the estimated output vector, and the estimated sensor fault vector are denoted by $\hat{y}(t)$ and $\hat{f}_s(t)$ respectively, the adaptation rule as follow:

$$ {}^{C}D_{t_0,t}^{\alpha}\hat{f}_s(t) = \Gamma D^T\big(y(t) - \hat{y}(t)\big) \tag{4} $$

Where $\Gamma = \Gamma^T > 0$ meaning the rate matrix of learning.

Consider fault tolerant control law (5), which is proved next, as follows:

$$ u(t) = -K\hat{x}(t) - K_f\hat{y}(t) \tag{5} $$

where state feedback gain is K the fault compensation gain is K_f , and state estimation error is defined as follow$e_x(t) = x(t) - \hat{x}(t)$, the error of fault estimation $e_f(t) = f_s(t) - \hat{f}_s(t)$, the estimation error of the output $e_y(t) = y(t) - \hat{y}(t) = Ce_x(t) + De_f(t)$.

The following theorem demonstrates that even in the presence of bias sensor defects, the proposed control law (5) is able to stabilize the states of the system in the (ML) sense.

Theorem 3.1 Examine the nonlinear (FOS) (1), the observer (2), the control law (5), the adaptation law (4), and assumption 1. The origin of errors and states in the closed-loop defective system is globally (ML) stable. $(e, \tilde{f}, x) = (0,0,0)$, if symmetric definite positive matrices P, matrices C and D, and ε and η are positive scalars can be found such that

$$ \begin{bmatrix} A^TP_o + P_0A - C^TW^T - WC + \eta L_f^2 I & P_0 \\ * & -\eta I \end{bmatrix} < 0 \tag{6} $$

$$ where\ W = LP_o $$

And

$$ \Big[\begin{matrix} A^T P_c + P_c A - B X - X^T B^T - B N C - C^T N^T B^T + \varepsilon L_f^2 I & I \\ * & - \end{matrix} \quad < 0 \tag{7} $$

where $X = P_cK\ and\ N = P_cK_f$ are auxiliary variables

Proof. The following equation may be obtained by rewriting the equation (2) of the observer based on control law (5):

$$ {}^{C}D_{t_0,t}^{\alpha}\hat{x}(t) = \big(A - BK - LC - BK_fC\big)\hat{x}(t) + f\big(\hat{x}(t)\big) + Ly(t) - \big(Bk_f + L\big)D\hat{f}_s(t) \tag{8} $$

The first equation in (1) will be as follows when control law (5) is used:

$$ {}^{C}D_{t_0,t}^{\alpha}x(t) = Ax(t) - \big(BK + BK_fC\big)\hat{x}(t) - BK_fD\hat{f}_s + f(x) \tag{9} $$

Next, by applying (8) and (9), the state estimation error $e(t)$ defined by the Caputo derivative is reduced to:

$$^{C}D_{t_0,t}^{\alpha}e_x(t) = {}^{C}D_{t_0,t}^{\alpha}x(t) - {}^{C}D_{t_0,t}^{\alpha}\hat{x}(t)$$

Now, substitute the system dynamics from equation (1) and the observer dynamics from equation (2):

$$\begin{aligned}^{C}D_{t_0,t}^{\alpha}e_x(t) = {}&[Ax(t) + Bu(t) + f(x(t))] \\ &- [A\hat{x}(t) + Bu(t) + f(\hat{x}(t)) + L(y(t) \\ &- \hat{y}(t))]\end{aligned}$$

Cancel the common term $Bu(t)$:

$$^{C}D_{t_0,t}^{\alpha}e_x(t) = A(x - \hat{x}) + (f(x) - f(\hat{x})) - L(y - \hat{y})$$

Substitute the output estimation error $y(t) - \hat{y}(t) = Ce_x(t) + De_f(t)$ and define $\tilde{f} = f(x) - f(\hat{x})$

$$^{C}D_{t_0,t}^{\alpha}e_x(t) = Ae_x + \tilde{f} - L(Ce_x + De_f)$$

Expand and collect terms:

$$^{C}D_{t_0,t}^{\alpha}e_x(t) = Ae_x - LCe_x + LDe_f + \tilde{f}$$

$$^{C}D_{t_0,t}^{\alpha}e_x(t) = (A - LC)e_x + LDe_f + \tilde{f}$$

The error of the state estimation using Caputo derivative:

$$^{C}D_{t_0,t}^{\alpha}e_x(t) = (A - LC)e_x + LDe_f + \tilde{f}(x, \hat{x}) \qquad (10)$$

Where $\tilde{f}(x, \hat{x}) = f(x) - f(\hat{x})$ satisfies the Lipschitz condition $\| \tilde{f}(x, \hat{x}) \| \leq L_f \| e_x \|$. This equation represents the state error dynamics and is used in the Lyapunov stability analysis.

In a similar way, we present below the derivation of the fault estimation, using Caputo derivative is$e_f(t) = f_s(t) - \hat{f}_s(t)$.

We start with the definition of the error of fault estimation:

$$^{C}D_{t_0,t}^{\alpha}e_f(t) = {}^{C}D_{t_0,t}^{\alpha}f_s(t) - {}^{C}D_{t_0,t}^{\alpha}\hat{f}_s(t)$$

By using fault estimation law from equation (4) then:

$$^{C}D_{t_0,t}^{\alpha}e_f(t) = {}^{C}D_{t_0,t}^{\alpha}f_s(t) - \Gamma D^T\big(y(t) - \hat{y}(t)\big)$$

Substitute the error of the output estimation $e_y(t) = y(t) - \hat{y}(t)$ with $Ce_x(t) + De_f(t)$,then:

$$^{C}D_{t_0,t}^{\alpha}e_f(t) = {}^{C}D_{t_0,t}^{\alpha}f_s(t) - \Gamma D^T\left(Ce_x(t) + De_f(t)\right)$$

Expand the term: Then the Caputo derivative of the error of fault estimation is:

$$^{C}D_{t_0,t}^{\alpha}e_f(t) = {}^{C}D_{t_0,t}^{\alpha}f_s(t) - \Gamma D^T Ce_x(t) + \Gamma D^T De_f(t)$$

the estimated output (3) in terms of true states and errors using, $e_x(t)$, and $e_f(t)$ is:

$$\begin{aligned}\hat{y} = C\hat{x} + D\hat{f}_s &= C(x - e_x) + D\big(f_s - e_f\big) \\ &= Cx + Df_s - Ce_x - De_f\end{aligned}$$

Substitute this into the control law (5):

$$u(t) = -K(x - e_x) - K_f(Cx + Df_s - Ce_x - De_f)$$

$$u(t) = -Kx + Ke_x - K_fCx - K_fDf_s + K_fCe_x + K_fDe_f$$

$$u(t) = -(K + K_fC)x + (K + K_fC)e_x + K_fDe_f - K_fDf_s \tag{11}$$

Substituting (11) into the system dynamics (1):

$${}^{C}D^{\alpha}_{t_0,t}x(t) = Ax + B[-(K + K_fC)x + (K + K_fC)e_x + K_fDe_f - K_fDf_s] + f(x(t))$$

$${}^{C}D^{\alpha}_{t_0,t}x(t) = Ax - B(K + K_fC)x + B(K + K_fC)e_x + BK_fDe_f - BK_fDf_s + f(x(t))$$

$${}^{C}D^{\alpha}_{t_0,t}x(t) = (A - BK - BK_fC)x + B(K + K_fC)e_x + BK_fDe_f - BK_fDf_s + f(x(t)) \tag{12}$$

Let the Lyapunov function, with the chosen composite Lyapunov function:

$$V = x^TP_cx + e_x^TP_oe_x + e_f^T\,\Gamma^{-1}e_f \tag{13}$$

Where $P_c = P_c^T > 0$ and $P_o = P_o^T > 0$ are positive definite matrices to be determined.

Taking the (CFD) of (13):

$${}^{C}D^{\alpha}_{t_0,t}V = {}^{C}D^{\alpha}_{t_0,t}(x^TP_cx) + {}^{C}D^{\alpha}_{t_0,t}(e_x^TP_oe_x) + {}^{C}D^{\alpha}_{t_0,t}(e_f^T\,\Gamma^{-1}e_f) \tag{14}$$

Now take the derivative of the First Term and using the product rule for Caputo derivatives (which follows the same rule as integer-order derivatives):

$${}^{C}D^{\alpha}_{t_0,t}(x^TP_cx) \le \left({}^{C}D^{\alpha}_{t_0,t}x\right)^TP_cx + x^TP_c\left({}^{C}D^{\alpha}_{t_0,t}x\right)$$

By substitute it in (12):

$$\begin{aligned}{}^{C}D^{\alpha}_{t_0,t}(x^TP_cx) \le{} & [(A - BK - BK_fC)x + B(K + K_fC)e_x \\ & + BK_fDe_f - BK_fDf_s + f(x(t))]^TP_cx \\ & + x^TP_c[(A - BK - BK_fC)x \\ & + B(K + K_fC)e_x + BK_fDe_f - BK_fDf_s \\ & + f(x(t))]\end{aligned}$$

$$\begin{aligned}{}^{C}D^{\alpha}_{t_0,t}(x^TP_cx) \le{} & x^T\Big[(A - BK - BK_fC)^TP_c \\ & + P_c(A - BK - BK_fC)\Big]x \\ & + 2x^TP_cB(K + K_fC)e_x \\ & + 2x^TP_cBK_fDe_f - 2x^TP_cBK_fDf_s \\ & + 2x^TP_cf(x(t))\end{aligned} \tag{15}$$

taking the derivative of the second term and substitute in (10):

$${}^{C}D^{\alpha}_{t_0,t}(e_x^TP_oe_x) = \left({}^{C}D^{\alpha}_{t_0,t}e_x\right)^TP_oe_x + e_x^TP_o\left({}^{C}D^{\alpha}_{t_0,t}e_x\right)$$

$$\begin{aligned}{}^{C}D^{\alpha}_{t_0,t}(e_x^TP_oe_x) & \\ ={} & [(A - LC)e_x + LDe_f + \tilde{f}(x,\hat{x})]^TP_oe_x \\ & + e_x^TP_o[(A - LC)e_x + LDe_f + \tilde{f}(x,\hat{x})]\end{aligned}$$

$$
\begin{aligned}
&{}^{C}D_{t_0,t}^{\alpha}(e_x^T P_o e_x) \\
&\quad = e_x^T[(A-LC)^T P_o + P_o(A-LC)]e_x \\
&\quad - 2e_x^T P_o L D e_f + 2e_x^T P_o \tilde{f}(x,\hat{x})^T
\end{aligned} \tag{16}
$$

now take the derivative for the last term and substitute (10):

$$
{}^{C}D_{t_0,t}^{\alpha}(e_f^T\, \Gamma^{-1} e_f) = \left({}^{C}D_{t_0,t}^{\alpha} e_f\right)^T \Gamma^{-1} e_f + e_f^T\, \Gamma^{-1}\left({}^{C}D_{t_0,t}^{\alpha} e_f\right)
$$

$$
\begin{aligned}
&{}^{C}D_{t_0,t}^{\alpha}(e_f^T\, \Gamma^{-1} e_f) \\
&\quad = [\,{}^{C}D_{t_0,t}^{\alpha} f_s - \Gamma D^T C e_x + \Gamma D^T D e_f]^T \Gamma^{-1} e_f \\
&\quad + e_f^T\, \Gamma^{-1}[\,{}^{C}D_{t_0,t}^{\alpha} f_s - \Gamma D^T C e_x + \Gamma D^T D e_f]
\end{aligned}
$$

Simplify using $\Gamma\Gamma^{-1} = I$

$$
\begin{aligned}
&{}^{C}D_{t_0,t}^{\alpha}(e_f^T\, \Gamma^{-1} e_f) \\
&\quad = 2e_f^T\, \Gamma^{-1}\, {}^{C}D_{t_0,t}^{\alpha} f_s - 2e_f^T D^T C e_x \\
&\quad - 2e_f^T D^T D e_f
\end{aligned} \tag{17}
$$

Apply Lemma 3 to each cross-term in (15), (16) and (17). And applying Schur complement lemma 4, the LMI condition for observer and fault estimation respectively are:

$$
\begin{bmatrix} A^T P_o + P_0 A - C^T W^T - WC + \eta L_f^2 I & P_0 \\ * & -\eta\, I \end{bmatrix} < 0
$$

Where $W = P_o L$

$$
\begin{bmatrix} A^T\, P_c + P_c\, A - B\, X - X^T\, B^T - B\, N\, C - C^T\, N^T\, B^T + \varepsilon\, L_f^2\, I & P_c \\ * & -\varepsilon I \end{bmatrix}
< 0
$$

Where $X = P_c K$ and $N = P_c K_f$ are auxiliary variables

The errors of the estimation origin $(e, \tilde{f}) = (0,0)$ is globally (ML) stable, according to theorem 2.1. Now using (8), we have:

$$
\begin{aligned}
{}^{C}D_{t_0,t}^{\alpha} x(t) &= (A - B(K + K_f C))x(t) + B(K + K_f C)e(t) \\
&\quad - B K_f D \hat{f}_s + f(x)
\end{aligned}
$$

Given that $(A - B(K + K_f C))$ is a Hurwitz matrix and that $e \to 0$ as $t \to \infty$, we infer that $x \to 0$ as $t \to \infty$. The proof is finished with this

Simulation and Results

A numerical example is studied, in this section, to back the theoretical results. A second-order system with a hypothetical continuous sensor fault is used:

$$
{}^{C}D_{t_0,t}^{\alpha} x(t) = \begin{bmatrix} -6 & 1 \\ 0 & -8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
$$

$$
y(t) = [1 \quad 0]\, x(t) + f
$$

$$
f(x(t)) = \begin{bmatrix} 0.005\ \ sinx_1(t) \\ 0.004\ \ cosx_2(t) \end{bmatrix}
$$

The matrix satisfies assumption 1.

$$
K = [26.1376 \quad 63.5281]
$$

This part uses the approach used in the preceding section to evaluate the viability of the closed-loop stability criteria (6). The MATLAB (LMI) Toolbox is used for this investigation, producing the following potential outcomes.

$$\eta = 0.1,\ \ \varepsilon = 0.1,\ L_f = 0.005\ ,\ K_f = 1.2\ ,$$

$$P_o = \begin{bmatrix} 0.2937 & -0.0978 \\ -0.0978 & 0.0503 \end{bmatrix},$$

$$W = \begin{bmatrix} -1.2623 \\ 1.6623 \end{bmatrix},\ A - LC = \begin{bmatrix} -25 & 1 \\ -70 & -8 \end{bmatrix}$$

$$\text{Then, } L = P_O^{-1}W = \begin{bmatrix} 19 \\ 70 \end{bmatrix},$$

and the observer (2) can be constructed.

$$P_c = \begin{bmatrix} 40.528 & 0.5228 \\ 0.5228 & 1.2706 \end{bmatrix},$$

$$\left(A - BK - BK_fC\right) = \begin{bmatrix} -6 & 1 \\ -27.3376 & -71.5281 \end{bmatrix}$$

Then,

$$X = KP_c = [1.0925 \quad 0.0944],$$

$$N = P_cK_f = \begin{bmatrix} 48.6336 & 0.6273 \\ 0.6273 & 1.5247 \end{bmatrix}$$

Let $\alpha = 0.95$, with the following initial conditions:

$$x_0 = [2 \quad -1.5]^T, \hat{x}_0 = [0 \quad 0]^T, \hat{f}_0 = 0,$$

The proposed fractional-order adaptive observer and (FTC) scheme is verified by simulation study on a second order Lipschitz nonlinear system of fractional order. In a healthy scenario $f = 0$, $u(t) = -K\hat{x}(t)$, the state estimation errors e_1 and e_2 in Fig. 1 converge smoothly to zero without overshoot, demonstrating the excellent damping characteristics of the observer. Fig. 2 demonstrates perfect state tracking with zero steady-state error, and Fig. 3 confirm the fault estimate remains at zero with no false alarms, validating the selectivity of the algorithm.

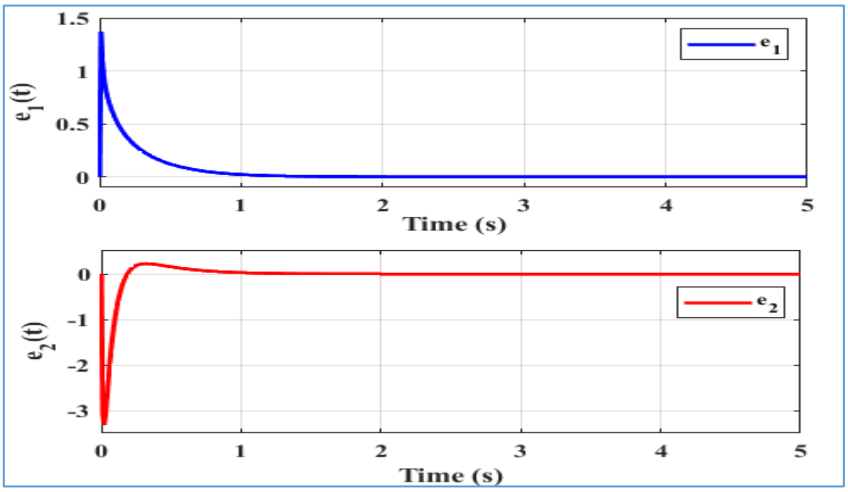

Fig 1: State estimation errors (e_1 & e_2) convergence to zero

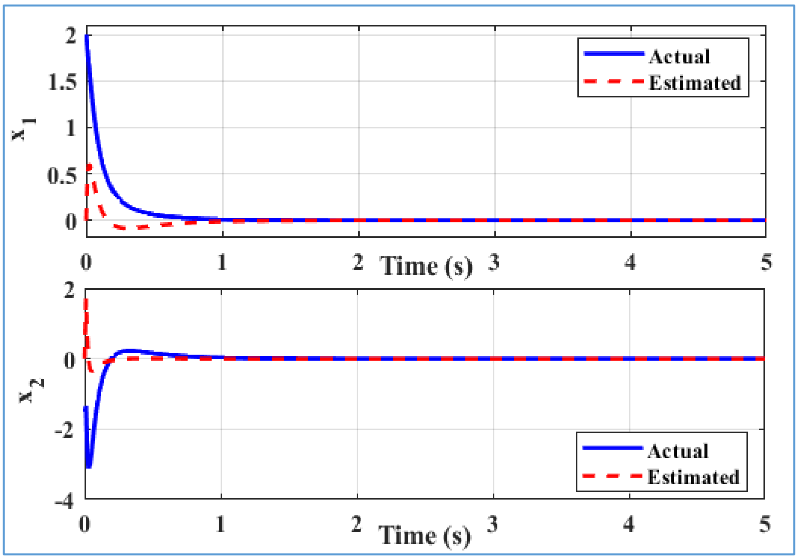

Fig 2: Zero convergence of the actual and estimated states.

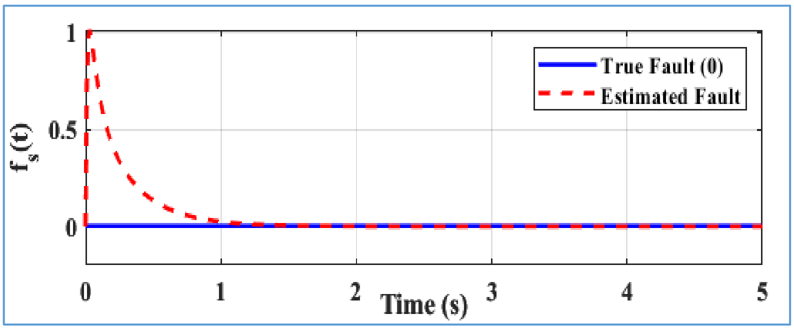

Fig 3(a): Sensor Fault: True vs Estimated

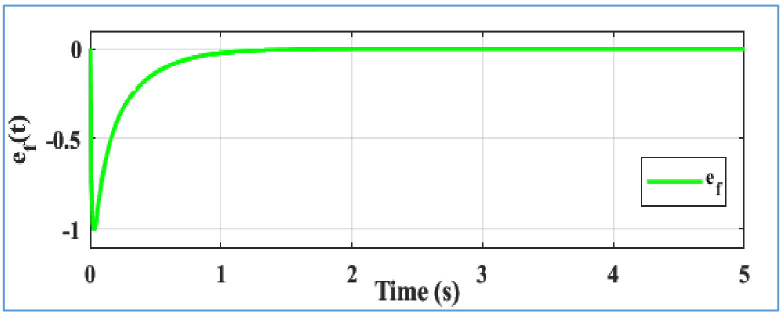

Fig 3(b): Fault estimation error converges to zero

In the faulty scenario ($f_s = 3$, ($u(t) = -K\hat{x}(t) - K_f\hat{y}(t)$), Fig. 4 demonstrates that the observer preserves the estimation accuracy in the face of severe measurement corruption. The transient profile is slightly different due to the dynamics of fault adaptation, but the ultimate convergence remains unaffected.

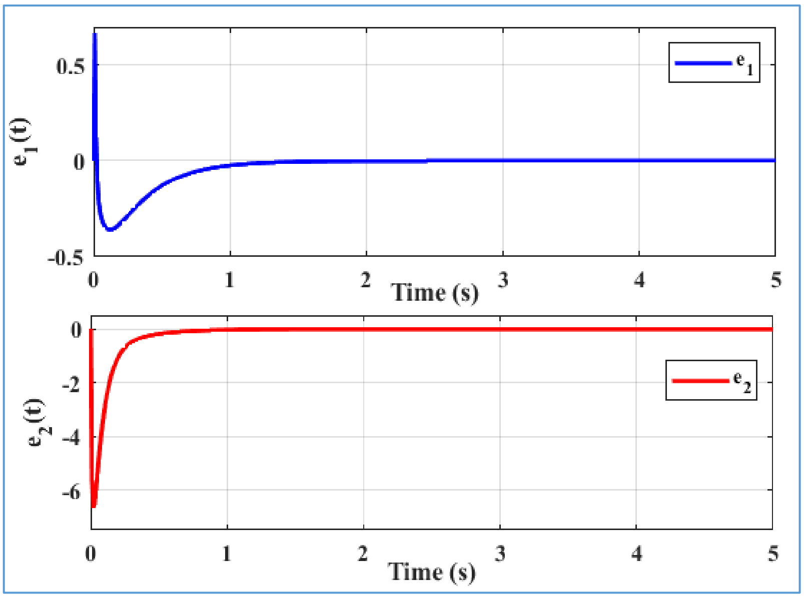

Fig 4: State estimation errors (e_1 & e_2) convergence to zero.

Fig. 5 exhibits the (FTC) law's success in stabilizing the actual states with zero overshoot and zero steady-state error despite the sensor reporting an erroneous offset of 3. Figs. 6 verify that the accurate fault reconstruction is accomplished with the estimated fault gradually changing from a totally mismatched initial guess ($\hat{f}_0 = 0$) to converge to the real value ($f_s = 3$), while the fault estimation error decays asymptotically to zero without chattering.

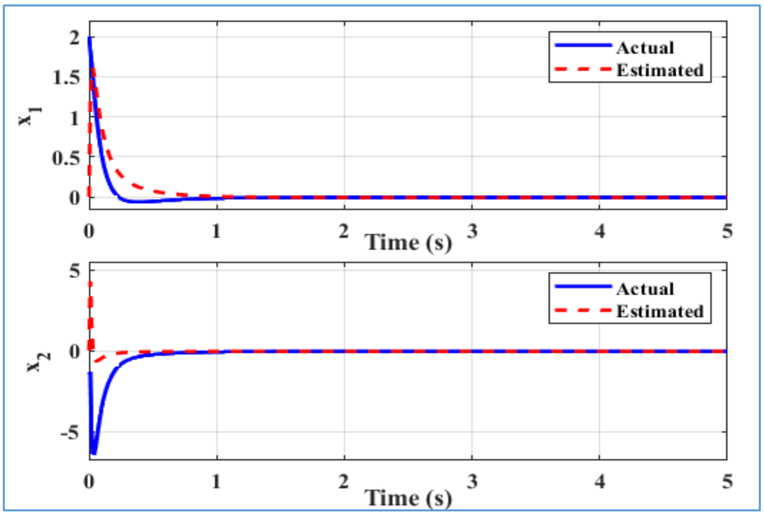

Fig 5: Zero convergence of the actual and estimated states

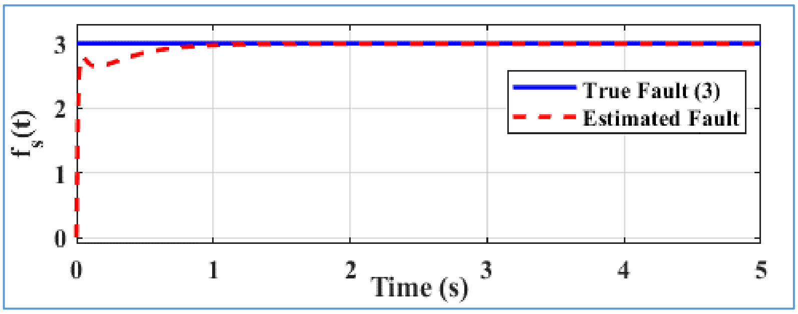

Fig 6(a): Sensor Fault: True vs Estimated

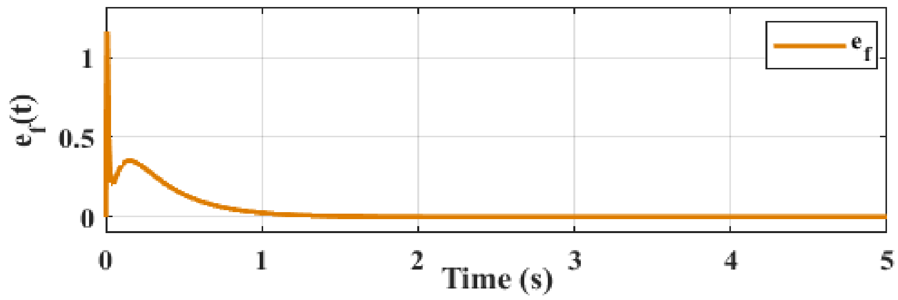

Fig 6(b): Fault estimation error converges to zero

In both scenarios, the comparative analysis verifies that the (ML) stability is preserved and that the adaptive observer can eliminate the steady-state offset through its integral action. This, in turn, verifies an accurate simultaneous state and fault estimation with robust asymptotic stabilization that complies with the theoretical guarantees.

Conclusions

In Caputo (FONS), this research offers a thorough framework for sensor fault estimation and (FTC). The proposed fractional-order adaptive observer can estimate the system states and the sensor defects concurrently and the active control law assures the closed-loop stability based on rigorous Lyapunov-based analysis. The theoretical approach is validated by simulation results which demonstrate accurate fault reconstruction and robust state estimation in the presence of severe measurement corruption and large initial errors. The (FTC) successfully stabilizes the system with ideal transient characteristics and zero steady-state error in healthy and faulty states. The comparison validates the reliability of the architecture, and shows that the proposed system maintains nominal performance under continuous sensor faults. These results provide a substantive contribution to fractional-order control systems by providing a unified framework that is theoretically sound and practically feasible for real-world engineering applications where sensor integrity cannot be guaranteed.

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Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.