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A CLASS OF POWER SERIES COLLOCATION MULTISTEP METHODS WITH LEGENDRE INTERPOLANT FOR THE INTEGRATION OF INITIAL VALUE PROBLEMS

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ABSTRACT

In this paper, we present a new approach for the derivation of a class of collocation linear multistep methods for the integration of some initial value problems of ordinary differential equations. The derivation of the discrete methods of orders p=2, 3, 4, and 5 for the un-economized collocation method for solving Initial Value Problems. By introducing the Legendre polynomial terms, the economized invariants of the methods are derived. The methods are all consistent and zero stable, thus they are convergent. For implementation as predictor-corrector pairs, the class of explicit methods of the Adams Bashforth methods are used as predictors while the new methods, which are implicit, are used as correctors. Python programming language and MATLab were used for the numerical computations. The methods exhibits the A-stability properties of the standard Trapezoidal method and performs better when implemented to solve Stiff initial value problems.

Keywords: Linear Multistep Methods, Legendre polynomials, integration, Collocation methods and Absolute stability

1. INTRODUCTION

A significant number of physical problems result in differential equations. Traditionally, solutions to these differential equations can be obtained using analytical methods. Nonetheless, solutions to certain differential equations are very difficult by any means other than an approximate solution by the application of numerical methods. These methods are thus classified into two: One-step and Multistep methods.

The general first order initial value problem is given as

$$y' = f(x, y) \tag{1}$$

where $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The continuous function f is conventionally solved by reducing it to a system of Ordinary Differential Equations (ODEs) and then applying the various methods available for solving systems of first order Initial Value Problems (IVPs).

Although there has been tremendous success with this approach, it has certain drawbacks. For instance, computer programs related with the methods are often complicated, particularly when incorporating subroutines to provide the methods with the starting values, resulting in longer computational time and more work.

Linear multistep methods are not self-starting hence, they need starting values from single-step methods like Euler's method and Runge-Kutta family of methods. The standard k-step linear multistep methods ($k \geq 2$) seek for single-step methods to start. This has greatly increased the computational costs and time in solving initial value problems (IVPs) of ODEs.

A lot of efforts have been devoted to the development of various methods for solving initial value problems directly without first reducing it to a system of first order Ordinary Differential Equations (ODEs). To mention but few are Heurici (1962), Lambert, (1973, 1976, 1991), Onumanyi (1999), and Simos (2002).

Definition 1: The general k-step numerical method
The general k -step linear multistep methods is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) \tag{2}$$

where α_j and β_j are uniquely determined and $\alpha_0 + \beta_0 \neq 0$, $\alpha_k = 1$.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad \alpha_k \neq 0 \tag{3}$$

The numerical method (2) is said to be consistent with the differential equation (1) if the truncation error defined by

$$T_n = \frac{\sum_{j=0}^{k} \left[\alpha_j y(x_{n+j}) - h\beta_j f(x_{n+j}, y(x_{n+j})) \right]}{h \sum_{j=0}^{k} \beta_j} \tag{4}$$

is such that for any $\varepsilon > 0$ there exists an $h(\varepsilon)$ for which $|T_n| < \varepsilon$ for $0 < h < h(\varepsilon)$, and any $k+1$ points $(x_n, y(x_n)),...,(x_n+k, y(x_n+k))$ on any solution curve in D of the initial value problem (1).

According to this condition, if a linear multistep method is consistent, then it has a simple root on the unit circle at $z = 1$; thus, the Root Condition is not violated by this root.

Definition 2: Order of the General Linear Multistep Methods
The general linear multistep method (1) is said to have order of accuracy p if p is the largest positive integer such that, for any sufficiently smooth solution curve in D of the initial value problem (1) there exist constants K and h_0 such that $|T_n| \leq Kh^p$ for $0<h<h_0$, for any $(k+1)$ points $(x_n, y(x_n)),\ldots, (x_{n+k}, y(x_{n+k}))$ on the solution curve.

From the definition above, the numerical method is of order p if and only if

$c_0 = c_j = \ldots = c_p = 0$ and $c_{p+1} \neq 0$ where (5)

In this case, $$T_n = \frac{C_{p+1}}{\sigma(1)h^p} y_{p+1}(x_n) + 0(h^{p+1})$$ (6)

The number $c_{p+1}/\sigma(1)$ is called the **Error Constant**.

Definition 3: Consistency

Assume that the numerical method is consistent with the differential equation if $c_0 = 0$ and $c_1 = 0$. (Süli & Mayers, 2003). By definition of consistency (see Dahlquist, 1974, Lambert, 1991), a linear multistep method is consistent if it satisfies the condition below:

$$\text{(i)} \quad \sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad (ii) \quad \sum_{\mathrm{j=0}}^{\mathrm{k}} \mathrm{j}\alpha_{\mathrm{j}} = \sum_{j=0}^{k} \beta_j \tag{7}$$

Definition 4: Stability

If a linear multistep method is convergent, then the zeroes of the first characteristic polynomial of the method

$p(r) = \alpha_0 + \ldots + \alpha_k r^k$ or equivalently, $$p(r) = \sum_{j=0}^{k} \alpha_j r^j \tag{8}$$

Theorem 1 (Root condition)

A linear multistep method is zero-stable if and only if the root condition is satisfied. (Süli & Mayers, 2003).

Where, $p(r) = \sum_{j=0}^{k} \alpha_j r^j$ satisfy the Dahlquist root condition: all zeroes r satisfy $|r| \leq 1$, and multiple zeroes r satisfy $|r| \leq 1$,.

2.0 DEVELOPMENT OF THE CONTINUOUS AND DISCRETE METHODS

In this paper two methods were developed considering both the continuous and discrete methods

2.1 Derivation of Method Using the Uneconomized Collocation Methods

Considering the initial value problem for ordinary differential equations (1).

The general s-stage implicit Runge-Kutta method given by

$$y_{n+1} = y_n + hF(t_n, y_n; h)$$

is defined by $y_{n+1} - y_n = hF(t_n, y_n; h)$

Let $y(x)=\sum_{r=0}^{k+1} a_r x^r , \quad k>0$ and the general form of y_n for k is,

$$y_n = a_0 + a_1 x_n + \ldots + a_k x_n^k + a_{k+1} x_n^{k+1} \tag{9}$$

Then,

$$y'(x) = \sum_{r=0}^{k+1} r a_r x^{r-1}$$

Therefore for k=1,

$$\begin{aligned} \bar{y}(x) &= a_0 x^0 + a_1 x^1 + a_2 x^2 \\ &= a_0 + a_1 x + a_2 x^2 \end{aligned} \tag{10}$$

consider $y_n = a_0 + a_1 x_n + a_2 x_n^2$ and the following collocation points where $y'(x) = f(x,y)$, $y(a)=\alpha$

$$\bar{y}'(x_{n+j}) = f_{n+k}; \text{ and } \bar{y}'(x_{n+k-1}) = f_{n+k-1} \tag{11}$$

$$f_n = a_1 + 2a_2 x_n$$
$$f_{n+1} = a_1 + 2a_2 x_{n+1}$$

This results into the general matrix equation

$$\begin{bmatrix} 1 & x_n & x_n^2 & \ldots & x_n^k & x_n^{k+1} \\ 1 & x_{n+1} & x_{n+1}^2 & \ldots & x_{n+1}^k & x_{n+1}^{k+1} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & x_{n+k+1} & x_{n+k+1}^2 & \ldots & x_{n+k+1}^k & x_{n+k-1}^{k+1} \\ 0 & 1 & 2x_{n+k-1} & \ldots & kx_{n+k-1}^{k-1} & (k+1)x_{n+k-1}^k \\ 0 & 1 & 2x_{n+k} & \ldots & kx_{n+k}^{k-1} & (k+1)x_{n+k}^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_{k-1} \\ a_k \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ . \\ . \\ . \\ y_{n+k-1} \\ f_{n+k-1} \\ f_{n+k} \end{bmatrix} \tag{12}$$

For *k=1*, the matrix system (12) becomes

$$\begin{bmatrix} 1 & x_n & x_n^2 \\ 0 & 1 & 2x_n \\ 0 & 1 & 2x_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+1} \end{bmatrix} \tag{13}$$

Solving the system (13) by Gaussian Elimination method, gives the following results for a_i for i=0,1,2

$$a_2 = \frac{1}{2h}(f_{n+1} - f_n),\ a_1 = f_n - 2x_n a_2,\ a_0 = y_n - x_n a_1 - x_n^2 a_2$$

By substituting the values of a_i for i=0,1,2 into equation (10), to have the continuous scheme

$$\bar{y}(x) = y_n + (x - x_n)a_1 + (x^2 - x_n^2)a_2 \tag{14}$$

Collocating on the grid point $x = x_{n+1}$, obtains the Trapezoidal rule

$$\bar{y}(x_{n+1}) = y_n + \frac{h}{2}\left[f_{n+1} + f_n\right] \tag{15}$$

Similarly **For *k*=2, *k*=3 and k=4** we obtain the third, fourth and fifth order methods in the following pairs:

$$\bar{y}(x_{n+2}) = \frac{1}{5}y_n + \frac{4}{5}y_{n+1} + \frac{4h}{5}f_{n+1} + \frac{2h}{5}f_{n+2}$$

$$y_{n+2} = \frac{1}{5}(y_n + 4y_{n+1}) + \frac{h}{5}\left[4f_{n+1} + 2f_{n+2}\right] \tag{16}$$

Similarly for For *k*=3, obtains the discrete method of the fourth order

$$\bar{y}(x_{n+3}) = \frac{6h}{17}f_{n+3} + \frac{18h}{17}f_{n+2} + \frac{9}{17}y_{n+2} + \frac{9}{17}y_{n+1} - \frac{1}{17}y_n$$

$$y_{n+3} = \frac{6h}{17}\left[f_{n+3} + 3f_{n+2}\right] + \frac{1}{17}\left(9y_{n+2} + 9y_{n+1} - y_n\right) \tag{17}$$

Similarly For *k*=4, obtains the fifth order method

$$\bar{y}(x_{n+4}) = \frac{12h}{37}f_{n+4} + \frac{48h}{37}f_{n+3} + \frac{8}{37}y_{n+3} + \frac{36}{37}y_{n+2} - \frac{8}{37}y_{n+1} + \frac{1}{37}y_n$$

$$y_{n+4} = \frac{12h}{37}\left[f_{n+4} + 4f_{n+3}\right] + \frac{1}{37}\left(8y_{n+3} + 36y_{n+2} - 8y_{n+1} + y_n\right) \tag{18}$$

2.2 The Derivation of the Economized Collocation Methods

Consider the initial value problem (1) given by

$$y'(x) = f(x, y)$$
$$y(0) = y_0; x_0 \le x \le b$$

The exact solution of the perturbed form of (1) is given by

$$y_k(x) = \sum_{j=0}^{k} a_j Q_j(x); \quad x \in [x_n, x_{n+k}],\ k > 0 \tag{19a}$$

where $Q_j(x) = x^j; j \ge 0$ is the power series. (19b)

From (1) and (19), and introducing the economized term τ we have

$$\sum_{j=0}^{k} a_j Q'_j(x) = f(x,y) + \tau P_K(x) \tag{20}$$

where $P_k(x)$ Legendre polynomial of degree *k* and valid in $x_n \le x \le x_{n+k}$ and τ is a parameter to be determined.

The Legendre Polynomial denoted by $P_k(x)$ of degree *k*,

$$P_k(x) = \sum_{j=0}^{k} C_j^{(k)} x^j = C_0 + C_1^{(k)} x + \ldots + C_k^{(k)} x^k \quad valid\ in\ x_n \le x \le x_{n+k} \tag{21}$$

This is obtained from the recurrence relation

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0 \tag{22}$$

Thus, using (22) to obtain $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ in the interval [-1,1].

Using the transformation equation, we obtain, $\bar{x} = \dfrac{(b-a)}{2}x + \dfrac{b+a}{2}$ and $x = \dfrac{2\bar{x} - (b-a)}{b-a}$

Consider the case when k is odd (i.e. *k*=1,3, and 5)

Case 1: For k=1

$$y(x) = a_0 + a_1 x$$

Using equation (22) for $P_i(x)$

$$P_1(x) = x$$

$$P_1(x_n) = \frac{2\bar{x} - (b+a)}{b-a} = \frac{2x_n - (x_{n+1} + x_n)}{x_{n+1} - x_n} = -1$$

$$P_1(x_{n+1}) = \frac{2\bar{x} - (b+a)}{b-a} = \frac{2x_{n+1} - (x_{n+1} + x_n)}{x_{n+1} - x_n} = 1$$

From equation (19b),

$$\begin{aligned} Q_0(x) = 1 &\Rightarrow Q'_0(x) = 0 \\ Q_1(x) = x &\Rightarrow Q'_1(x) = 1 \end{aligned} \tag{23}$$

Putting equation (23) into equation (20),

$$a_1 = f(x,y) + \tau P_1(x) \tag{24}$$

Collocating (24) at $x_{n+j};\ j=0(1)$ and interpolate at $x=x_n$, obtain a system of three equations with $a_j,\ (j=0(1))$ and parameter τ

$$y_n = a_0 + a_1 x_n$$

$$f(x,y) = a_1 - \tau P_1(x)$$
$$f_n = a_1 + \tau$$
$$f_{n+1} = a_1 - \tau$$

which can be presented in the matrix system below

$$\begin{bmatrix} 1 & x_n & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \tau \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+1} \end{bmatrix}$$

Solving for the coefficients $a'_j s$ and τ and by directed substitution of $a_j(j=0,1)$ in (19a, 22) obtain the continuous scheme

$$\tau = \frac{1}{2}\left[f_{n+1} - f_n\right]$$
$$a_1 = f_n - \tau$$
$$a_0 = y_n - a_1 x_n$$

$$\bar{y}(x) = a_0 + a_1 x = y_n + (x - x_n)a_1 \tag{25}$$

Collocating (24) at the grid point $x = x_{n+1}$, obtain the newly proposed discrete method

$$y_{n+1} = y_n + \frac{h}{2}\left[3f_n - f_{n+1}\right] \tag{26}$$

Similarly for Case 2: For *k*=3 using the Legendre Polynomial, we obtain the values of other functions as thus

$P_3(x_{n+1}) = \frac{11}{27}$, $P_3(x_{n+2}) = -\frac{11}{27}$ and $P_3(x_{n+3}) = 1$

By collocating at the grid point, $x = x_{n+3}$, we obtain the newly proposed discrete method as

$$y_{n+3} = y_{n+2} + \frac{h}{40}\left[f_n - \frac{19}{3}f_{n+1} + \frac{89}{3}f_{n+2} + \frac{47}{120}f_{n+3}\right]$$

$$y_{n+3} = y_{n+2} + \frac{h}{120}\left[3f_n - 19f_{n+1} + 89f_{n+2} + 47f_{n+3}\right] \tag{27}$$

Similarly for Case 3: For k=5 using the Legendre Polynomial, we obtain the values of other colocation points:

$$P_5(x_{n+1}) = \frac{474}{3125}, \quad P_5(x_{n+2}) = -\frac{961}{3125}, \quad P_5(x_{n+3}) = \frac{961}{3125}, \quad P_5(x_{n+4}) = -\frac{477}{3125} \text{ and } P_5(x_{n+5}) = 1$$

By collocating at the grid point $x = x_{n+5}$, we obtain the newly proposed discrete method as

$$y_{n+5} = y_{n+4} + \frac{h}{90720}\left[31161 f_{n+5} + 83721 f_{n+4} - 37914 f_{n+3} + 18006 f_{n+2} - 4719 f_{n+1} + 465 f_n\right]$$

$$y_{n+i} = y_{n+i-1} + \frac{h}{90720}\left[c_1 k_5 + c_2 k_4 - c_3 k_3 + c_4 k_2 - c_5 k_1 + c_6 k_0\right] \tag{28}$$

3.0 ANALYSIS OF THE METHOD

3.1 Order and Error Analysis of the Methods

Using the definition of the order of the general linear multistep method (**Definition 5**), the order and error constant for the method at k=1 (15) Since $C_3 \neq 0$, then the method is of order two and the error constant is $-\frac{1}{12}$.

Following the same procedure, we obtain the order and error constants for the other methods.:

3.2 Convergence Analysis of the Uneconomized Collocation Methods

From definition of convergent, the necessary and sufficient conditions for a linear Multistep method to be convergent are that it be consistent and zero-stable as seen in the definition above. Hence, test for the consistency and zero-stability of the methods we have the following.

For k=1 (15) since the method is both consistent and zero stable, it is convergent, k=2 (16) since the method is both consistent and zero stable, it is convergent, k=3 (17) since the method is both consistent and zero stable, it is convergent, k=4 (18) since the method is both consistent and zero stable, it is convergent

3.3 Order and Error Analysis of the Economized Collocation Methods

Consider the case when k equals an odd number (i.e. k=1,3, and 5) For k=1 (26) Since $C_2 \neq 0$, then the discrete scheme is of order one (1), and the error constant is 1, k=3 (27) Since $C_4 \neq 0$, then the discrete scheme is of order 3 and the error constant is $-\frac{1}{60}$.For k=5 (28) Since $C_6 \neq 0$, then the discrete scheme is of order five (5), and the error constant is $-\frac{1}{73}$.

3.4 Convergence Analysis of the Economized Collocation Methods

For the economized method to be convergent, it must be consistent and zero-stable as seen in the definition 8. By testing the consistency and zero-stability of the methods.

For k=1 (26) of the economized method is both consistent and zero-stable therefore it converges

For k=3 (27) of the economized method is both consistent and zero-stable therefore it converges

For k=5 (28) of the economized method is both consistent and zero-stable therefore it converges

Table 1: Analysis for the Uneconomized Methods

K	Order	Error Constant
1	2	-1/12
2	3	-1/30
3	4	-3/170
4	5	-2/185

Table 2: Analysis for the Economized Methods

K	Order	Error Constant
1	1	-1/24
3	3	-1/60
5	5	-1/73

4.0 NUMERICAL EXAMPLES

In attempting to implement the methods for the integration of initial value problems, we need to use the predictor-corrector approach and hence we need explicit multistep methods for each newly derived method. Since equations (27) is of order three and (28) is of order five, hence some equations of equivalent orders are used respectively as the predictor to solve the following initial value problems:

Problem 1 (Non-Stiff)

Solve the initial value problem

$$f(x,y) = -\frac{y}{2+2x} \tag{29}$$

with initial values $y(x_0) = 1; x_o = 0$ and the exact solution is given as $y(x) = (1+x)^{-0.5}$

See numerical results using equations (27) and (28) in Table 1 and Table 2 respectively.

Problem 2 (Stiff)

Solve the initial value problem

$$f(x,y) = -100xy \tag{30}$$

with initial values $y(x_0) = 1; x_o = 0$ and the exact solution is given as $y(x) = e^{-50x^2}$

See numerical results using equations (27) and (28) in Table 3 and Table 4 respectively.

For the Python Program see Appendix I,

Table 3: Error for Problem 1 using equations (27)

h = 0.04		h = 0.02				h = 0.01							
n	Error	N	Error	n	Error	n	Error	n	Error		Error	N	Error
0	0.00000000e+00	0	0.00000000e+00	26	7.79222483e-03	0	0.00000000e+00	26	2.57113201e-03	51	4.06929638e-03	76	4.82134760e-03
1	3.58947205e-10	1	1.18361987e-11	27	7.95593043e-03	1	3.80140364e-13	27	2.64630540e-03	52	4.10830342e-03	77	4.84352612e-03
2	6.42725984e-10	2	2.23479013e-11	28	8.11220690e-03	2	7.38187289e-13	28	2.71957528e-03	53	4.14641808e-03	78	4.86521796e-03
3	2.08033382e-03	3	5.57661139e-04	29	8.26143000e-03	3	1.44546071e-04	29	2.79099880e-03	54	4.18366306e-03	79	4.88643408e-03
4	3.98131592e-03	4	1.08983689e-03	30	8.40395268e-03	4	2.85685764e-04	30	2.86063103e-03	55	4.22006038e-03	80	4.90718513e-03
5	5.69670712e-03	5	1.59425643e-03	31	8.54010669e-03	5	4.23012082e-04	31	2.92852502e-03	56	4.25563137e-03	81	4.92748150e-03
6	7.23426363e-03	6	2.07067465e-03	32	8.67020409e-03	6	5.56380754e-04	32	2.99473191e-03	57	4.29039673e-03	82	4.94733330e-03
7	8.61552820e-03	7	2.52095161e-03	33	8.79453863e-03	7	6.85930058e-04	33	3.05930102e-03	58	4.32437652e-03	83	4.96675036e-03
8	9.85925132e-03	8	2.94682356e-03	34	8.91338700e-03	8	8.11794238e-04	34	3.12227988e-03	59	4.35759021e-03	84	4.98574227e-03
9	1.09815790e-02	9	3.34989348e-03	35	9.02701004e-03	9	9.34102239e-04	35	3.18371436e-03	60	4.39005668e-03	85	5.00431835e-03
10	1.19963858e-02	10	3.73163614e-03	36	9.13565376e-03	10	1.05297746e-03	36	3.24364871e-03	61	4.42179426e-03	86	5.02248770e-03
11	1.29156454e-02	11	4.09341007e-03	37	9.23955037e-03	11	1.16853803e-03	37	3.30212563e-03	62	4.45282074e-03	87	5.04025917e-03
12	1.37497335e-02	12	4.43646840e-03	38	9.33891916e-03	12	1.28089708e-03	38	3.35918636e-03	63	4.48315337e-03	88	5.05764137e-03
13	1.45076778e-02	13	4.76196852e-03	39	9.43396738e-03	13	1.39016298e-03	39	3.41487071e-03	64	4.51280892e-03	89	5.07464273e-03
14	1.51973623e-02	14	5.07098076e-03	40	9.52489100e-03	14	1.49643962e-03	40	3.46921711e-03	65	4.54180366e-03	90	5.09127141e-03
15	1.58256967e-02	15	5.36449611e-03	41	9.61187543e-03	15	1.59982657e-03	41	3.52226272e-03	66	4.57015339e-03	91	5.10753541e-03
16	1.63987571e-02	16	5.64343323e-03	42	9.69509619e-03	16	1.70041934e-03	42	3.57404341e-03	67	4.59787348e-03	92	5.12344250e-03
17	1.69219035e-02	17	5.90864464e-03	43	9.77471954e-03	17	1.79830957e-03	43	3.62459387e-03	68	4.62497883e-03	93	5.13900027e-03
18	1.73998785e-02	18	6.16092239e-03	44	9.85090305e-03	18	1.89358519e-03	44	3.67394764e-03	69	4.65148394e-03	94	5.15421610e-03
19	1.78368899e-02	19	6.40100313e-03	45	9.92379613e-03	19	1.98633065e-03	45	3.72213713e-03	70	4.67740288e-03	95	5.16909722e-03
20	1.82366817e-02	20	6.62957267e-03	46	9.99354052e-03	20	2.07662701e-03	46	3.76919370e-03	71	4.70274935e-03	96	5.18365065e-03
21	1.86025934e-02	21	6.84727015e-03	47	1.00602708e-02	21	2.16455220e-03	47	3.81514767e-03	72	4.72753665e-03	97	5.19788325e-03
22	1.89376110e-02	22	7.05469182e-03	48	1.01241147e-02	22	2.25018106e-03	48	3.86002841e-03	73	4.75177772e-03	98	5.21180170e-03
23	1.92444107e-02	23	7.25239436e-03	49	1.01851936e-02	23	2.33358558e-03	49	3.90386431e-03	74	4.77548514e-03	99	
24	1.95253964e-02	24	7.44089809e-03	50	1.02436230e-02	24	2.41483495e-03	50	3.94668285e-03	75	4.79867113e-03	100	
25	1.97827319e-02	25	7.62068970e-03			25	2.49399574e-03		3.98851067e-03				
									4.02937352e-03				

Table 4: Error for Problem 1 using equations (28)

h = 0.04		h = 0.02				h = 0.01							
n	Error	n	Error	N	Error	N	Error	n	Error		Error	n	Error
0	0.00000000e+00	0	0.00000000e+00	26	4.89388015e-02	0	0.00000000e+00	26	3.02055272e-02	51	1.02640137e-05	76	6.03752158e-07
1	1.67971997e-06	1	6.64008859e-09	27	1.40853092e-02	1	2.60156341e-11	27	2.57905105e-02	52	8.95312423e-06	77	1.18686391e-06
2	9.61186878e-06	2	5.87766091e-08	28	4.09872641e-02	2	2.52553423e-10	28	2.12088453e-02	53	5.09355219e-06	78	1.08641698e-06
3	1.60902833e-01	3	6.79581082e-02	29	8.08093630e-02	3	1.92085594e-02	29	1.67999241e-02	54	5.41952461e-07	79	3.53405919e-07
4	1.83852511e-01	4	1.21002149e-01	30	6.85022847e-02	4	3.73440532e-02	30	1.28150060e-02	55	2.90566898e-06	80	6.00167004e-07
5	4.75249632e-02	5	1.49264538e-01	31	6.50586833e-03	5	5.35227743e-02	31	9.41156432e-03	56	4.20999483e-06	81	1.22202491e-06
6	1.95125220e-01	6	1.46386291e-01	32	1.08008647e-01	6	6.68792633e-02	32	6.65826959e-03	57	3.35581035e-06	82	1.13033274e-06
7	4.09517265e-01	7	1.13280206e-01	33	1.59806079e-01	7	7.67790210e-02	33	4.54845410e-03	58	1.17605220e-06	83	3.45077485e-07
8	4.29744731e-01	8	5.73842513e-02	34	8.96696719e-02	8	8.27907459e-02	34	3.01913116e-03	59	1.12429717e-06	84	6.97701469e-07
9	1.38458944e-01	9	8.70071563e-03	35	1.04405415e-01	9	8.47167326e-02	35	1.97233004e-03	60	2.51453206e-06	85	1.37489725e-06
10	4.16771973e-01	10	6.99895320e-02	36	3.09631611e-01	10	8.26051361e-02	36	1.29567275e-03	61	2.53162436e-06	86	1.23667834e-06
11	9.20949518e-01	11	1.12667399e-01	37	3.30656244e-01	11	7.67445646e-02	37	8.79693436e-04	62	1.38570228e-06	87	2.97629521e-07
12	8.18182196e-01	12	1.27630670e-01	38	2.30056932e-02	12	6.76404510e-02	38	6.30264423e-04	63	2.28106959e-07	88	9.24006729e-07
13	3.80598904e-01	13	1.13068473e-01	39	5.25525409e-01	13	5.59750845e-02	39	4.75483603e-04	64	1.49898940e-06	89	1.67329077e-06
14	2.44906040e+00	14	7.52401003e-02	40	9.08476303e-01	14	4.25546626e-02	40	3.67325980e-04	65	1.87589908e-06	90	1.40718682e-06
15	3.72713526e+00	15	2.69566521e-02	41	5.75850828e-01	15	2.82479416e-02	41	2.79118151e-04	66	1.28783237e-06	91	1.70469550e-07
16	1.17639380e+00	16	1.61236548e-02	42	6.88989020e-01	16	1.39218309e-02	42	2.00353786e-04	67	1.27064931e-07	92	1.34919596e-06
17	7.47279551e+00	17	4.05343877e-02	43	2.23587171e+00	17	3.79534011e-04	43	1.30488774e-04	68	9.77672427e-07	93	2.17329983e-06
18	1.87158703e+01	18	3.98476819e-02	44	2.49690359e+00	18	1.16934637e-02	44	7.31605501e-05	69	1.48006947e-06	94	1.63302753e-06
19	1.76289263e+01	19	1.74862082e-02	45	5.52439050e-02	19	2.17710875e-02	45	3.18483335e-05	70	1.17432332e-06	95	1.29745586e-07
20	2.00886669e+01	20	1.40221575e-02	46	5.03856131e+00	20	2.95128466e-02	46	7.44755849e-06	71	2.74568058e-07	96	2.11084407e-06
21	1.03903749e+02	21	3.77868243e-02	47	8.65996493e+00	21	3.47728743e-02	47	2.29916383e-06	72	7.14492858e-07	97	2.96500428e-06
22	1.72726756e+02	22	4.00667332e-02	48	4.81364122e+00	22	3.75919476e-02	48	1.98059880e-06	73	1.26592734e-06	98	1.86665666e-06
23	3.58459214e+01	23	1.76330578e-02	49	9.67127811e+00	23	3.81742737e-02	49	3.01311635e-06	74	1.10221290e-06	99	8.03884956e-07
24	5.73965495e+02	24	1.87980360e-02	50	2.75138326e+01	24	3.68522750e-02	50	8.04103637e-06	75	3.34033507e-07	100	3.46215247e-06
25	1.61620180e+03	25	4.80336761e-02			25	3.40436499e-02						

Table 5: Error for Problem 2 using equations (28)

h = 0.04		h = 0.02				h = 0.01							
n	Error	n	Error	N	Error	N	Error	n	Error		Error	n	Error
0	0.00000000e+00	0	0.00000000e+00	26	3.37307945e-07	0	0.00000000e+00	26	3.41278122e-02	51	2.26093763e-06	76	3.34174257e-13
1	1.67971997e-06	1	6.64008859e-09	27	1.21208012e-06	1	2.60156341e-11	27	2.61836738e-02	52	1.35142738e-06	77	1.61188289e-13
2	9.61186878e-06	2	5.87766091e-08	28	3.45150600e-06	2	2.52553423e-10	28	1.98889106e-02	53	7.99792131e-07	78	7.76917458e-14
3	4.86752256e-01	3	8.35270211e-01	29	6.26524518e-07	3	9.55997482e-01	29	1.49571832e-02	54	4.68646918e-07	79	3.75035097e-14
4	2.78037300e-01	4	7.26149037e-01	30	4.83709710e-06	4	9.23116346e-01	30	1.11364602e-02	55	2.71896348e-07	80	1.81781079e-14
5	4.17885017e-04	5	6.04368608e-01	31	1.57235834e-06	5	8.84875504e-01	31	8.20924497e-03	56	1.56190883e-07	81	8.87245197e-15
6	7.80867694e-02	6	4.91078297e-01	32	6.82451210e-06	6	8.37171171e-01	32	5.99125935e-03	57	8.88402220e-08	82	4.37360965e-15
7	1.20914195e-01	7	3.76408594e-01	33	5.41178181e-06	7	7.84436623e-01	33	4.32904469e-03	58	5.00349760e-08	83	2.18353531e-15
8	9.97479496e-02	8	2.78716692e-01	34	9.44223277e-06	8	7.27755009e-01	34	3.09688640e-03	59	2.79034699e-08	84	1.10671747e-15
9	1.46775654e-01	9	1.98707138e-01	35	1.28433428e-05	9	6.68451915e-01	35	2.19340066e-03	60	1.54090512e-08	85	5.70408120e-16
10	2.28293221e-01	10	1.35730193e-01	36	1.34233355e-05	10	6.07873106e-01	36	1.53804988e-03	61	8.42640705e-09	86	2.99169738e-16
11	3.44747420e-01	11	8.91928916e-02	37	2.80418227e-05	11	5.47284134e-01	37	1.06778282e-03	62	4.56328151e-09	87	1.59630145e-16
12	4.84390292e-01	12	5.63598730e-02	38	2.05831204e-05	12	4.87831733e-01	38	7.33932011e-04	63	2.44739782e-09	88	8.65529169e-17
13	1.29330594e+00	13	3.41642737e-02	39	6.10135086e-05	13	4.30511289e-01	39	4.99447120e-04	64	1.30003368e-09	89	4.76047524e-17
14	2.31995054e-01	14	1.99125588e-02	40	3.68478761e-05	14	3.76145943e-01	40	3.36499495e-04	65	6.84014534e-10	90	2.65038736e-17
15	4.59223983e+00	15	1.11653016e-02	41	1.36213341e-04	15	3.25376064e-01	41	2.24461078e-04	66	3.56520587e-10	91	1.49046618e-17
16	7.42950037e+00	16	6.00273016e-03	42	8.33426208e-05	16	2.78658465e-01	42	1.48238146e-04	67	1.84108153e-10	92	8.44917749e-18
17	5.62988423e-01	17	3.10203829e-03	43	3.13487670e-04	17	2.36274224e-01	43	9.69263525e-05	68	9.42122748e-11	93	4.81977052e-18
18	2.39262726e+01	18	1.54722974e-03	44	2.38214837e-04	18	1.98343495e-01	44	6.27462102e-05	69	4.77843889e-11	94	2.76269471e-18
19	7.77100814e+01	19	7.37523841e-04	45	7.31206302e-04	19	1.64845517e-01	45	4.02158965e-05	70	2.40288991e-11	95	1.58943453e-18
20	2.22505339e+02	20	3.35997365e-04	46	7.89248720e-04	20	1.35641923e-01	46	2.55195886e-05	71	1.19842607e-11	96	9.17030841e-19
21	8.14766852e+02	21	1.51343790e-04	47	1.65012696e-03	21	1.10501561e-01	47	1.60330612e-05	72	5.93093788e-12	97	5.30255958e-19
22	3.93237976e+03	22	6.47779690e-05	48	2.75448445e-03	22	8.91252230e-02	48	9.97303893e-06	73	2.91429105e-12	98	3.07154279e-19
23	2.30356415e+04	23	2.36177411e-05	49	3.21291639e-03	23	7.11689669e-02	49	6.14198177e-06	74	1.42290278e-12	99	1.78183713e-19
24	1.56929747e+05	24	1.06034119e-05	50	9.42729996e-03	24	5.62650338e-02	50	3.74508293e-06	75	6.90997228e-13	100	1.03499265e-19
25	1.22226161e+06	25	6.26375872e-06			25	4.40397132e-02						

5 GRAPHS OF RESULTS

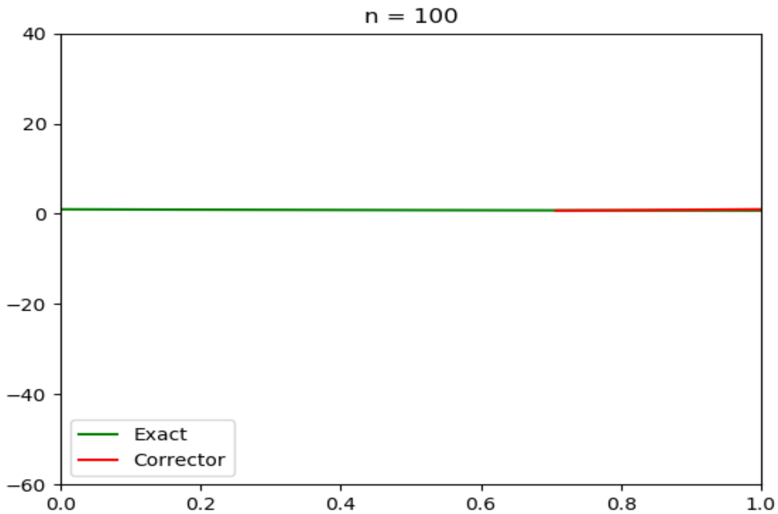

Fig 1: Graph of the Error y(x) for Problem 1 using equations (27)

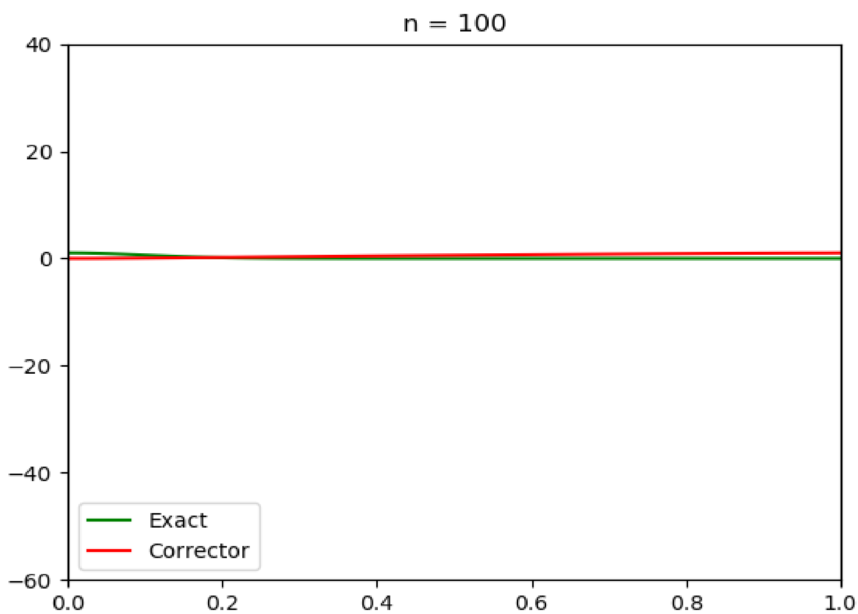

Fig 2: Graph of the Error y(x) for Problem 2 using equations (27)

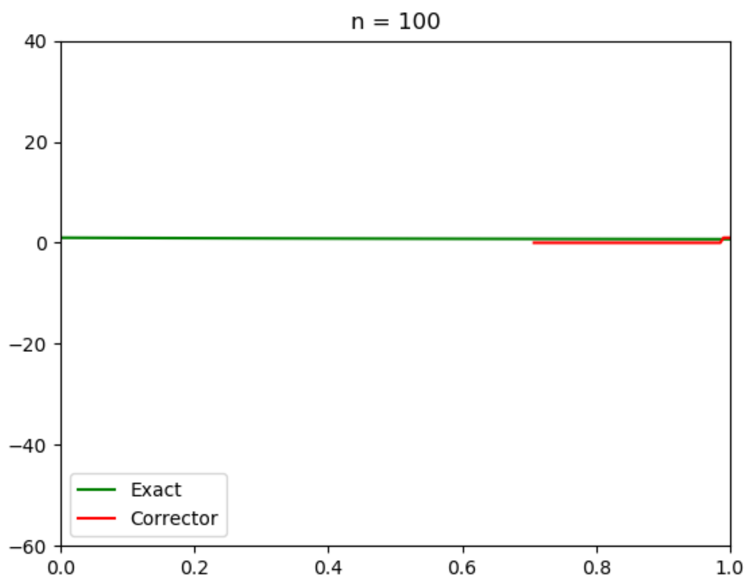

Fig 3: Graph of the Error y(x) for Problem 1 using equations (28)

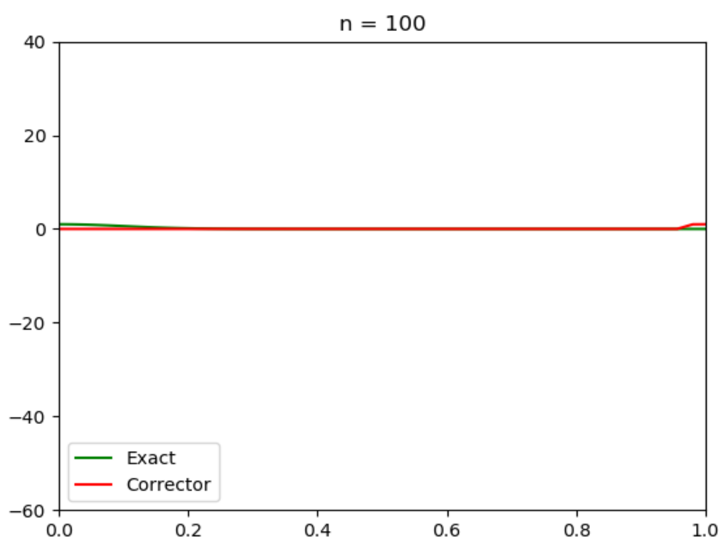

Fig 4: Graph of the Error y(x) for Problem 2 using equations (28)

6.0 DISCUSSION OF THE RESULTS

This research attempt the possibility of developing multistep methods for initial value problems, which has been demonstrated through power series approximation and r-order of power series continuous form of Implicit Runge-Kutta collocation method.

Table 1 explains the analysis of the uneconomized methods including the order and the error constant while table 2 gives a defined analysis of the economize method in consideration with the order and error constant. Table 3 shows the absolute errors for different values of h using equation 27 to solve problem 1. Table 4 explains the error values for problem 1 at different values of h using equation 28. Table 5 is the error value for problem 2 at different values of h using equation 28. Figure 1 explains that the error value y(x) for problem 1 using equation 27 by exact and corrector is the same with a little difference at 0.8 and above, whose problem 1 is use with equation 22. Figure 2 the graph shows that the error value at exact and corrector is not the same with a little as it is used for problem 2 with equation 27. Figure 3 equation 28 is used to explain the same value y exact and correct but a little bit different at 0.7 while figure 4 was to explain the graph of problem 2 using equation 28 as show the exact and corrector with an almost the same outcome. The figures and tables above gives a clear picture of a method that is accurate and convergence with less error.

7.0 CONCLUSION

In conclusion, the first method (15) at collocating on the grid point yields the standard Trapezoidal rule. This therefore is an indication that the proposed methods discussed in this research can be recommended for solving stiff initial value problems leveraging on the properties of the Trapezoidal method with guaranteed accuracy and A-stability properties.
Also, it was discovered that the error constant is less than that of the Adams method of the same order thus claiming better accuracy. In addition, it has been established that evaluating the continuous scheme at all the grid points yields a block of different methods of the same order. Thus making the method self-starting.

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