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Measure Synchronization in a Coupled Helmholtz Resonator Hamiltonian

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Competing Interests.

The authors declare no competing interests.

ABSTRACT

Background: Measure synchronization (MS) is a weak form of synchronization that is observed in coupled Hamiltonian systems, where the trajectories share the same phase space without perfectly locking.
Objective: In this study, MS was investigated in a pair of nonlinearly coupled Helmholtz resonator described by a quartic Hamiltonian.
Methods: By varying the coupling strength k, the distinct dynamical regimes such as quasiperiodic unsynchronized states, quasiperiodic measure synchronized (QPMS), chaotic unsynchronized states, and chaotic measure synchronized states (CMS) were identified. We characterized these transitions between these states using, average bare energy, Lyapunov exponent, phase portraits, power spectra, and wavelet transforms.
Results: The results reveal that MS emerges when the coupling strength exceeds a critical threshold, leading to energy redistribution and alignment in the system dynamics despite their chaotic or quasiperiodic nature.
Conclusion: This work contributes to a broader understanding of synchronization in Hamiltonian systems and highlights the role of nonlinear coupling in the dynamics of collective behavior.

Keywords: Measure Synchronization, oscillator, Hamiltonian, Lyapunov exponent

1. Introduction

The concept of synchronization has been a fundamental phenomenon observed across a wide range of natural and engineered system, where two or more interacting systems achieve coordinated behavior through phase, frequency or amplitude alignment. It was first reported by Huygens in the 17th century when he noticed the synchronization of pendulum clock mounted on the same wall. It has become a central topic in nonlinear science, with applications in physics, biology, and engineering. Synchronization can manifest in various forms ranging from complete synchronization, phase synchronization, lag synchronization, generalized synchronization, anti-synchronization, projective synchronization, and measure synchronization (Vincent and Kolebaje, 2020; Ghosh, 2023). Measure synchronization (MS), a weak form of synchronization was first reported by Hampton and Zanette, (1999) and it occurs in coupled Hamiltonian systems. MS differs from all other forms of synchronization in that it happens in the absence of dissipation. MS is not defined by the asymptotic merging of two or more trajectories into one; rather, it occurs when the trajectories share the same phase space domain although the trajectories intersect but do not merge. The occurence of MS has been reported in coupled Hamiltonian systems such as Huygens's non-dissipative two-pendulum clock system (Tian *et al.*, 2018), φ^4 systems (Wang *et al.*, 2002; Wang *et al.*, 2003), Duffing oscillators (Vincent, 2005), pendulum systems suspended from a common beam (Tian *et al.*, 2019), Bosonic-Josephson junctions (Tian et al, 2010; Tian *et al.*, 2013), Pullen Edmond potential (Gupta *et al.*, 2017; De *et al.*, 2018; Gupta *et al.*, 2019), camphor rotors (Qiu *et al.*, 2025). There are various methods that have been employed in literature to explore the phenomenon of MS analytically and quantitatively. MS has been explored using phase space analysis (Vincent, 2005; Gupta *et al.*, 2017; De *et al.*, 2018), Lyapunov exponents (Hampton and Zanette, 1999; Tian *et al.*, 2013; Tian *et al.*, 2019), order parameters (Hampton and Zanette, 1999; Tian

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et al, 2010; Gupta *et al.*, 2017; De *et al.*, 2018), Poincare section analysis (Tian *et al.*, 2013; Tian *et al.*, 2019; Qiu *et al.*, 2025), energy characteristics (Tian *et al.*, 2010; Qiu *et al.*, 2010; Gupta *et al.*, 2017; De *et al.*, 2018), stability analysis of fixed point (Tian *et al.*, 2010), frequency spectrum analysis (Gupta *et al.*, 2017; De *et al.*, 2018; Gupta *et al.*, 2019), phase coherence dynamics (Tian *et al.*, 2018), and wavelet analysis (De *et al.*, 2018; Gupta *et al.*, 2019).

The Helmholtz resonator is a classic physical system used to model acoustic resonance, it can also be studied as a nonlinear oscillator (Alamo *et al.*, 2018; Lu *et al.*, 2022; Omoteso *et al.*, 2024). The system describes a two-degree-of-freedom nonlinear oscillator with a specific form of potential energy. The kind of potential that defines the system is non-integrable and supports a wide range of dynamical behavior such as periodicity, quasiperiodicity, and chaos. It has application in vibration analysis, molecular physics, and acoustics (Ingard and Ising, 1967; Lu *et al.*, 2022; Omoteso *et al.*, 2024). This potential function defines the energy landscape of the coupled Helmholtz-like resonator system, where x, and y represents the generalized coordinates for two interacting resonators. The quadratic term $1/2\,(x^2+y^2)$ represents the standard harmonic oscillator potential i.e it is the restoring forces that try to bring the system back to equilibrium. The cubic term $1/3\,(x^3+y^3)$ introduces asymmetry and nonlinearity which can lead to complex dynamics including chaos where α controls the strength of this nonlinearity. The quartic term $1/4\,(x^4+y^4)$ adds a hardening or softening effect depending on the sign and magnitude of β. It also helps to bound the potential at large displacements, keeping the system stable.

In this paper, MS in the coupled Helmholtz resonator Hamiltonian using the quartic coupling, analyzing how coupling strength, energy variation and nonlinearity influence the emergence and stability of synchronized states was investigated. Various types of synchronization behavior including quasiperiodic, chaotic, synchronized, and unsynchronized states are observed and quantitatively analyzed as the coupling strength is varied revealing transitions between these states. We characterize the transition to MS through average bare energy, Lyapunov exponent, power spectrum, and wavelet analysis.

2. The Model

Consider the Helmholtz resonator potential (Omoteso *et al.*, 2024) given by:

$$V(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{3}\alpha(x^3+y^3) + \frac{1}{4}\beta(x^4+y^4) \quad (1)$$

The Hamiltonian of the system can be be written as

$$H = \frac{1}{2}\left(P_x^{\,2}+P_y^{\,2}\right) + \frac{1}{2}(x^2+y^2) - \frac{1}{3}\alpha(x^3+y^3) + \frac{1}{4}\beta(x^4+y^4) + \frac{k}{2}x^2y^2 \quad (2)$$

where P_x and P_y are the momentum of oscillators x and y respectively, α and β are the parameters of the system and k is the coupling strength. The system of interest consists of two non-linearly coupled oscillators x and y governed by the Hamiltonian defined by Equation (2). Since the system is conservative, the total energy of the system can be represented as,

$$E = E_x + E_y + E_{xy}, \quad (3)$$

where the bare energies of oscillators x, and y are given by E_x, and E_y respectively, and the interaction energy of the system

$$E_x = \frac{1}{2}\left(P_x^{\,2}+x^2\right) - \frac{1}{3}\alpha\, x^3 + \frac{1}{4}\beta\, x^4$$

$$E_y = \frac{1}{2}\left(P_y^{\,2}+y^2\right) - \frac{1}{3}\alpha\, y^3 + \frac{1}{4}\beta\, y^4 \quad (4)$$

is given by E_{xy}. $E_{xy} = \frac{k}{2}x^2y^2$

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Applying Hamilton canonical equation to Eq. (2), the equation of motions are:

$$\dot{x} = P_x$$

$$\dot{P}_x = -x + \alpha x^2 - \beta x^3 - kxy^2 \quad (5)$$

$$\dot{y} = P_y$$

$$\dot{P}_y = -y + \alpha y^2 - \beta y^3 - kx^2 y$$

3. Measure Syncronization

Throughout our investigation, we will keep the initial condition as (0.0; 0.2; 0.0; 0.6), the parameters of the system $\alpha = 1.2$ and $\beta = 1.5$ as a constant. The coupling strength k will be switched on for MS dynamical phase transitions in the Helmholtz Hamiltonian.

3.1 Average Bare Energy and Lyapunov Exponent

To characterize the various MS transition phase, we calculate the average bare energy of each oscillator. The average bare energy for coupled Hamiltonian systems can be employed in characterization of MS transitions, and the average energy for each oscillator is computed as follows: It is calculated as

$$\langle E_{x,y} \rangle = \frac{1}{T}\int_0^T E_{x,y}(t)\, dt \qquad (6)$$

Equation (6) has been used by several authors in calculating the average energy during MS transitions (Vincent, 2005; Gupta et al., 2017). Figure 1 shows the average bare energy and Lyapunov exponent of the Helmholtz resonator.

In Figure 1(a), at $k = 0$ (no coupling), there is a noticeable variation in energy between oscillators x and y. The average energy of x denoted as $\langle E_{x,} \rangle$ is higher than that of oscillator y denoted as $\langle E_y \rangle$. As the coupling k increases, there is a redistribution of energy between the oscillators due to the interaction between them. For $0 < k \le 1.1$, the average energy of both oscillators approach each other, this is indicative of energy sharing or energy equipartition due to coupling. Beyond $k = 1.1$, the energies remain close but not perfectly equal such that the difference between the energies $\langle E_{x,} \rangle - \langle E_y \rangle \approx 0$ indicating that although the oscillators are not perfectly synchronized, they are in the quasiperiodic measure synchronized state (QPMS). This regime is characterized by quasiperiodic dynamics where the system retains its individual oscillatory nature but shows MS.

Chaotic unsynchronized state persists beyond $k = 6.6$, the energy values of the two oscillators fluctuate wildly and show no clear pattern of synchronization. This regime is characteristics indicative of chaotic dynamics where coupling is strong enough to induce irregular energy exchange but not synchronization. Chaotic measure synchronization (CMS) occurs at $k \ge 11.72$. Here, the average bare energy of both oscillators approximates each other but not exactly zero. This shows that the system has entered a completely synchronized state in terms of energy measure, despite the presence of chaotic dynamics.

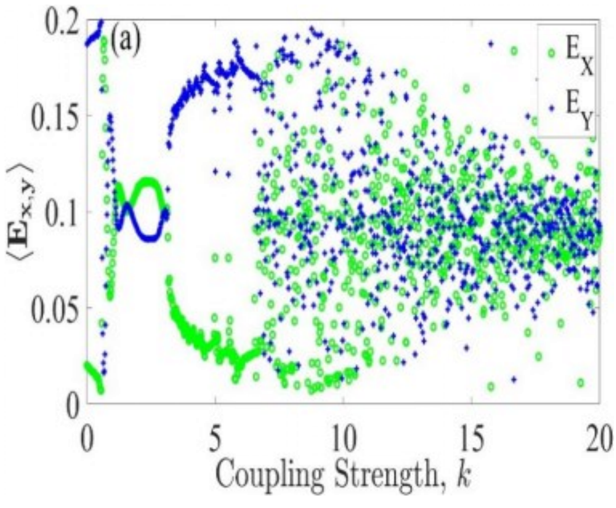

Figure 1(a): Average bare energy variation of oscillators x and y with coupling strength k from 0 to 20.

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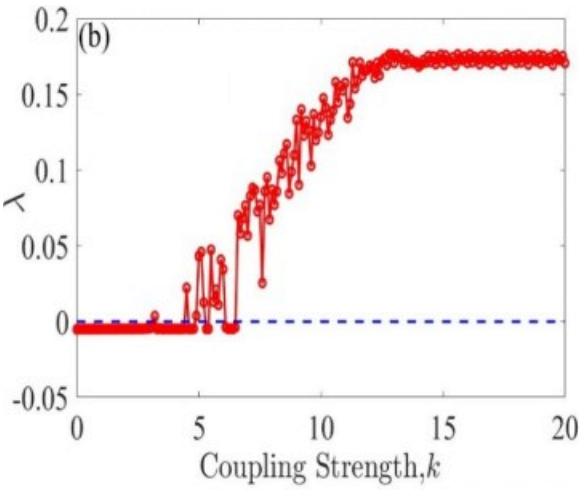

Figure 1(b): Largest Lyapunov exponent for the oscillators corresponding to Figure 1(a).

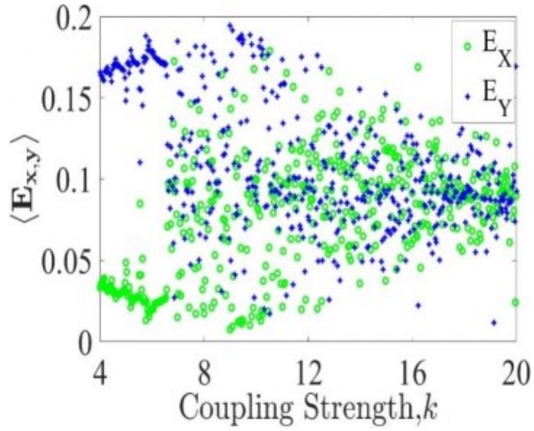

Figure 2: Average bare energy variation with coupling strength k from 4 to 20 showing the transition from quasiperiodic unsynchronized state to chaotic unsynchronized state to chaotic measure synchronized state.

The plot of the Largest Lyapunov Exponent (λ) against the coupling strength k is presented in Figure 1(b). The Largest Lyapunov Exponent (LLE) is a key indicator of the system's stability. It can be less than zero ($\lambda < 0$), equals to zero ($\lambda = 0$), and greater than zero ($\lambda > 0$) referring to periodic, quasiperiodic, and chaotic motion respectively. The LLE is zero or slightly negative for $k = 0$, this corresponds to regular periodic behaviour in the absence of coupling. A small positive spike which occurs at $k = 3.2$ shows a brief weak chaotic regime. Frequent transitions between quasiperiodic and chaotic regimes are observed for $4.4 \leq k \leq 6.5$. In this k-values regime λ fluctuates between positive and zero, this is an indication of intermittent dynamics where the system occassionally bursts into chaos and then returns to quasiperiodic behavior. At $k \geq 6.5$, λ becomes persistently positive. The system remains chaotic for all higher values of k without reversion to periodic or quasiperiodic states irrespective of further increase in coupling strength.

3.2 Phase Portrait

The dynamical transition of the system is observed using phase space domain analysis, revealing different MS states. Here, oscillator x is denoted by color green while oscillator y color blue. When the system is decoupled i.e $k = 0$, each oscillators follow distinct periodic trajectories with oscillator y covering oscillator x as shown in Figure 3(a). Upon introducing a coupling of 0.4, the trajectories of both oscillators broaden and give rise to two separate phase space domains, as illustrated in Figure 3(b). With a further increase in coupling to $k = 2.0$, both oscillators begin to share an overlapping phase space domain, as shown in Figure 3(c), indicating a quasiperiodic measure synchronized state (QPMS) in which the oscillators possess identical measures. At $k = 3.5$, Figure 3(d) displays a return to a quasiperiodic unsynchronized state, where both oscillators remain confined within bounded regions of phase space while maintaining their distinct identities. Although the oscillators are not phase-locked (no exact matching of their trajectories), they occupy approximately equal regions in phase space. In Figure 3(e), a chaotic unsynchronized state is reported for $k = 8.0$. Despite chaotic motion, both oscillators have their trajectories covering the same area in phase space. A closer look reveals that, while the two share the same measure, they do not totally overlap. There are also

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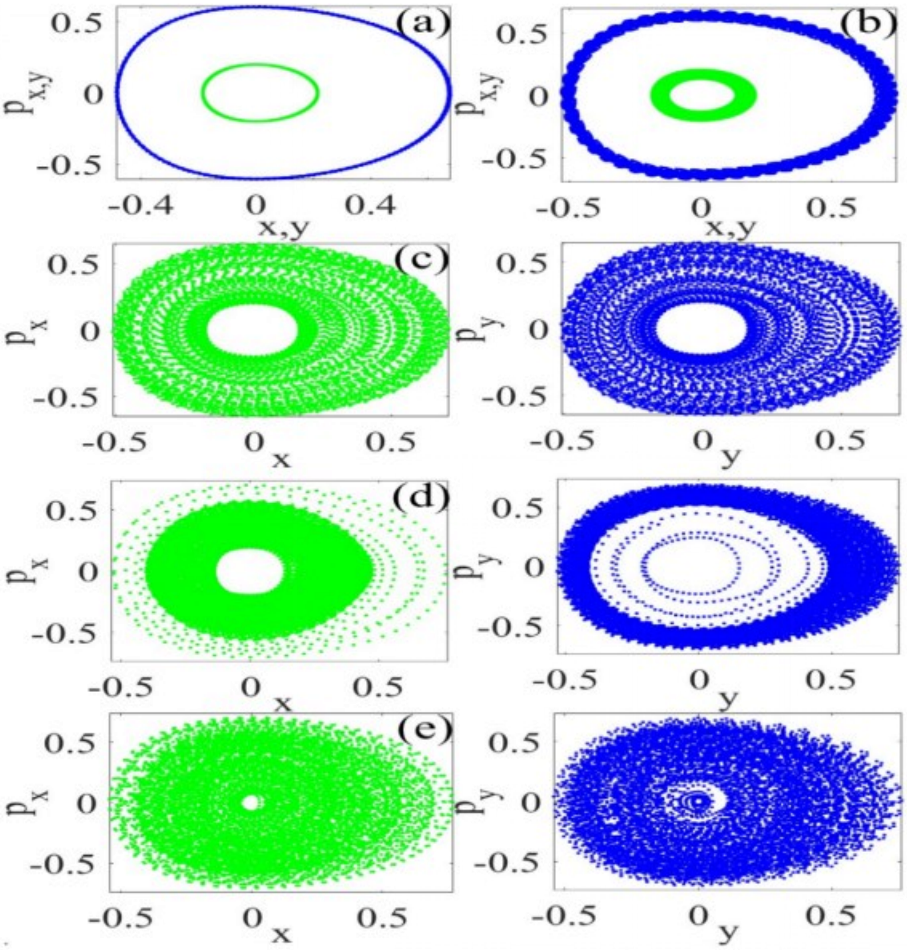

Figure 3: Phase space diagram for (a) no coupling, $k = 0$; (b) quasiperiodic unsynchronized state, $k = 0.4$; (c) QPMS state, $k = 2.0$; (d) quasiperiodic unsynchronized state, $k = 3.5$; (e) CMS state, $k = 8.0$.

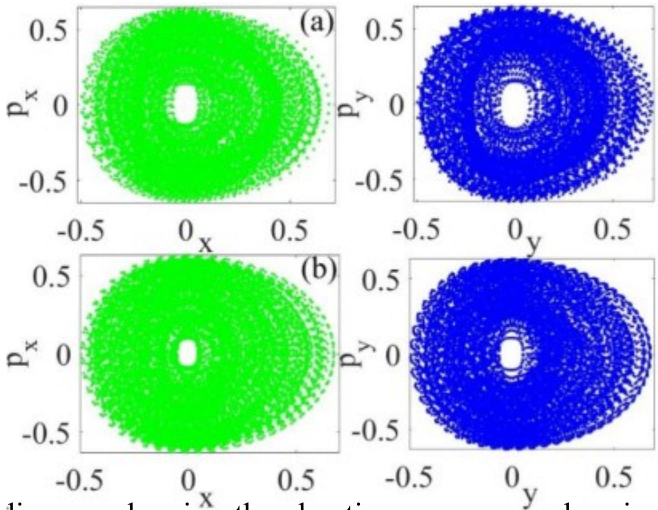

Figure 4: Phase space diagram showing the chaotic measure synchronized (CMS) state for (a) $k = 19$, and (b) $k = 30$ respectively.

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minor differences between the inner parts of the two phase-space domains.

Figure 4 shows two distinct values of the coupling strength k, illustrating the behavior of the system in the CMS state. With a coupling strength of $k=19$, and $k=30$, the phase portraits exhibit complete CMS for both oscillators. This means that the invariant measures are completely synchronized. The behavior further reinforces average bare energy (Figure 2(a)) and Lyapunov (Figure 2 (b)) suggesting that once the system enters the CMS regime, it sustains the state across a broad range of higher coupling strengths.

3.3 Power Spectrum

In Figure 5, different power spectra were calculated using the Fast Fourier Transform (FFT) at different values of the coupling strength k. The power-frequency plot illustrates the transition from periodic to chaotic dynamics and the emergence of MS as the coupling strength increases. The power spectra of the x and y components of the oscillators are represented in green and blue lines, respectively.

In Figure 5(a) with $k=0$, the oscillators exhibit periodic motion, characterized by single, sharp peaks at their respective frequencies. The absence of coupling results in distinct spectral lines for both oscillators indicating no synchronization.

This is like what was observed in the phase space domain analysis at no coupling in Figure 3(a). As the coupling strength increases to $k=0.4$ in Figure 5(b), the system evolves in a quasiperiodic unsynchronized state. Several spectral peaks emerge due to the presence of incommensurate frequencies, with noticeable differences between the spectra of both oscillators, reflecting the absence synchronization. At $k=2$ as shown in Figure 5(c), the oscillators exhibit quasiperiodic synchronized behavior. The dominant spectral peaks of both oscillators now align, signifying a frequency locking phenomenon and the onset of QPMS, despite the persistence of multiple frequency components.

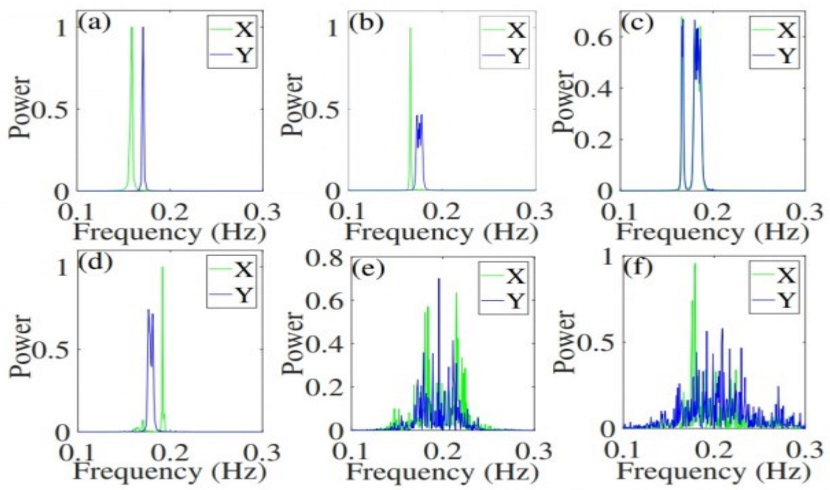

Figure 5: Power spectrum analysis to show the various MS transition states (a) no coupling, $k=0$; (b) QP unsynchronized, $k=0.4$; (c) QPMS, $k=2.0$; (d) QP unsynchronized, $k=3.5$; (e) and (f) CMS, $k=10$ and $k=20$ respectively.

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Interestingly, for $k = 3.5$ in Figure 3(d), the system re-enters a quasiperiodic unsynchronized regime, evidenced by the non -overlapping spectral peaks. This indicates that synchronization is not solely dependent on increasing the coupling strength but also influenced by the underlying system dynamics. For higher coupling strengths, $k = 10$ and $k = 20$, as illustrated in Figures 3(e) and (f), the oscillators exhibit chaotic dynamics characterized by broad and continous spectra. Notably, the power spectra of both oscillators are remarkably similar with overlapping frequency components. This signifies the chaotic measure synchronized state (CMS), where although the trajectories are chaotic and unpredictable, the statistical properties of both oscillators, including their energy distributions over the frequency domain, become nearly identical due to strong coupling.

3.4 Wavelet Analysis

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \quad (7)$$

$$C_f(a,b;\psi(t)) = \left|\langle\psi_{a,b}(t)|f(t)\rangle\right| = \left|\frac{1}{\sqrt{a}}\int \bar{\psi}\left(\frac{t-b}{a}\right) f(t)\, dt\right| \quad (8)$$

Figure 6 shows the CWT plots representing the two systems showing the wavelet power spectrum over time (x-axis) and scale (y-axis). The color represents the wavelet power: blue (low) to red (high). In Figure 6 (a), the two wavelet plots are nearly identical in structure and intensity for $k = 2$. Though the systems are not phase-locked, they share the same distribution of time spent across states indicative of the QPMS state. For the quasiperiodic unsynchronized state $k = 3.5$, similar frequency is present but there is a visible mismatch in timing and structure between the two oscillators. The oscillators share similar spectral features but are not measure synchronized as shown in Figure 6 (b).

In the chaotic unsynchronized regime for $k = 8$ as seen in Figure 6(c), the figure shows broad and disordered wavelet power patterns, characteristics of chaotic dynamics. The system behaves chaotically and does not exhibit measure synchronization. Despite the presence of chaos, the oscillators exhibit CMS for $k = 20$ in Figure 6(d) meaning both oscillators explore the same regions of state space.

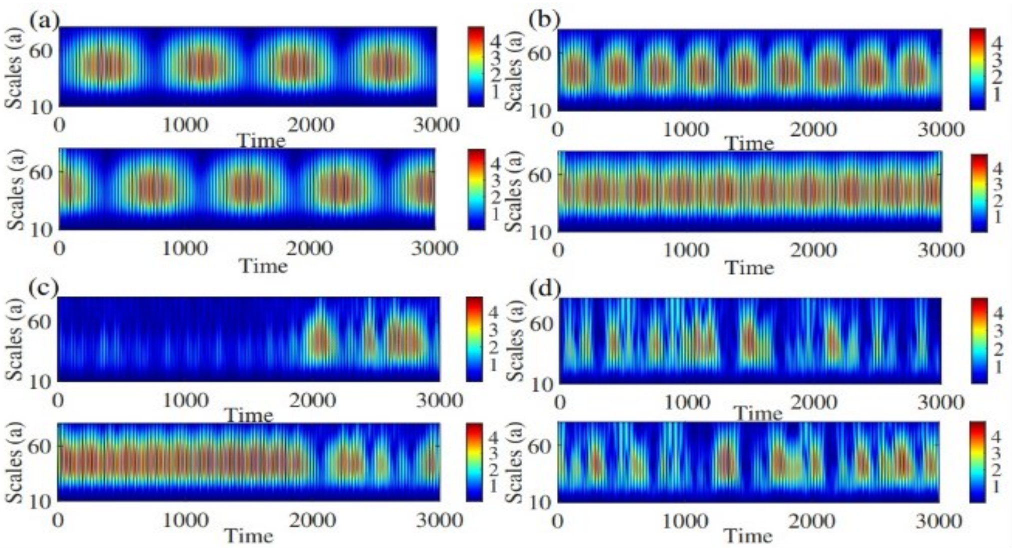

Figure 6: CWT spectra for two nonlinearly coupled HR Hamiltonian (a) QPMS state $k = 2.0$; (b) QP unsynchronized state $k = 3.5$; (c) chaotic unsynchronized $k = 8.0$; (d) chaotic synchronized $k = 19$.

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4. Conclusion

In this study, the dynamical phase transitions leading to measure synchronization in a coupled Helmholtz resonator system governed by a nonlinear Hamiltonian comprising quadratic, cubic, and quartic potential terms are explored.By analyzing the average bare energy, Lyapunov exponents, phase portraits, power spectra, and wavelet analysis, it was observed that the system undergoes a sequence of transitions from periodic to quasiperiodic, and eventually to chaotic dynamics as the coupling strength increases. The emergence of quasiperiodic measure synchronization (QPMS) occurs around $k \approx 1.1$, characterized by the convergence of average energies and overlapping invariant measures. With further increases in the coupling strength, the system experiences intermittent dynamics before reaching a fully developed chaotic measure synchronized (CMS) state at $k > 11.72$. These findings highlight the nontrivial path to MS, underscoring the critical role of coupling strength and nonlinear interactions in shaping collective behavior.

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References

Alamo Vargas, V., Gourdon, E., & Ture Savadkoohi, A. (2018). Nonlinear softening and hardening behavior in Helmholtz resonators for nonlinear regimes. *Nonlinear Dynamics, 91*(1), 217–231.

De, S., Gupta, S., Janaki, M. S., & Sekar Iyengar, A. N. (2018). Frequency and wavelet based analyses of partial and complete measure synchronization in a system of three nonlinearly coupled oscillators. *Chaos, 28*(11), 113108.

Ghosh, A. (2023). Measure synchronization in interacting Hamiltonian systems: A brief review. *Chaos, Solitons & Fractals, 177*, 114237.

Gupta, S., De, S., Janaki, M., & Iyengar, A. (2017). Exploring the route to measure synchronization in non-linearly coupled Hamiltonian systems. *Chaos, 27*, 113103.

Gupta, S., De, S., Janaki, M. S., & Iyengar, A. S. (2019). Using wavelet analysis to investigate synchronization. *Physical Review E, 100*(2), 022218.

Hampton, A., & Zanette, D. H. (1999). Measure synchronization in coupled Hamiltonian systems. *Physical Review Letters, 83*, 2179–2182.

Ingard, U., & Ising, H. (1967). Acoustic nonlinearity of an orifice. *The Journal of the Acoustical Society of America, 42*(1), 6–17.

Lu, Q., Li, X., Zhang, X., Lu, M., & Chen, Y. (2022). Perspective: Acoustic metamaterials in future engineering. *Engineering, 17*, 22–30.

Omoteso, K. A., Ozioko, O., Bagdasar, O., Roy-Layinde, T. O., & Diala, U. H. (2024). Numerical analyses of acoustic vibrational resonance in a Helmholtz resonator. *Nonlinear Dynamics*, 1–21.

Qiu, H., Tian, J., & Fu, L. (2010). Collective dynamics of two-species Bose–Einstein–condensate mixtures in a double-well potential. *Physical Review A, 81*, 043613.

Qiu, H., Yue, G., Fan, H., Liu, X., & Tian, J. (2025). Measure synchronization transition and its critical behavior in coupled camphor rotors. *Chaos, 35*(3).

Tian, J., Chen, Z., Qiu, H., & Xi, X. (2018). Measure synchronization in a Huygens's non-dissipative two-pendulum clocks system. *Journal of the Korean Physical Society, 72*, 208–214.

Tian, J., Li, B., & Qiu, H. (2019). Measure synchronization and phase coherence in a high dimensional Hamiltonian system. *Journal of the Korean Physical Society, 74*(8), 764–770.

Tian, J., Li, B., Liu, T., & Qiu, H. (2019). Measure synchronization and clustering in a coupled pendulum system suspended from a common beam. *Chaos, 29*, 093131.

Tian, J., Qiu, H., Chen, Z., & Chen, Y. (2013). Poincaré section analysis to measure synchronization in coupled Hamiltonian systems. *Modern Physics Letters B, 27*(05), 1350036.

Tian, J., Qiu, H., Wang, G., Chen, Y., & Fu, L. (2013). Measure synchronization in a two-species Bosonic Josephson junction. *Physical Review E, 88*, 032906.

Tian, J., Qiu, H., & Chen, Y. (2010). Nonlocal measure synchronization in coupled Bosonic Josephson junctions. *Chinese Physics Letters, 27*(7), 070501.

Vincent, U. E. (2005). Measure synchronization in coupled Duffing Hamiltonian systems. *New Journal of Physics, 7*, 209.

Vincent, U. E., & Kolebaje, O. (2020). Introduction to the dynamics of driven nonlinear systems. *Contemporary Physics, 61*(3), 169–192.

Wang, X., Li, H., & Hu, K. (2002). Partial measure synchronization in Hamiltonian systems. *International Journal of Bifurcation and Chaos, 12*(5), 1141–1148.

Wang, X., Zhan, M., Lai, C. H., & Gang, H. (2003). Measure synchronization in coupled Hamiltonian systems. *Physical Review E, 67*(6), 066215.