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Fuzzifying The Subgroups of The Cartesian Product of The Generalized Quaternion Group and The Cyclic Group

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ABSTRACT

This research investigates the number of distinct fuzzy subgroups in the Cartesian product of the generalized quaternion group and the cyclic group of order four. Building on prior foundational works and researches , this study extends the characterization of fuzzy subgroups within complex group structures. The generalized quaternion group, known for its non-commutative properties, combined with the cyclic group, creates a rich algebraic environment for examining subgroup behaviors. By employing both theoretical and computational methods, the research identifies and classifies the distinct fuzzy subgroups, revealing new insights into their structures, cardinality, and interrelations. The results are crucial for expanding the mathematical understanding of fuzzy group theory, especially within the realm of finite p-groups. In addition to providing a comprehensive count of the distinct fuzzy subgroups, this project contributes to the ongoing exploration of fuzzy subgroup structures in algebra, with potential applications in cryptography, group theory, and algorithmic mathematics. The findings offer significant progress in understanding the intersection of fuzzy logic and group theory, while also proposing new directions for future work in higher-order generalized quaternion and cyclic groups.

Keywords. Finite p-Groups, Generalized quaternion group, cyclic group, Fuzzy subgroups, Inclusion-Exclusion Principle, Maximal subgroups

1. Introduction

Indeed, a common method of investigation in algebra is to break up a complex structure into simpler substructures. The hope is that my repeated application of this procedure, one will eventually arrive at structures that are easy to understand. It may then be possible, in some sense, to synthesis these substructures to reconstruct the original one. While it is rare for the procedure just described to be brought to such a perfect state of completion, the analytic synthetic method can yield valuable information and suggest new concepts. Group theory in general, considers how far this can be used. In the recent few years, the problem of classifying the fuzzy subgroups of a finite group has experienced very rapid and dynamic developments. One particular case or the other have been treated by several papers such as the finite abelian as well as the non-abelian groups. In [15], the number of distinct fuzzy subgroups of a finite cyclic group of square-free order has been determined. Others have dealt with the number for cyclic groups of order p^nq^m, where p and q are prime numbers. Moreover, a recurrence relation has now been indicated which can successfully be used to count the number of distinct fuzzy subgroups for some classes of finite abelian as well as non abelian groups. The ideas now further extended and continuously been applied severally to other various algebraic structures. The algebraic structures Group, Ring, Field and Vector spaces are important innovations in Mathematics. Most of the theoretical concepts of Mathematics are based on the theorems related to these algebraic structures. Initially many mathematicians developed theorems related to all these algebraic structures. In 20th century most of the researchers introduced the theorems on the algebraic structures with Fuzzy and Intuitionistic fuzzy sets. Recently in 21st century the researchers concentrated on Neutrosophic sets and introduced the algebraic structures like Neutrosophic Group, Neutrosophic Ring, Neutrosophic Field, Neutrosophic Vector spaces and Neutrosophic Linear Transformation. In the current scenario of relating the spaces with the structures, we have introduced the concepts of Neutrosophic topological vector spaces. In this article, the study of Neutrosophic Topological vector spaces has been initiated. Some basic definitions and properties of classical vector spaces are generalized in Neutro-sophic

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environment over a Neutrosophic field with continuous functions. Neutrosophic linear transformations and their properties are also included in Neutrosophic Topological Vector spaces. The fuzzy set generally identifies the topological relationship between two vague spatial objects. Indeterminacy can arise at any point in the modeling process, and the fuzzy model is unable to deal with this. Given that the Pythagorean fuzzy is better equipped to deal with indeterminacies than the fuzzy set, we advocated using Pythagorean fuzzy modeling to determine the topological relation between two spatial objects. In[9], more contributions were made by Subhankar Jana et al to the expanding study of Pythagorean fuzzy topological spaces by introducing core, fringe, outer, regular open set, regular closed set, double connectedness and homeomorphism in Pythagorean fuzzy environment. Some algebraic models such as the pythagorean fuzzy 9-intersection matrix were also proposed to find topological relations between any two Pythagorean fuzzy sets in a Pythagorean topological space. The inbuilt capability of the PFS to handle indeterminacy establishes the proposed model as the potential tool to find topological relations between two indeterminate spatial objects. Extensions to this can also be found in [10]. Moreover, in the course of breaking up complex group structure into simpler substructures, subgroups are formed. Recently, fuzzy multi-sets have come to the forefront of scientists' interest and have been used in algebraic structures such as multi-groups, multirings, anti-fuzzy multigroup and (α,γ)-anti-fuzzy subgroups. In this paper, we first summarize the knowledge about the algebraic structure of fuzzy multi-sets such as (α,γ)-anti-multi-fuzzy subgroups. In a way, the notion of anti-fuzzy multigroup is an application of anti-fuzzy multi sets to the theory of group.The concept of anti-fuzzy multigroup is a complement of an algebraic structure of a fuzzy multi set that generalizes both the theories of classical group and fuzzy group. Now , [13] has been able to highlight the connection between fuzzy multi-sets and algebraic structures from an anti-fuzzification point of view. It was then clarified that the new concept of homomorphism as a bridge among set theory, fuzzy set theory, anti-fuzzy multi sets theory and group theory.In [12], the Pythagorean and Fermatean Fuzzy Subgroups (FFSG) was studied in the context of T-norm and S-conorm functions. There, the extensions of fuzzy subgroups, specifically "Pythagorean Fuzzy Subgroups (PFSG)" and "FFSG" was examined alongside their properties. By incorporating new approach, multiple options were introduced for selecting the minimum and maximum values. Other results were also obtained relating to Pythagorean and FFSG by using the T-norm and S-conorm including other important properties associated with them.In [7], Muhammad Waseem Asghar, et al defined some of the algebraic properties of the concept of η-fuzzy cosets and were able to show that every fuzzy subgroup is an η-fuzzy subgroup.The study of the η-fuzzy normal subgroup and the quotient group was also initiated with respect to the η-fuzzy normal subgroup in line with some of their various group theoretical properties. An attempt to demonstrate the interval-valued fuzzy code have been made [8] by extending the concept of an interval-valued fuzzy set. The operations of the interval-valued fuzzy code were clarified. The interval-valued fuzzy soft code is introduced, and various other related properties were investigated through the use the set of integers modulo two given by: $Z_2 = \{0,1\}$.

In [11], some essential scopes to study the characterizations of the antineutrosophic subgroup and antineutrosophic normal subgroup were given in line with several other theories and properties which are essential for the analysis of their mathematical framework. In [14] , E. Kungumaraj , et al extended the work of fuzzy and intuitionistic fuzzy vector

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spaces which were introduced in fuzzy and intuitionistic fuzzy environments. In [6] Vimala Jayakumar, et al introduced a new concept called hyperlattice-ordered group with practical examples and some properties. It was additionally explored, the features of algebraic hyperstructures as well as the fuzzy algebraic hyperstructures. Furthermore, the relationship between the fuzzy hyperlattice ordered group and the hyperlattice-ordered group have also been investigated.

This work begins with a comprehensive background to the study, outlining the importance of investigating fuzzy subgroups in algebra. In this context, we establish the research problem, emphasizing the gaps in classifying fuzzy subgroups for certain finite groups. The aim and objectives of the research, as well as the methodology used, are presented to provide clarity on the study's direction.Essential preliminaries, such as group theory and fuzzy logic concepts, are also introduced to set the stage for the subsequent discussions.

In the course of this research, we have been able to explore the historical background of logical fuzziness, tracing its evolution and application in modern algebra. We also considered the fundamental properties of two key group structures. In this section, we present computations detailing the distinct fuzzy subgroups that arise from the Cartesian product of Z_4 and Z_{2n} when $n\geq 2$. And finally, the work is also extended to the Computations of the Cartesian product of $Q_{2n}\times Z_4$ for $n\geq 3$.

1.1. Background to the study

The fundamental concept of fuzzy sets was introduced by Zadeh in 1965 to represents information possess- ing non-statistical uncertainties. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer science, control engineering, information science, coding theory, group theory, real analysis, measure theory etc.The first publication in fuzzy set theory by Zadeh (1965) and then by Klaua (1965) showed the intention of the authors to generalize the classical set. In classical set theory, a subset of a set can be defined by its characteristic function $X_A: X \rightarrow \{0, 1\}$ which is defined by :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is in } A \\ 0, & \text{if } x \text{ is not in } A \end{cases}$$

The mapping may be represented as a set of ordered pairs $\{(x, X_A(x))\}$ with exactly one ordered pair present for each element of X. The first element of the ordered pair is an element of the set X and the second is its value in $\{0,1\}$ under X_A. The value '0' is used to represent non-membership and the value '1' is used to represent membership of the element in A. The truth or falsity of the statement "x is in A" is determined by the ordered pair. The statement is true, if the second element of the ordered pair is '1', and the statement is false, if it is '0'. Fuzzy set theory is an extension of classical set theory where elements have varying degrees of membership. A logic based on the two truth values, True and False, is sometimes inadequate when describing human reasoning.

Fuzzy logic uses the whole interval between 0 (false) and 1(true) to describe human reasoning. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval [0,1], with the value of $\mu_A(x)$ at representing the "grade of membership" of x in A. In 1971, Rosenfeld first introduced the concept of fuzzy subgroups, which was the first fuzzification of any algebraic structure and shows that many results in group theory can be extended in an elementary manner to develop the theory of fuzzy group. Gu *et al* (1994) studied the theory of fuzzy groups and developed the concept of M-fuzzy groups. This was an attempt to study M-fuzzy subgroups and its level M-subgroups. The study of classifying the fuzzy subgroups of a finite group has enjoyed a rapid development since its first emergence. This project focus on classifying the number of distinct fuzzy subgroups of the nilpotent group formed by the Cartesian product of the quaternion group of order 2^n and a cyclic group of order four (i.e.$Q_{2n}\times Z_4$). Our utmost aim and desire is in general to find the number of distinct fuzzy

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subgroups of $Q_2n\times Z_2m$ for the cases : $m=n$ and m not equal to n (i.e. $m>n, m<n$).

2. Research Methodology

In this work, one of the essential roles in solving counting problems is played by adopting the Inclusion- Exclusion Principle. The process leads to some recurrence relations from which the solutions are finally computed with ease. In the process of our computation, the use of GAP (Group Algorithm and Programming) [16] was actually applied. In the actual sense, some elements of fuzzy logics come to play in the process of computing certain numbers of subgroups. And so representation, management, manipulation and processing of data and information are very imperative in this fast growing, developing and advancing world, most especially, the non statistical aspect and the uncertainties. This process allows us to obtain explicit formulas of the number of fuzzy subgroups of the Cartesian product $Q_2n\times C_4$ for some positive integers n, where Q_2n and C_4 denote the Quaternion group of order 2^n and the cyclic group of order 4.This requires that we already known the number of maximal subgroups of the Cartesian product, their types and their intersections, then the recurrence relations can be inferred which permit us to determine explicitly the required result.

2.1. Preliminaries

Suppose that $(G,.,e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element λ □$S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are satisfied:

1. $\lambda(ab)\geq\{\lambda(a),\lambda(b)\}, \forall a,b\in G.$
2. $(a^{-1})\geq\lambda(a)\forall a\in G$

And since $(a^{-1})^{-1}=a$, we have that $\lambda(a^{-1})=\lambda(a)$, for any $a\in G$. Also, we have $\lambda(e)=sup\lambda(G)$. (Marius [3]).For subgroups, the set $FL(G)$ consisting of all fuzzy subgroups of G forms a lattice with respect to the usual ordering of fuzzy set inclusion, called the *fuzzy subgroup lattice of* G. For each $\mu\in[0,1]$, we define the level subset:

$${}_{\lambda}G_{\mu}=\{x\in G|\lambda(x)\geq\mu\}.$$

Now, we will describe the method that will be used in counting the chains of an arbitrary finite group G. Let $M_1,M_2,...,M_k$ be the maximum subgroups of G and denoted by $h(G)$ the number of chains of subgroups of G ended in G. The technique to compute $h(G)$ is obtained on applying the Inclusion-Exclusion Principle. Let C be the set of chains in G of type

$$H_1\subset H_2\subset\ldots\subset H_r=G, \qquad (1)$$

C' be the set of chains in G of type

$$H_1\subset H_2\subset\ldots\subset H_r\neq G$$

The Inclusion – Exclusion equation has been used by other researchers to obtain the number of finite subgroups of some finite p-groups and their Cartesian products. We will also apply it here to obtain the h($Q_2n\times Z_4$), (the number of fuzzy subgroups of the Cartesian product of the quaternion group of order $2^n$and a cyclic group of order 4).

3. Literature Review

3.1. Historical background of logical fuzziness

The first important person to introduce fuzzy thinking was Buddha who lived in India about 500 BC and founded a religion called Buddhism. His philosophy was based on the thought that the world is filled with contradictions, that almost everything contains some of its opposite, i.e. things can be right and wrong at the same time. About 200 years later, the Greek philosopher and scientist Aristotle developed binary logic which was contrary to Buddha's. Aristotle thought that the world was made up of opposites, for example good versus bad, right versus wrong, dry versus wet, active versus passive, etc. Everything has to be A or not A, it can't be both. Aristotle's logic was accepted by the Greek scholars and later got spread all over Europe; first by the Romans and then through to the Christian world. Aristotle's binary logic became the base and cornerstone of science; if something got proven with logic, it was and still is accepted as scientifically correct. Buddha's logic was rejected by most scholars, particularly those who subscribed to Christianity. In 1964, Professor Lotfi A. Zadeh, a US mathematician, electrical engineer, computer scientist, artificial intelligence researcher, started wondering, if there wasn't a better logic to use in machinery. He had the

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idea that if you could tell an air-conditioner to work a little faster when it gets hotter, or similar problems, it would be much more efficient than having to give a rule for each temperature. That was the day fuzzy logic, the way we know it today, was born. Zadeh introduced fuzzy logic in a way that appealed to computer scientists and mathematicians. Many mathematicians in the US and Western Europe started to accept fuzziness in mathematics. Indeed nowadays, with fuzzy logic you can tell an air-conditioner to slow down as soon as it gets chilly or to work faster when it gets hotter. In this way fuzzy logic has application in many 2 more machineries such as washing machines, microwave ovens, generally in robotics and computer programming. Engineers, philosophers, psychologists, and sociologists soon became interested in applying fuzzy logic into their sciences. In 1987, the first subway system was built which worked with a fuzzy logic-based automatic train operation control system in Japan. It was a great success and resulted in a fuzzy boom. Universities as well as industries got interested in developing the new ideas using fuzzy logic. Many mathematicians began fuzzifying the classical mathematics and extending it. Thus nowadays we have Fuzzy Topology, Fuzzy Group Theory, Fuzzy Ring Theory, Fuzzy Vector Spaces, in fact almost any mathematical concept can be fuzzified.

3.2. Classifying the fuzzy subgroups of a finite *p*-group

In the recent few years, the problem of classifying the fuzzy subgroups of a finite group has experienced a very rapid and dynamic development. One particular case or the other have been treated by several papers such as the finite abelian as well as the non-abelian groups. In (Akgual, 1988) (cited [5]), the number of fuzzy subgroups of a finite cyclic group of square-free order which are distinct has been determined, while ((Bhattachachanya,1997), (Blackburn,1960), (Blackburn,1965) and (Gagola,1999) (cited [5])) deal with this number for cyclic groups of order $p^n q^n$ where p and q are prime numbers. Moreover, in (Gegang, 2005), a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups, namely: the arbitrary finite cyclic groups and finite elementary abelian p-groups. The explicit formula obtained gave rise to an expression of a well-known central Delannoy numbers (Cossey, 1982). Sequel to the discovery of the existence of p-groups, various work have been done while researches are continuously being carried out from day to day by a number of eminent personalities.

Joseph L. Lagrange, in 1771 had a theorem accredited to him based on finite Group. Meanwhile, he did not prove this theorems all he did, essentially, was to discuss some special cases.The first complete proof of the theorem was provided by Abbati. In 1872, the Normegran mathematician L. Sylow had a collection of theorems on finite group named after him. Moreover, the Sylow theorems have been proved in a number of different ways, and the history of the proofs themselves are the subjects of many papers. Wielandt in 1959 used combinatorial arguments to show part of the Sylow theorems. Frattini had his argument on Sylow subgroups of a normal subgroup which was slightly generalized by Burnside as Burnside's fusion theorem. Others are Brauer, Gorenstein and J. L. Alperin. Some of the current and dynamic researches are being carried on some categories of p-groups by Y.Berkovich and Zvonimir Janko who are from Israel and Germany respectively. Computing the number of distinct fuzzy subgroups of the Cartesian product of the quaternion group of order 2^n and a cyclic group of order 4(i.e., $h(Q_2n\times Z_4)$ was inspired in [1] where $h(Q_2n\times Z_2)$ was computed. This work is based on a further research upon computation of $h(Q_2n\times Z_2)$.

The following theorem was used in [2] citing M. Tarnauceanu (2011) to compute the number of distinct fuzzy subgroups of the Cartesian product of the quaternion group of order 2^n and a cyclic group of order 2 ($h(Q_2n\times Z_2)$).

Theorem 2.1:[3] The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

$$h(Z_{p^n})=2^n$$

$$h(D_{2^n})=2^{2n-1}$$

$$h(\phi_{2^n})=2^{2n-2}$$

$$h(S_{2^n})=3.2^{2n-2}$$

$$h(Z_p\times Z_{p^{n-1}})=h(M_{p^n})=2^{n+1}[2+(n-1)p]$$

Theorem2.2: Let $G=Q_2n\times Z_2$, the nilpotent group obtained by taking the Cartesian product of the generalised quaternion group of order 2^n, and a cyclic group of order 2

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Then, the number of distinct fuzzy subgroups of G is:

$h(G)=n.2^{2n}-2^{n=1}$, for $n>3$.

4. Basic concept of the generalized Quaternion Group and Cyclic group of order four.

The Quaternion group and cyclic group discussed in this research are parts of the family of the finite p-group. More specifically, they are members of the 2-group (finite p-group when p =2).A 2-group plays essential roles in the study of general p-group, in fact, the detailed knowledge about the existence of 2-group leads to the generalization into the p-group as a whole.

5. Generalized quaternion 2-groups

By a generalized quaternion 2-group we mean a group of order 2^n for some natural number $n \geq 3$, defined by the presentation

$$Q_{2^n}=\{a,b \mid a^{2^{n-2}}=b^2, a^{2^{n-1}}=1, b^{-1}ab=a^{-1}\}$$

Also, these groups are the unique finite non cyclic p-groups all of whose abelian subgroups are cyclic, or equivalently the unique finite non cyclic p-groups possessing exactly one subgroup of order p.

The generalized quaternion group Q_2n (for $n\geq 3$) is an extension of the standard quaternion group Q_8, which has 8 elements. The generalized quaternion group Q_2nconsists of 2^nelements and is defined as a non-abelian group where every element either squares to 111 or generates an order of 4.The quaternion groups can be viewed as a specific type of non-commutative 2^n-order groups that generalize properties of rotations in three-dimensional space.

For Q_8 where n=3, the elements are usually expressed and written as:

$$Q_8=\{\pm 1, \pm i, \pm j, \pm k\}$$

Where $i^2=j^2=k^2=ijk=-1$.The relations satisfy:

$$i^4=1, i^2=j^2=k^2=ijk$$

For the generalized form, the group has 2^n elements and retains many structural similarities with the smaller quaternion groups. The core relations in the generalized version include:

$$x^n=1, y^2=x^{n-1}, xy=yx^{-1}$$

Here, x is an element of order2^{n-1}, and y is an element of order

Table1: Multiplication Table for Q_8 To illustrate, the multiplication table of Q_8 is as follows

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

Group Properties:

Non-abelian: This is a key feature. Q_2n is not commutative.

Order : Q_2n has 2^n elements.

Subgroups : It has cyclic subgroups, and its center consists of the identity element and -1.

Application : Quaternion groups of ten arise in the study of symmetries, particularly in three dimensional relations and in the representation theory of certain finite groups.

6. Cyclic Group of Order four

6.1. Definition and Structure:

A cyclic group is a group that can be generated by a single element, meaning all elements in the group are powers of this generator. For the cyclic group of order 4, denoted by C_4, the group consists of 4 elements.

In its most basic form, C_4 is given by:

$$C_4=\{e,a,a^2,a^3\}$$

where e is the identity element and the generator a satisfies $a^4=e$. The group is abelian, meaning that the operation (often denoted multiplicatively or additively) is commutative: Also, we have $a^i \cdot a^j=a^{i+j}$

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Group properties:
Abelian: This is a commutative group.
Order: The group has 4 elements.
Subgroups : It has subgroups of orders 1, 2, and 4.
Cyclic Nature : Since it's cyclic, all elements are powers of the generator.

Table 2 : Multiplication Table for C_4:

	e	a	a_2	a_3
e	e	a	a_2	a_3
a	a	a_2	a_3	e
a_2	a_2	a_3	e	a
a_3	a_3	e	a	a_2

7. Cartesian Product of $Q_{2}n$ and C_4

The Cartesian product of two groups G and H, denoted by $G\times H$, is the set of all ordered pairs (g,h), where where $g\in G$ and $h\in H$, equipped with the group operation defined component-wise:

$g_1,h_1) \cdot (g_2,h_2)=(g_1 \cdot g_2,h_1 \cdot h_2)$

For the general quaternion group $Q_2n$and the cyclic group C_4, the Cartesian product $Q_2n\times C_4$has the following properties:

Structure of $Q_2n\times C_4$

Order : The order of the Cartesian product $Q_2n\times C_4$ is the product of the orders of Q_2n and C_4:

Order $(Q_2n\times C_4)^n=2\times4=2^{n+2}$

Thus, for $Q_8\times C_4$, the group has order $8\times4=32$.

Elements : The elements of $Q_2n\times C_4$are pairs (q,c), where $q\in Q_2n$and $c\in C_4$. The group operation combines the quaternion multiplication in $Q_2n$with the cyclic multiplication in C_4.

Non-Abelian Nature: Since Q_2n is non-abelian, the Cartesian product $Q_2n\times C_4$will also be non- abelian, even though C_4isabelian. The non-commutative nature arises from the quaternion component.

Applications

The Cartesian product $Q_2n\times C_4$ can be applied in fields that study combined symmetries, such as higher- dimensional rotations and crystallography, where the quaternion group's role in 3D rotations is combined with the cyclic symmetry in time or other cyclic operations.

8. The computations of the number of distinct fuzzy subgroups for $Z_4\times Z_2n$, $n>1$

Here, we are going to use equation (1.2) to compute the number of distinct fuzzy subgroups. The list of maximal subgroups are all determined by the use of GAP 16]. $Z_4\times Z_2n,n>1$ is the Cartesian product of the cyclic groups of order $2^n$and cyclic group of order 4.

Lemma4.1:[6] Let G be abelian such that $G=Z_4\times Z_4$. Then $h(G)=2h(Z_2\times Z_22)=48$.

Proof: G has three maximal subgroups in which each of them is isomorphic to $Z_2\times Z_22$. Hence, we have that

$$\frac{1}{-}2h(G) = 3h(Z_2\times Z_22)-3h(Z_2\times Z_22)+h(Z_2\times Z_22)=h(Z_2\times Z_2)$$

$$h(G)=2h(Z_2\times Z_2)$$

Applying Theorem 2.1,i.e $h(Z_p\times Z_pn-1)=2^{n-1}[2+(n-1)p]$ where $p=2$ and $n=3$,we have

$$h(Z_2\times Z_22)=24.\Rightarrow h(Z_4\times Z_4)=48$$

The above lemma takes care of the case where $n=2$, the following theorem will cover the other cases when $n\geq3$.

Theorem4.1: Let G be abelian such that $G=Z_4\times Z_2n$ for $n\geq3$.Then $h(G)=2^{n+1}(n^2+5n-8)$.

Proof:

When $n=3$;$G=Z_4\times Z_23$.This group has 3 Maximal subgroups which are isomorphic to $(Z_4\times Z_22)$,$(Z_2\times Z_23)$,and$(Z_2\times Z_22)$. Applying Equation (1.2),we have;

$$h(Z_4\times Z_{23})=2h(Z_4\times Z_{22})+2h(Z_2\times Z_{23})^2-2h(Z_2\times Z_{22})^2 \quad (3)$$

From the previous lemma and theorems, we have;

$$h(Z_4\times Z_23)=2(48)+4(64)-4(24)=256.$$

When n=4;$G=Z_4\times Z_24$.This group also possessed 3 Maximal subgroups which are isomorphic to $(Z_4\times Z_23)$,$(Z_2\times Z_24)$, *and*

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$(Z_2\times Z_23)$. Applying Equation (1.2),we have,

$h(Z_4\times Z_24)=2h(Z_4\times Z_23)^2+2h(Z_2\times Z_24)-2h(Z_2\times Z_23)^2$ (4)

From the previous lemma and theorems, we have;

$h(Z_4\times Z_24)=2(256)+4(160)-4(64)=896.$

Whenn=5; $G=Z_4\times Z_25$ .The 3 Maximal subgroups here are which are isomorphic to $(Z_4\times Z_24)$,$(Z_2\times Z_25)$, *and* $(Z_2\times Z_24)$. Applying Equation (1.2),we have;

$h(Z_4\times Z_25)=2h(Z_4\times Z_24)^2+2h(Z_2\times Z_25)-2h(Z_2\times Z_24)^2$ (5)

From the previous lemma and theorems, we have;

$h(Z_4\times Z_25)=2(896)+4(384)-4(160)$
$=2688.$

When n=6; $G=Z_4\times Z_26$.This group has 3 Maximal subgroups which are isomorphic to $(Z_4\times Z_25)$,$(Z_2\times Z_26)$, *and* $(Z_2\times Z_25)$. Applying Equation (1.2),we have;

$h(Z_4\times Z_26)=2h(Z_4\times Z_25)^2+2h(Z_2\times Z_26)-2h(Z_2\times Z_25)^2$ (6)

From the previous lemma and theorems, we have;

$h(Z_4\times Z_26)=2(2688)+4(896)-4(384)$
$=7424.$

We can easily notice a pattern following from equations 4.1,4.2,4.3 and 4.4. Hence we have for $n=k$

$h(Z_4\times Z_2k)=2h(Z_4\times Z_2k-1)+2^2h(Z_2\times Z_Zk)-2^2h(Z_2\times Z_2k-1)$ (7)

Using the previous lemma and theorems, we have;

$h(Z_4\times Z_2k)=2h(Z_4\times Z_2k-1)+2^2[2^k(2+2k)-2^{k-1}(2+(n-1)2]$

$=2h(Z_4\times Z_2k-1)+2^{k+2}(k+2)$

We therefore have the recursive relation

$h(Z_4\times Z_2n)=2h(Z_4\times Z_2n-1)+2^{n+1}(n+2)$ for $n\geq 3$. (8)

$=2[h(Z_4\times Z_2n-2)+2^{n+1}(n+1)]+2^{n+2}(n+2).$

$=2h(Z_4\times Z_2n-3)+2^{n+2}[n+(n+1)+(n+2)]$

$=2h(Z_4\times Z_2n-4)+2^{n+2}[(n-1)+n+(n+1)+(n+2)]$

$=2h(Z_4\times Z_2n-4)+2^{n+2}[(n+2)+(n+1)+n+(n-1)]$

Following this pattern, we have;

$h(Z_4\times Z_2n)=2[h(Z_4\times Z_2n-k)+2^{n+2}[(n+2)+(n+1)+n+(n-1)+\ldots+(n-k+3)]$

Let $n-k=2$,which implies $k=n-2$. Hence,

$(Z_4\times Z_2n)=2^{n-2}\ h(Z_4\times Z_22)+2^{\ n+2}\ [(n+2)+(n+1)+n+(n-1)+\ldots+5]$

$=2^{n-2}(48)+2^{n+2}1\ [\ \ (n-2)(n+7)]\ 2$

$=2^{n+1}(6)+2^{n+1}(n^2+5n-14)=2^{n+1}(n^2+5n-8)$

We therefore have : $h(Z_4\times Z_2n)=2^{n+1}(n^2+5n-8)$ for $n\geq 3$.Combining lemma4.1 and theorem 4.1, we obtain the number of distinct fuzzy subgroups for $Z_4\times Z_2n$, $n>1$.

10. The computations of the number of distinct fuzzy subgroups for : $Q_2n\times Z_4$, $n\geq 3$.

We are also going to use equation (1.2) to compute the number of distinct fuzzy subgroups. The list of maximal subgroups is all determined by the use of GAP [16]. $Q_2n\times Z_4,n\geq 3$ is the Cartesian product of the generalized quaternion 2-groups and cyclic group of order 4.

Lemma 5.1:[6] Let G be abelian such that $G=Q_23\times Z_4$.Then $h(G)=120$.

Proof: From Gap [16] we obtained the maximal subgroups, and then using equation 1.2, we have; 1

$=1/2h(Q_23\times Z_4)=h(Q_23\times Z_2)+3h(Z_4\times Z_4)-14h(Z_4\times Z_2)+3h(Z_4)+8h(Z_2\times Z_2)$ 2

$1/2h(Q_23\times Z_4)=88+3(48)-14(24)+3(4)+8(8)=60$

Which implies

$h(Q_23\times Z_4)=2(60)=120$

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The above lemma takes care of the case where n=3, the following theorem will cover the other cases when $n>3$

Theorem 5.1: Let G be abelian such that $G=Q_2n\times Z_4$, for $n>3$.Then

$$h(G)=2^n[2^{n-2}(11n-18)-(4n+3)]-2^{n+1}\quad \Sigma_{i=1}^{n-3}(i)2$$

Proof:

When n=4; $G=Q_24\times Z_4$, Using it's maximal subgroups and equation 2.1,we get;

$$=h(Q_24\times Z_4) = 2h(Q_23\times Z_4)+h(Q_24\times Z_2)+2h(Q_23)+h(Z_23)+h(Z_23\times Z_4)-4h(Z_23\times Z_2)\ 2$$

$$—4h(Q_23\times Z_2)-4h(Z_22)-2h(Z_22\times Z_4)+8h(Z_22\times Z_2)(9)$$

$$h(Q_24\times Z_4)=4h(Q_23\times Z_4)+2h(Q_24\times Z_2)+4h(Q_23)+2h(Z_23)+2h(Z_23\times Z_4)-8h(Z_23\times Z_2)$$

$$—8h(Q_23\times Z_2)-8h(Z_22)-4h(Z_22\times Z_4)+16h(Z_22\times Z_2)(10)$$

From the previous lemma and theorems, we have;

$$h(Q_24\times Z_4)=4(120)+2(992)+4(16)+2(8)+2(256)-8(64)-8(176)-8(4)-4(48)+16(24)$$

$$=1296.$$

When n=5; $G=Q_25\times Z_4$, Using it's maximal sub-groups and equation 2.1,we get;

$$=h(Q_25\times Z_4) = 2h(Q_24\times Z_4)+h(Q_25\times Z_2)+2h(Q_24)+h(Z_24)+h(Z_24\times Z_4)-4h(Z_24\times Z_2)\ 2$$

$$—4h(Q_24\times Z_2)-4h(Z_23)-2h(Z_23\times Z_4)+8h(Z_23\times Z_2)(11)$$

$$h(Q_25\times Z_4)=4h(Q_24\times Z_4)+2h(Q_25\times Z_2)+4h(Q_24)+2h(Z_24)+2h(Z_24\times Z_4)-8h(Z_24\times Z_2)$$

$$—8h(Q_24\times Z_2)-8h(Z_23)-4h(Z_23\times Z_4)+16h(Z_23\times Z_2)(12)$$

From the previous lemma and theorems, we have;

$$h(Q_25\times Z_4)=4(1296)+2(5056)+4(64)+2(16)+2(896)-8(160)-8(992)-8(8)-4(256)+16(64) =8096$$

$$=h(Q_26\times Z_4) = 2h(Q_25\times Z_4)+h(Q_26\times Z_2)+2h(Q_25)+h(Z_25)+h(Z_25\times Z_4)-4h(Z_25\times Z_2)\ 2$$

$$—4h(Q_25\times Z_2)-4h(Z_24)-2h(Z_24\times Z_4)+8h(Z_24\times Z_2)(13)$$

$$h(Q_26\times Z_4)=4h(Q_25\times Z_4)+2h(Q_26\times Z_2)+4h(Q_25)+2h(Z_25)+2h(Z_25\times Z_4)-8h(Z_25\times Z_2)—8h(Q_25\times Z_2)-8h(Z_24)-4h(Z_24\times Z_4)+16h(Z_24\times Z_2)(14)$$

From the previous lemma and theorems, we have;

$$h(Q_26\times Z_4)=4(8096)+2(24448)+4(256)+2(32)+2(2688)-8(384)-8(5056)-8(16)-4(896)+16(160) =43072.$$

$$=h(Q_27\times Z_4)=2h(Q_26\times Z_4)+h(Q_27\times Z_2)+2h(Q_26)+h(Z_26)+h(Z_26\times Z_4)\quad 2—4h(Z_26\times Z_2)-4h(Q_26\times Z_2)-4h(Z_25)-2h(Z_25\times Z_4)+8h(Z_25\times Z_2)(15)$$

$$=h(Q_27\times Z_4)=4h(Q_26\times Z_4)+2h(Q_27\times Z_2)+4h(Q_26)+2h(Z_26)+2h(Z_26\times Z_4)$$

$$—8h(Z_26\times Z_2)-8h(Q_26\times Z_2)-8h(Z_25)-4h(Z_25\times Z_4)+16h(Z_25\times Z_2)(16)$$

From the previous lemma and theorems, we have;

$$h(Q_27\times Z_4)=4(43072)+2(114432)+4(1024)+2(64)+2(7424)-8(896)-8(24448)-8(32)-4(2688)+16(384) =212608.$$

We can easily notice a pattern following from equations 5.2, 5.4,5.6 and 5.8. Hence we have for $n=k$

$$h(Q_2k\times Z_4)=4h(Q_2k-1\times Z_4)+2h(Q_2k\times Z_2)+4h(Q_2k-1)+2h(Z_2k-1)+2h(Z_2k-1\times Z_4)—8h(Z_2k-1\times Z_2)-8h(Q_2k-1\times Z_2)-8h(Z_2k-2)-4h(Z_2k-2\times Z_4)+16h(Z_2k-2\times Z_2)(17)$$

Using the previous theorems and lemma, we have;

$$h(Q_2k\times Z_4)=4h(Q_2k-1\times Z_4)+2[k(2^{2k})-2^{k+1}]+2[2^{2(k-1)-2}]+2[2^{k-1}]+2[2^k((k-1)^2+5(k-1)-8)]-8[2^{k-1}(2+(k-1)2)]-8[(k-1)2^{2(k-1)}-2^k]—8[2^{k-2}]-4[2^{k-1}((k-2)^2+5(k-2)-8)]$$

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$+16[2^{k-2}(2+(k-2)2)](18)$

This is then simplified to get;

$h(Q_2k\times Z_4)=4h(Q_2k-1\times Z_4)+2^k[2^{k-2}(9)+4k-1]$

We therefore have the recursive relation;

$h(Q_2n\times Z_4)=4h(Q_2n-1\times Z_4)+2[2^{nm-2}(9)+4n-1]$ for $n\geq 4$.

From the above recursive relation, we will simplify further to obtain the desired result.

$h(Q_2n\times Z_4)=4h(Q_2n-1\times Z_4)+2[2^{nm-2}(9)+4n-1]$

$h(Q_2n\times Z_4)=4h(Q_2n-2\times Z_4)+(4)2\ [2\ (9)+4n-5]+2[2\ (9)+4n-1]$

$h(Q_2n\times Z_4)=4h(Q_2n-2\times Z_4)+(4)2\ [2\ (9)+4n-5]+2[2\ (9)+4n-1]$

$h(Q_2n\times Z_4)=4h(Q_2n-3\times Z_4)+(4)2\ [2\ (9+4n-9]+(4)2\ [2\ (9)+4n-5]$

$+2^n[2^{n-2}(9)+4n-1]$

$h(Q_2n\times Z_4)=4h(Q_2n-4\times Z_4)+(4)2\ [2\ (9)+4n-13]+(4)2\ [2\ (9)+4n-9]$

$+(4)2^{n-1}[2^{n-3}(9)+4n-5]+2^n[2^{n-2}(9)+4n-1]$

This can be rearranged as;

$h(Q_2n\times Z_4)=4h(Q_2n-4\times Z_4)+2[2\ (9)+4n-1]+(4)2\ [2(9)+4n-5]$

$+(4^2)2^{n-2}[2^{n-4}(9)+4n-9]+(4^3)2^{n-3}[2^{n-5}(9)+4n-13]$

Following this pattern, we obtain;

$h(Q_2n\times Z_4)=4h(Q_2n-k\times Z_4)+2[2\ (9)+4n-1]+(4)2\ [2\ (9)+4n-5]+(4)2\ [2\ (9)$

$+4n-9]+(4^3)2^{n-3}[2^{n-5}(9)+4n-13]+\ldots(4^{k-1})2^{n-(k-1)}[2^{n-(k+1)}(9)+4n-(4k-3)]$

There are k terms in the series. Now, we proceed to expand each bracket and then collect like terms to obtain;

$h(Q_2n\times Z_4)=4h^k(Q_2n-k\times Z_4)+[2^{2n-2}(9)+2^{2n-2}(9)+2^{2n-2}(9)+\ldots+2^{2n-2}(9)]+[2^{n+2}(n)\ 2^{n+3}(n)+2^{n+4}(n)+2^{n+5}(n)+\ldots+2^{n+k-1}(n)]+[2^n(1)+2^{n+1}(5)+2^{n+2}(9+2^{n+3}(13)+\ldots+2^{n+k-1}(4k-3)$

Which implies;

$h(Q_2n\times Z_4)=4^kh(Q_2n-k\times Z_4)+2^{2n-2}(9k)+(n)2^{n+1}\Sigma 2^i-2^n\Sigma(4i-3)2^{i-1}$

$h(Q_2n\times Z_4)=4^kh(Q_2n-k\times Z_4)+2^{2n-2}(9k)+[(n)2^{n+1}+(3)2^{n-1}]\Sigma 2^i-2^{n+1}\Sigma(i)2^i$

Since $^k\ 2^i=2(2^k-1)$, we have;

$h(Q_2n\times Z_4)=4^kh(Q_2n-k\times Z_4)+2^{2n-2}(9k)+2^n(4n+3)(2^k-1)-2^{n+1}\ \Sigma(i)2^i$

Since the quaternion with least order is Q_23, we will let $n-k=3$ so that $k=n-3$, and we now have;

$h(Q_2n\times Z_4)=4^{n-3}h(Q_23\times Z_4)+2^{2n-2}(9(n-3))+2^n(4n+3)(2^{n-3}-1)-2^{n+1}\ \Sigma(i)2^i$

then ; $h(Q_2n\times Z_4)=4^{n-3}h(Q_23\times Z_4)+2^{2n-3}(22n-51)-2^n(4n+3)-2^{n+1}\ \Sigma(i)2^i$

Recall that

$h(Q_23\times Z_4)=120\quad n-3$

$h(Q_2n\times Z_4)=2^{2n-6}(120)+2^{2n-3}(22n-51)-2^n(4n+3)-2^{n+1}\ \Sigma(i)2^i$

$h(Q_2n\times Z_4)=2^{2n-3}(15+22n-51)-2^n(4n+3)-2^{n+1}\ \Sigma(i)2^i$

$h(Q_2n\times Z_4)=2^{2n-2}(11n-18)-2^n(4n+3)-2^{n+1}\ \Sigma(i)2^i$

We therefore have, $h(Q_2n\times Z_4)=2^n[2^{n-2}(11n-18)-(4n+3)]-2^{n+1}\ \Sigma(i)2^i$ for $n\geq 4$.

Combining lemma 5.1 and theorem 5.1, we obtain the number of distinct fuzzy subgroups for $Q_2n\times Z_4, n\geq 3$.

11. Conclusion

In this context, our utmost aim and desire is in general to find the number of distinct fuzzy subgroups of $Q_2n\times Z_2m$ for the cases : $m=n$ and $m\neq n(m>n, m<n)$

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