Anchor University Journal of Science and Technology (AUJST)
A publication of the Faculty of Science Applied and Health Science, Anchor University Lagos

URL: journal.aul.edu.ng

In AJOL: https://www.ajol.info/index.php/aujst

Vol. 5 No 2, December 2024, Pp. 278-289

ISSN: 2736-0059 (Print); 2736-0067 (Online)

Development of A Generalised E-Calculator for The Mathematical Analysis of the General Linear Multistep Methods

[1]Fatokun J. Oladele*, [1]Adaramola D. Ibukun, [1]Okoro S. Ifeanyi
[1]Department of Mathematics, Anchor University Lagos, Nigeria
***Corresponding author Email**:
jfatokun@aul.edu.ng

Submitted : 29 October 2024

Accepted: 30 December, 2024

Competing Interests.

The authors declare no competing interests.

ABSTRACT

In this paper, we appreciate the use of a simple and powerful programming language to establish a computational mathematical tool for the analysis of convergence, stability and order of accuracy of the general Linear Multistep Methods

$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}$ where α_j and β_j are coefficients of y_{n+j} and f_{n+j}

respectively.

This replaces the more rigorous pencil and paper calculation. The code was applied to several Adams-like Multistep methods and the results are consistent with the known computations in literature. Python language was used for the coding and the final product is in the class of robust online calculator/app with capacity of carrying out the mathematical analysis of the k-step (k=1,2,...,N, for N) linear multistep methods for both explicit or implicit types.

Keywords: Linear Multistep Methods, e-calculator, convergence, zero-stability, order of accuracy.

1. Introduction

Differential equation is one of the significant topics in the field of Mathematics especially in dealing with science (engineering) issues. Other diverse fields differential equation arises are Economics, Medicine, Psychology, Operation research and even in areas such as Biology and Anthropology. Considering the vast usefulness of differential equations, methods are being proposed to find the solution of differential equations, with the availability of various solutions. Ordinary differential equations can be solved by analytical and numerical methods.

To anticipate the difficulty of more complicated calculations posed by the analytical resolution, numerical method is used. With the development of science and technology especially in the field of computers, the resolution of numerical differential equations began to take place - see Archibald, Fraser & Grattan-Guinness, 2004.. Numerical method is the key computer aspect that is used to understand differential equations. This can be done by using the available powerful software packages manipulated to solve a variety of mathematical problems using different mathematical operations.

Computer programs like FORTRAN, MATLAB and Python among others can be used to come up with algorithms and afterwards use them to find the numerical approximation to the desired solution of the differential equation.

Numerical methods rely heavily on computers to perform the numerical computations because they are capable of performing analytic formulas. The different numerical methods will be combined so as to achieve the best interpretation of the solution analytically and numerically. Essentially, the goal of any numerical method is to approximate a solution to the problem so as close to the actual solution as possible. Thus, the order of accuracy of a numerical method is considered of great importance in the analysis of the basic properties of the method.

According to (Mahooti, 2018) LMM is used for the numerical solution of ordinary differential equations.

https://dx.doi.org/10.4314/aujst.v5i2.12

Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution. According to G. Dahlquist, the two results below were proposed and established, and represent an important bound for the order of convergence and for the A-stability of a linear multi-step method. First Dahlquist barrier theorem states that A zero-stable and linear q-step multi-step method cannot attain an order of convergence greater than q + 1 if q is odd and greater than q + 2 if q is even. If the method is also explicit, then it cannot attain an order greater than q (Hairer, Nørsett & Wanner 1993, Thm III.3.5). Second Dahlquist barrier theorem states that There are no explicit A-stable and linear multistep methods. The implicit ones have order of convergence at most 2. The trapezoidal rule has the smallest error constant amongst the A-stable linear multistep methods of order 2. There are a lot of applications in the generalized linear multi-step method;

In NASA, Error behavior of multi-step methods applied to unstable differential systems is a common application. "It is shown that the global error in a multi-step numerical method is the solution to a difference equation initial value problem, and the approximate solution is given for several popular multi-step integration formulas. Inspection of the solution leads to the formulation of four criteria for integrator appropriate to unstable problems (Brown, 1977)."

There is the Analysis of Time Filters in Multi-step Methods. "Time filters are modular and parallel. Their effect on stability of the overall process has been tested in numerous simulations, but never analyzed. Stability is proven herein for the Crank-Nicolson Leapfrog (CNLF) method with the Robert-Asselin (RA) time filter and for the Crank-Nicolson Leapfrog method with the Robert-Asselin-Williams (RAW) time filter for systems by energy methods. We derive an equivalent multistep method for CNLF+RA and CNLF+RAW and stability regions are obtained (Nicholas, 2017).Traditional methods, such as a moment method, weighted centroid calculation method is simple but has a big error, especially in the condition of a low SNR. Gaussian method has a high positioning accuracy, but the computational complexity. Analysis of the energy distribution in star image, a location method for star target centroids based on multi-step minimum energy difference is proposed (Wang, Han, Sun, 2013)."

1.1. THE GENERALIZED LINEAR MULTI-STEP METHOD

Single-step methods (such as Euler's method) refer to only one previous point and its derivative to determine the current value. Methods such as Runge–Kutta take some intermediate steps (for example, a half-step) to obtain a higher order method, but then discard all previous information before taking a second step. Euler method and Heun's method or modified Euler's method are other types of the one-step methods.

Multi-step methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, multi-step methods refer to several previous points and derivative values. In the case of linear multi-step methods, a linear combination of the previous points and derivative values is used. Here, integration of the normalized two-body problem from t0 = 0 to t = 86400(s) for an eccentricity of e = 0.1 is implemented. The types of this method are Adams-Moulton, Adams-Bashfort and Backward differentiation formulas (BDF). Analysis of Linear Multi-Step Methods must fulfil the following criteria's; Consistency and order and Stability and convergence.

Numerical Methods (single-step and multi-steps) are for solving First Order Ordinary Differential Equations.

In a study by Fatokun (2006), a FORTRAN code was used to implement the JOFAT method following the spirit of Adams Predictor-corrector approach. The multi-step methods which use data from multiple prior data points for explicit methods or include the current target in implicit methods. The two can be used to advantage as a predictor-corrector pair. Treating the independent and dependent variables as vector quantities allows systems of differential equations to be approached using these same methods. Higher order differential equations can also be recast as systems of first-order equations. Shooting methods provide a good approach to (two-point) boundary value problems. The

https://dx.doi.org/10.4314/aujst.v5i2.12

The second initial condition (typically the slope) is an unknown and we solve for that unknown to ensure the final point is on target. In research from Ahamad and Charan (2019, p. 237). Presently, there are a lot of stand-alone programs for the Linear Multi-step Methods (LMM) to model the efficiency and accuracy of the Ordinary Differential Equations (ODE) problems. There is a need to develop a generalized computer code that achieves the goal of numerical methods and would be versatile for various linear multi-step methods; highly efficient in terms of accuracy; and accessible to all users.

2. METHODOLOGY

2.1. Theoretical Framework

The basis for this research work centers on the following fundamental definitions:

2.1.1 Linear Multi-step Methods (LMM)

Multi-step methods make use of information about the solution and its derivative at more than one point in order to extrapolate to the next point (University of Connecticut, 2017). One specific class of multi-step methods is based on the principle of numerical integration. The General Linear Multi-step Methods is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

where; α_j and β_j are constant coefficients of y_{n+j} and f_{n+j} respectively.

If;

$B_k = 0$, LMM is explicit

$B_k \neq 0$, LMM is implicit

2.1.2 Convergence of Multi-step Methods

The Dahlquist Theorem states that a linear Multi-step method is said to be convergent if and only if it is consistent and zero stable"

(i) Consistency

A LMM is said to be consistent if it satisfies the following two conditions

$$\sum_{j=0}^{k} \alpha_j = 0 \quad and \quad \sum_{j=0}^{k} \left[j\alpha_j - \beta_j \right] = 0$$

(ii) **Zero Stability**

A LMM is said to be zero stable if the absolute root of its characteristic polynomial is less than or equal to One.

That is $|r| \leq 1$

2.1.3 Order and Error Constant of Linear Multi-step Methods

The method of obtaining order and error constant for first and second order Linear Multi-step Methods were extended to derive a similar method for obtaining the order and error constant of the third order linear multi-step methods. Specifically the method is meant for the LMM schemes on grid and off grid points to determine their Order and Error constants easily.

In the past, efforts have been made by eminent scholars to derive method of obtaining Order and Error constants especially the first and second linear multi-step methods .Though there exist Taylor's series expansion approximation used for obtaining order and error constants which is too cumbersome compared to this proposed approach (University of Reading, 2015).

Since most of life and physical problems can be modeled into differential equations, we need some numerical Algorithms to obtain its approximate solution. Also we need to analyze the order and error constants of the Algorithms to be used.

The three Linear multi-step methods demonstrated with this proposed method confirmed the accuracy of the method of obtaining the Order and error constants of the third order LMM schemes. This new approach is less cumbersome and simple in its implementation.

2.1.4 Python

The programming language to be used to implement this computer code is python. Presently, there are more than a thousand programming languages in use. According to Guido van Rossum, the inventor of Python Language, "Python is a lot easier than to teach to students programming and teach them C or

https://dx.doi.org/10.4314/aujst.v5i2.12

C++ or Java at the same time because all the details of the languages are so much harder[1]".

Python for Back-End Web Development

In a web application, all the code used to interact with the user and create what the user sees is called the front-end part of the application.

Python is used to code the behind-the-scenes functionality of the application, the part that powers all the functionality of the application but that you don't see directly on screen.

It handles the server-side of the application, interacting with all the necessary databases when the user requests data. It returns the requested data to the user to make the application run as expected.

2.2 Methodology of Software

The Adams' Method Calculator is taken as the application methodology of the generalized software for the general multi-step methods focusing on the Adams' method requires a cycle of defined processes for creating high quality software. This chapter explains different methods diagrammatically on how the project is built from different people's perspective.

Software projects follow a methodology of clearly defined processes which refers to Software Development Life Cycle (SDLC) to ensure the end product is effective, efficient and of high quality. A SDLC model is a conceptual framework that identifies and describes all activities in a software development project from planning to maintenance and the structured flow from one phase to another phase. SDLC model is also called Software Development Process Model.

The SDLC model that is suitable for this project is the waterfall method. This method is a basic model that involves all the phases in the SDLC model. Waterfall method is also known as the linear sequential model, in which the output of a previous phase determines the input of the next phase.

2.3 Requirement Analysis

This involves defining all the prerequisites in planning and developing this application software. Being a small project that does not involve a stakeholder(s) the necessary requirements that are needed for a desired solution by the user are gathered and documented properly. The documentation includes a detailed description of;

The functional requirements:

•The input fields should accept values of different formats that are in float or integer type.

•The system should display an error message for wrong inputs.

•A system that represents a mathematical model

•The result displayed should be easily comprehended and understood by the user

•Being a web-based system it should be easily accessible by anyone Non-functional requirements include:

•The response time of the system.

•The usage of the system is -usability

•The maintenance of the system-maintainability

2.4 System and Software Design

System Design allocates the requirements to the software system by establishing system architecture. The architectural pattern is used to describe the system design and how it functions.

Software Design identifies and describes the fundamental software system abstractions and their relationships. The Unified Modeling Language is used to describe the software The reason for this is that it is possible to look at a system from many different viewpoints. Different stakeholders like designers, analysts, coders or even the customer play a part in visualizing the program in the software development process. There are two types of the UML Diagram, the

https://dx.doi.org/10.4314/aujst.v5i2.12

the structural diagram and the behavioral diagram which have

2.4.1 Architectural Pattern of Software

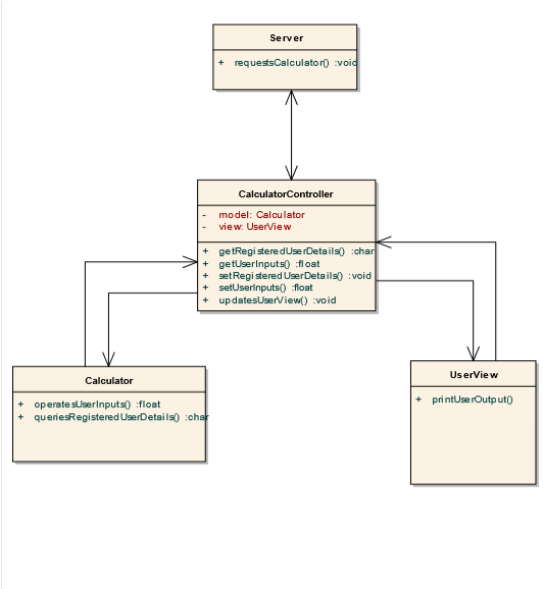

Figure 1 – Model View Controller

Architectural Patterns are similar to software desing pattern but explains it better. The archiectural Pattern suitable for this project is Model View Controller (MVC)

Model – handles the data and the functionality of the architecture.

View – Handles the presentation or display of the data to the user

Controller –This handles the input from the user and tells the Model and User view what to do.

1. Server- User requests for a calculator and request goes to the server. Server collects the user's input and sends it to the controller.

2. Controller – Collects the user's input and sends it to the model. The controller is called the Calculator Controller.

3. Model- The model is the calculator which queries its database and carries out operations on the user's input. Model sends results to the controller after the operation.

4. Controller – Collects the output from the model and sends it to the user's view.

5. User View – The user view displays the output in a designed representation and outputs to the controller. The controller sends the result to the user through the server.

If any process is unsuccessful - The user view receives an error message as an output to the user through the server.

Web Frameworks

The web framework for this project is Django. Django is a "high-level Python Web framework that encourages rapid development and clean, pragmatic design."

This study focuses on the web development application which implements the generalized computer code for the Linear Multi-step Methods.

2.4.2 UNIFIED MODEL LANGUAGE

UML is a way of specifying, visualizing, constructing, and documenting a software program using a collection of diagrams. The modeling tool used for displaying the UML Diagrams below is Enterprise Architect.

A. USE CASE DIAGRAM

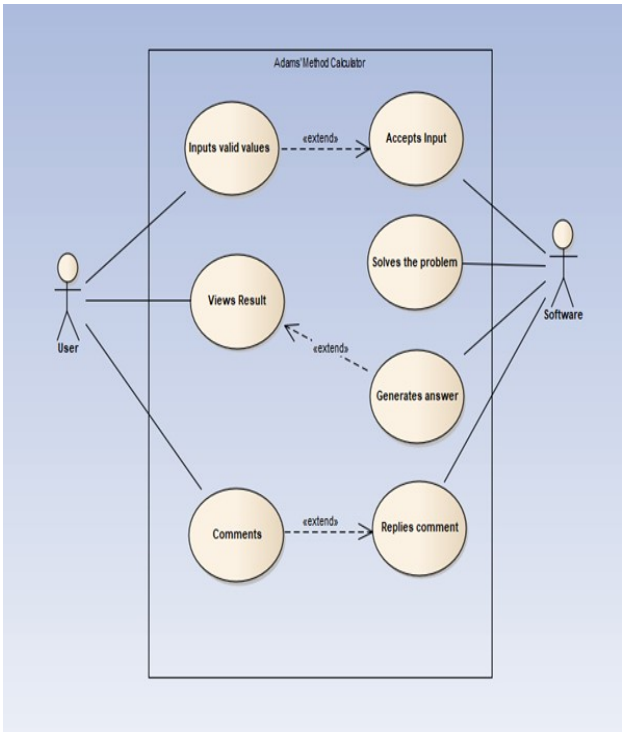

Figure 2. Use Case Diagram

The Use Case Diagram is a type of Behavioral Diagram under the types of UML which represents the functional requirements and the relationship among use cases, actors and systems.

https://dx.doi.org/10.4314/aujst.v5i2.12

DESCRIPTION:

Use Cases;

Inputs Valid Values
Solves a problem
Generates answer
Replies comment

Actors;

-User:

Inputs the valid values from the problem in the system.
View Result from the system.

-Software:

Accepts the user's input.
Checks for validity of input.
Gives an error message for an invalid input.
Solves the problem.
Generates the answer to the user.

B. SEQUENCE DIAGRAM

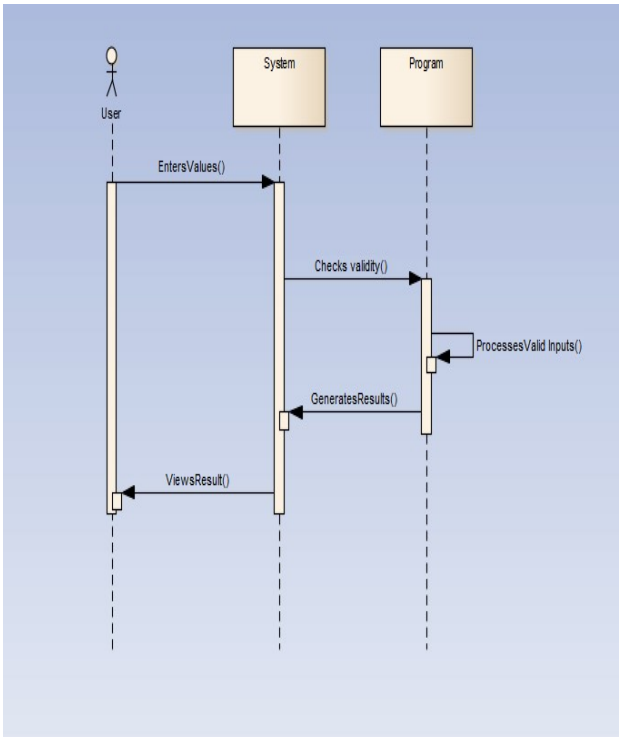

Figure 3. Sequence Diagram

The Sequence diagram is a type of interaction diagram that details how interactions are carried out and shows elements as they interact over time and they are organized according to object (horizontally) and time (vertically). The Use Case diagram has to be drawn before the sequence diagram in order to decide what interactions should be included.

DESCRIPTION:

The flow of the sequence is described as;
- The user enters values from the problem question.

- The system checks for validity of the entered values
- If valid, the program solves the problem.
- If invalid, the program gives an error message.
- The system generates the results.
- The user can view the results.

Activity diagram is used to describe the dynamic aspects of the system and behavior of the system diagrammatically. Activity diagram is similar to the data flow diagram or flowchart. It comprises of an initial state at the beginning then a path, usually drawn with an arrow line, which shows the movement from one action state to another. The diagram ends at the final state, which is described as an arrow pointing to a filled circle nested inside another circle. The flow of this diagram is partitioned into swimlanes, which are groups of related activities into a column (Figure 4).

DESCRIPTION:

Swimlanes;

User's End
Calculation Process
System

All the activities carried out by the user is partitioned in the first swinlane, the software only takes the input and carries out the calculation on the values and it is in the middle swinlane. The last swinlane is the system which collects result from the software and presents the result to the user. The flowchart ends on the user's end because the user views the result of the problem (Figure 5).

https://dx.doi.org/10.4314/aujst.v5i2.12

C. ACTIVITY DIAGRAM

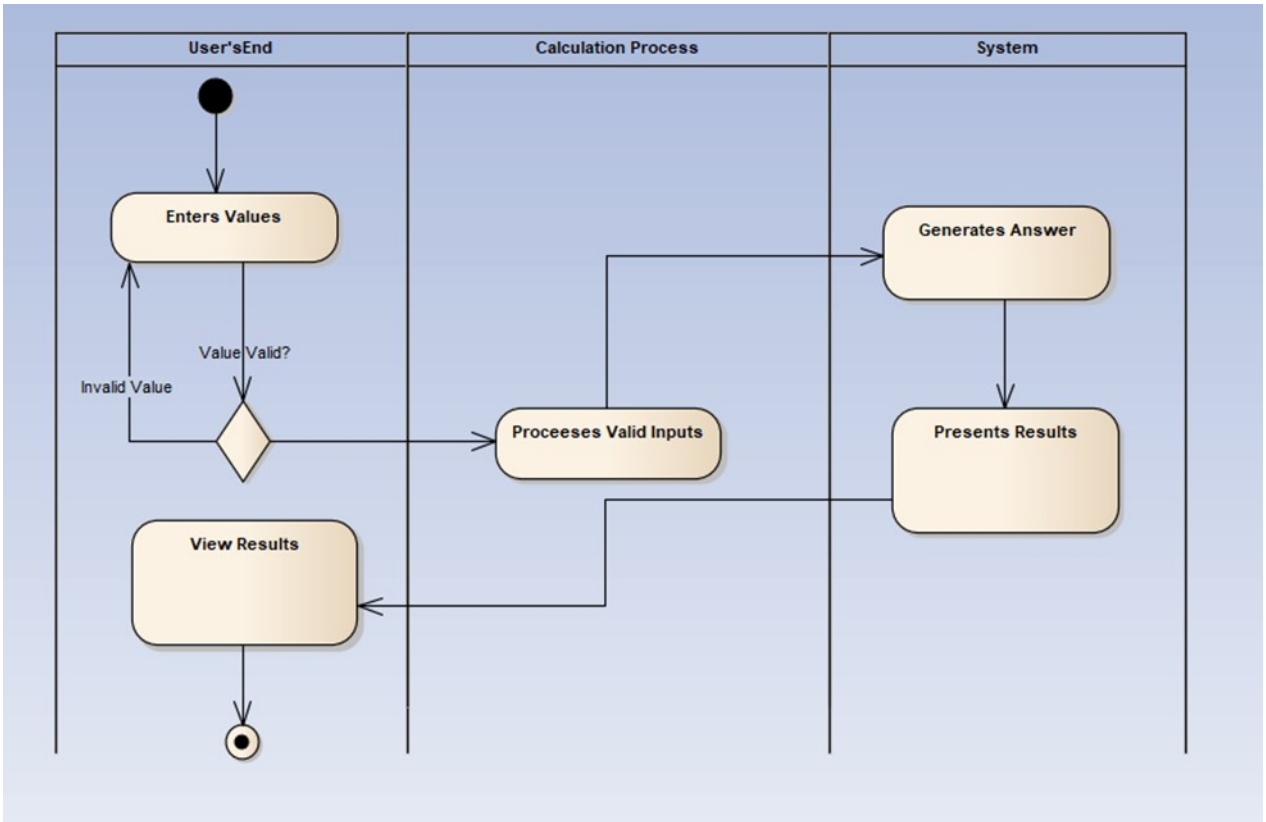

Figure 4. Activity Diagram

D. CLASS DIAGRAM

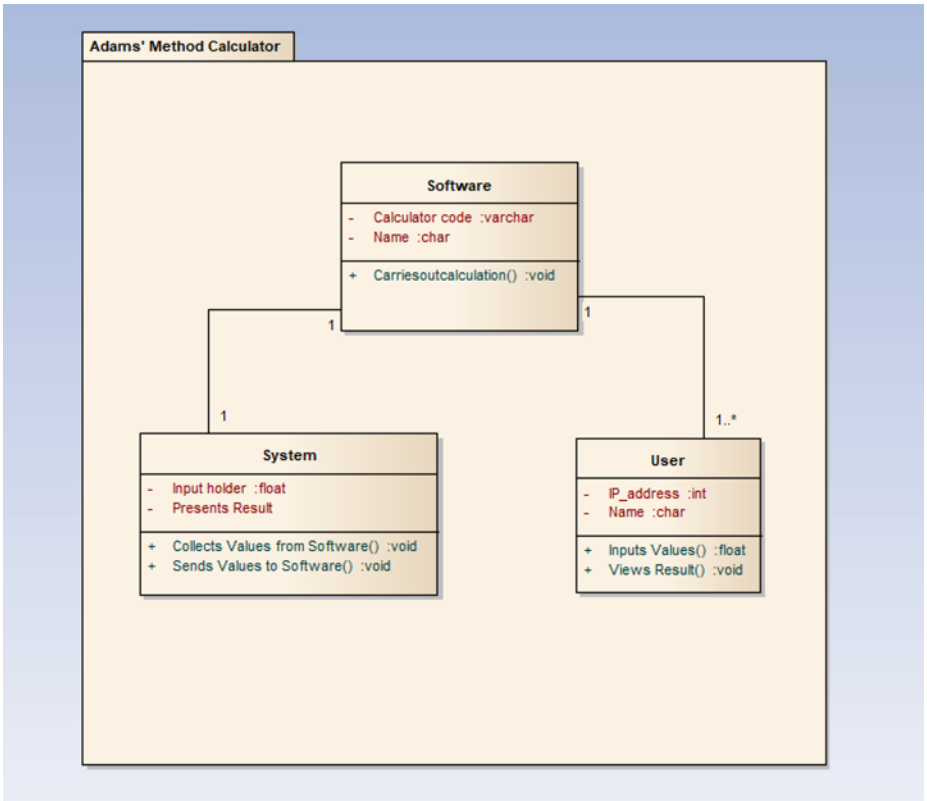

Figure 5. Class Diagram

https://dx.doi.org/10.4314/aujst.v5i2.12

Class diagram is a type of the structural diagram in the UML diagram types. It shows the classes in a system, its attributes, and operations of each class and the relationship between each class. The diagram is sectioned into three parts, the name of the class at the top, the attributes of the in the middle and their operations at the bottom.

DESCRIPTION:

Classes;

- Software

The attributes of the software are the name and the code written in python language.

The operation carried out is the calculation process on the values entered.

The software has a relationship of one program code to one user or many users as well as the system that is one program attending to many users at the same time.

-User

The IP address and the name of the user specify the attributes of the user class.

The two operations of the user are simply to input values and view the result.

The relationship is many users can use the software on different devices at a time.

-System

Input holders and Result section are the attributes of the system.

These input fields collect the entered values and sends to the software program.

The relationship between the system and software and also the system and the user is one-to-one because the software can only work the input values of a user at a time on a system.

3.0 IMPLEMENTATION AND RESULT

The LMMs Calculator implementation process involves the remodeling of the available system requirements into a working system that is effective which implies the system is producing the right output and efficient indicates the system uses the inputs in a right way and also the system should be user friendly.

3.1 COMPUTATIONAL TOOLS, PROGRAMS AND ENVIRONMENT

Implementing the Adams' Method Calculator software required the basic building blocks or computational tools for the web-based system which are Hyper Text Markup Language (HTML), Cascading Style Sheet (CSS)/Bootstrap. It was implemented using Python programming language and the python framework, Django. The Integrated Development Environment (IDE) used for the language was Thonny.

3.2 REQUIREMENT FOR IMPLEMENTATION

A system with the following hardware and software properties was used;

Processor: Intel(R) Core(TM) i5 – 3380M CPU @ 2.90GHz 2.90GHz

Installed memory (RAM): 6.00 GB

System Type: 64-bit Operating System, x64-based processor

Software System: A Windows 10 Pro OS

3.2.1 Testing and documentation:

Examples of LMMs:

1. $Y_{n+4} - Y_{n+3} = h/24 \{55F_n+3 -59F_{n+2} +37F_{n+1} - 9F_n\}$
2. $Y_{n+3} - Y_{n+2} = h/24 \{9F_{n+3} +19F_{n+2} - 5F_{n+1} + F_n\}$
3. $Y_{n+2} - 4/5Y_{n+1} - 1/5Y_n = h/5 \{2F_{n+2} +4F_{n+1} \}$
4. $Y_{n+3} - 1/17(9Y_{n+2} + 9Y_{n+1} - Y_n) = 6h/17 \{ F_{n+3} + 3F_{n+2}\}$
5. $Y_{n+4} - 1/37(8Y_{n+3} + 36Y_{n+2} - 8Y_{n+1} + Y_n) = 12h/37 \{F_{n+4} + 4F_{n+3}\}$
6. $Y_{n+5} - Y_{n+4} + h/90720(31161F_{n+5} + 83721F_{n+4} - 37914F_{n+3} + 18006F_{n+2} - 4719F_{n+1} + 465F_n)$

https://dx.doi.org/10.4314/aujst.v5i2.12

The General Linear Multi-step Methods is given by

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

where; α_j and β_j are constant coefficients of y_{n+j} and f_{n+j} respectively.

A sample question: $Y_{n+3} - Y_{n+2} = h/24\{9F_{n+3} + 19F_{n+2} - 5F_{n+1} + F_n\}$(Figure 6)

Enter the highest subscript of Y, that is = 3

Enter the highest subscript of F, that is = 3 (Figure 7)

The coefficients of α_j ; $\alpha_0 = 0$; $\alpha_1 = 0$; $\alpha_2 = -1$; $\alpha_3 = 1$;

The coefficients of β_j ; $B_0 = 1/24$; $B_1 = -5/24$; $B_2 = 19/24$; $B_3 = 9/24$

These values are entered into the computer program, which accepts both integer and float values (Figure 8).

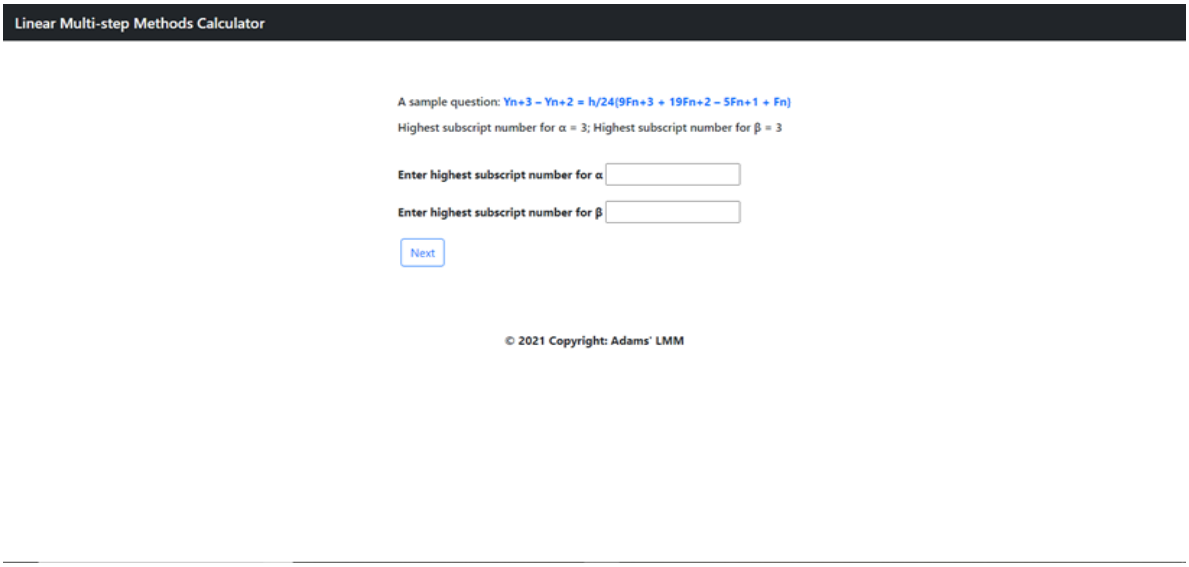

Figure. 6-Input Field

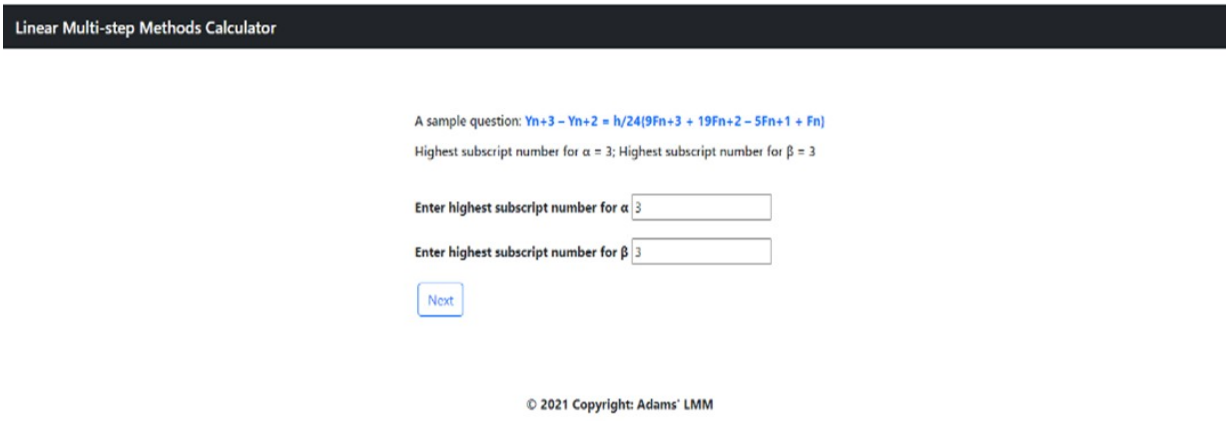

Figure 7. Values in Input Field

https://dx.doi.org/10.4314/aujst.v5i2.12

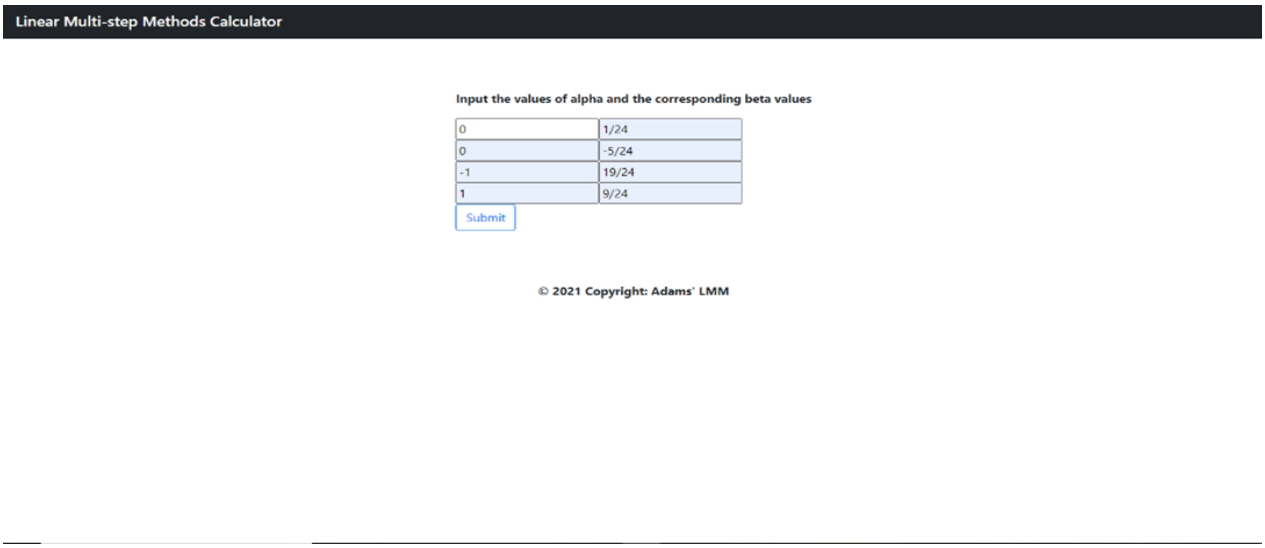

Figure 8. Coefficients in Integer Form

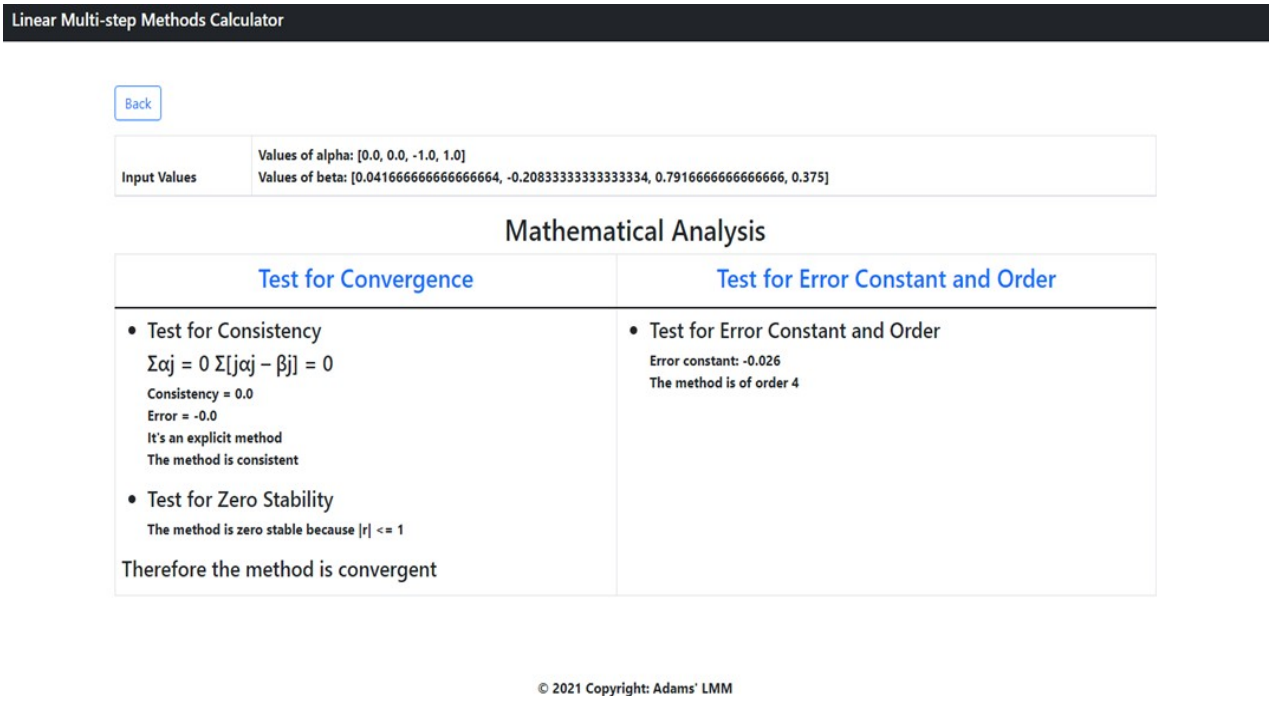

Figure 9. Result of Integer Input Values

The values entered by the user are in the value type, integer and fraction. This shows the program accepts integer values from the user. The result is shown in Figure 9. The next set of values entered is in the float value type format, using the sample question above. The program accepts the float values and also converts the fraction into float values and eventually gives a result (Figure 10). The values entered by the user are in the value type, float and fraction. This shows the program accepts float values from the user. The result is shown in Figure 11.

3.2.2 Limitation of System

The major limitation of the program is the values being entered to the program. If the values are not in the right format, that is an integer or a float value. The program will give an error to the user. Therefore, concentration of the student is required when entering the values into the computer program.

https://dx.doi.org/10.4314/aujst.v5i2.12

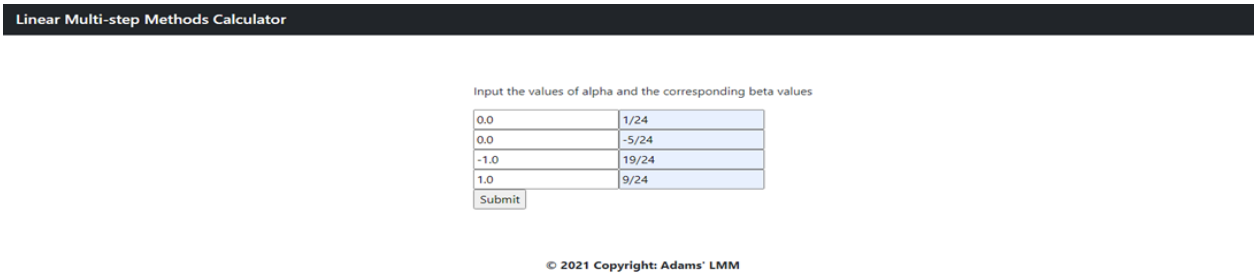

Figure 10. Coefficients in Float Form

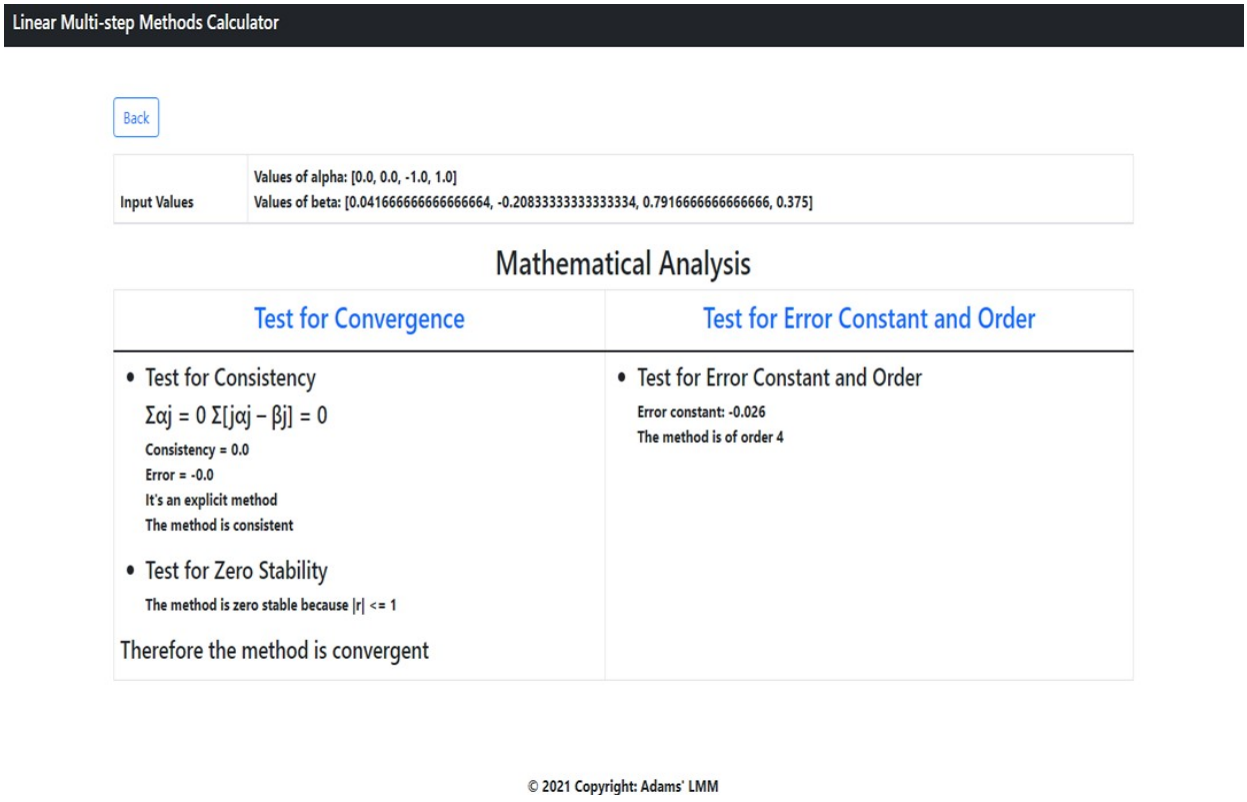

Figure 11. Result of Float Input Values

4. CONCLUSION

We have developed a generalized computer code that calculates and analyses the convergence, order and error constants of any given linear multi-step methods, with integer steps which is highly efficient in terms of accuracy, and accessible to all users. This calculator is web-based therefore it can be accessed by anyone who visits the website without the need for logging in to use the calculator.

REFERENCES

Ahamad, N., & Charan, S. (2019). Study of Numerical Solution of Fourth Order Ordinary Differential Equations by

https://dx.doi.org/10.4314/aujst.v5i2.12

by fifth order Runge-Kutta Method. International Journal of Scientific Research in Science, Engineering and Technology, 230–238. https://doi.org/10.32628/ijsrset196142

Asif Arefin, M. (2020). A Comparative Exploration On Different Numerical Methods For Solving Ordinary Differential Equations. Journal Of Mechanics Of Continua And Mathematical Sciences, 15(12), 1–11. https://doi.org/10.26782/jmcms.2020.12.00001

Badmus, A. M., & Yahya, Y. A. (2014). New Algorithm of Obtaining Order and Error Constants of Third Order Linear Multi-Step Method (LMM). New Algorithm of Obtaining Order and Error Constants of Third Order Linear Multi-Step Method (LMM), Volume 02(Issue 06), 1–5. https://www.ajouronline.com/

Bhartendu (2021). Adams Bashforth Moulton Method (https://www.mathworks.com/matlabcentral/fileexchange/63034- adams- bashforth -moulton-method), MATLAB Central File Exchange. Retrieved March 2, 2021.

Brown, R. L. (1978). Error behaviour of multistep methods applied to unstable differential systems. Applied Mathematical Modelling, 2(2), 115–118. https://doi.org/10.1016/0307-904x(78)90048-3

Duo Wang, YanLi Han, and Tengfei Sun "Star sub-pixel centroid calculation based on multi-step minimum energy difference method", Proc. SPIE 8907, International Symposium on Photoelectronic Detection and Imaging 2013:Infrared Imaging and Applications, 89075M (11 September 2013); https://doi.org/10.1117/12.2038813.

Fatokun J. O (2006) Continuous Approach for Deriving Self-Starting Multistep Methods for initial value problems in Ordinary Differential Equations. International Journal of Engineering and Applied Sciences 2 (3):50508,2007.URL:http://www.medwelljournals.com

Fatokun J.O and Aimufua, G.I.O (2007): Implementation of an order Seven Self-Starting Multistep Methods Using Scilab and Fortran Codes. International Journal of Soft Computing 2(2):320-324, 2007. URL: http://www.medwelljournals.com

Filbet, F., & Prouveur, C. (2016). High order time discretization for backward semi-Lagrangian methods. Journal of Computational and Applied Mathematics, 303, 171–188.https://doi.org/10.1016/j.cam.2016.01.024

Guido van Rossum Quotes. (n.d.). BrainyQuote.com. Retrieved March 4, 2021, from BrainyQuote.comWebsite: https://www.brainyquote.com/quotes/guido_van_rossum_242964 [1]

https://www.freecodecamp.org/news/what-is-python-used-for-10-coding-uses-for-the-python-programming-language/

ProQuest Dissertations And Theses; Thesis (Ph.D.)--University of Pittsburgh, 2017; Publication Number: AAT 10645835; ISBN: 9780355190892; Source: Dissertation Abstracts International, Volume: 79-01(E), Section: B.; 65 p.

Wikipedia contributors. (2021, March 1). Linear multi-step method. In Wikipedia, The Free Encyclopedia. Retrieved 03:17, March 4, 2021, from https://en.wikipedia.org/w/index.php?title=Linear_multistep_method&oldid=1009682032

https://dx.doi.org/10.4314/aujst.v5i2.12