ABSTRACT
This paper describes the Lagrange based formulation of bit-assignment for multi-hypothesis frames over number of pictures in a group for motion compensated video compression. Theoretical insight on bit assignment is accomplished by applying Lagrange multiplier and its cost function for a real-time computational analysis. Video signals are compressed at optimal bit allocation for qualitative reconstruction at the decoder (receiver) depending upon the number of hypothesis used in coding the signal. This has made multi-hypothesis motion compensation (MHMC) a great process in the prediction of the actual signal through the combination of more than one motion compensated prediction (MCP). Optimal Bits were allocated to multi-hypothesis frames using MATLAB to optimize the number of hypothesis frames per group of pictures (GOP) decoded for an improved signal compression. Simulation results show that an optimal bit assignment of the range 0.089 bpp to 0.159 bpp was realized with eight numbers of hypotheses frames per (GOP). This implies that varying number of hypotheses actualizes optimal bit assignment for motion compensation. This shows a qualitative signal compression performance analysis for high technology motion pictures in communication services and storage processes. It also addresses a better prediction due to greater time correlation between hypotheses and helps in the improvement of coding and reconstruction quality of video signals.

Key words: Bit-rate, frame, Group of Pictures (GOP), Motion Compensation, multi-hypothesis, Video Compression.

INTRODUCTION
In recent years, video compression algorithms have improved tremendously with applications in multimedia such as digital storage and communication, for video games and internet network systems etc. Multi-hypothesis motion compensation (MHMC) has been greatly used in the prediction of the current frame from previous frames with the application of motion vectors and quantization error in coded bit stream for a qualitative real-time video compression analysis. Therefore, bit assignment (bit allocation) for multi-hypothesis frames is important for video data storage and network bandwidth. Additionally, video compression issues is addressed with improved coding standard like H.263 and H.264, which utilizes MHMC with the following codec characteristics: 3D variable length coding (VLC) table, syntax based arithmetic coding (SAC) with about 5% less bits at the same signal to noise ratio (SNR) - Inter frames is approximately 3 to 4% while Intra frames is approximately 10%, unrestricted motion vector (UMV) mode, rate-constrained mode decision, entropy coding model adjustment (CABAC), advanced prediction mode (AP), multi-hypothesis motion estimation issues and other negotiable options like error robustness, Scalability Mode etc (Flierl and Gird, 2003). This made MHMC preferable for low bit rate applications. An optimal hypothesis selection and predictor coefficient selection algorithm was proposed (Flierl et al., 1998). A rate-constrained mode decision algorithm was also proposed in Flierl et al. (2002). The collective utilization of MHMC (2hypotheses), variable block-size coding, and long-term memory (10 reference frames) proved that 30% bit-rate for the same video quality is achieved through computational simulation (Flierl et al., 2002). Rate-distortion optimization (RDO) has been used to overcome the problem of minimal distortion at high rate to improve the performance of codec in video compression system (Li et al., 2009).
Outstandingly, mathematical models are now developed to characterize the error propagation effect caused by packet losses in MHMC coder (Kung et al., 2004a, 2004b). Error resilience property of the 2HMCP, which presented a decoder distortion model (given channel conditions) and a simplified encoder distortion model has also been addressed (Lin and Wang, 2002). Optimal bit allocation for picture ordering has been presented in Sam (2012). Lastly, bit allocation for MHMC has also been presented in closed form expression in Sam (2015).

**BACKGROUND THEORY**

As shown in Figure (1), all standard hybrid video systems like Motion Picture Expert Group 1, 2 and 4 (MPEG-1, MPEG-2, MPEG-4), International Telecommunication recommendation ITU-T Rec. H.263 and H.264 encodes set of image signals to reduce spatial and temporal redundancy by dividing the frames of video signal into group of pictures (GOP). Thereafter, classifying and ordering each frame into I, P or B frames (intra, inter and Bi-directional frames). Image data in each frame are split into regions (blocks) for the prediction of a block in the current frame, using reference frames as the previously decoded frames (last I or P-frames). M and N represent the distance between core pictures and the number of core pictures respectively. Therefore, $M \times N = Q$ which means the total number of pictures in a group of pictures.

**Multi-Hypothesis Motion Compensation (MHMC)**

Multi-hypothesis motion compensation simply involves the application of motion vectors (block bases) to an image to synthesize the transformation to the next image through efficient and accurate measurement of the displacement field between two frames (spatial-temporal displacement). It has found many applications in video coding such that coders employ motion-compensated prediction signals that are superimposed to predict the original frame for easy reconstruction at the decoder. Therefore, motion vectors MVs are applied to describe the transformation from one 2-D image to another, usually from adjacent video frames in a video sequence. Successive video frames may contain the same objects (still or moving). Therefore, it examines the movement of objects in an image sequence and tries to obtain vectors representing the estimated motion. As a means of exploiting the temporal redundancy, it is a key part of video coding and compression. It may relate to the whole image or specific parts such as rectangular blocks, arbitrarily shaped patches or even per pixel. Motion vectors MVs may be represented by model or many other models that can approximate the motion of a real video camera, such as rotation and translations in all three dimensions (3D) and zoom.

**Methods for Motion Compensation**

**Bi-directional Motion Compensation**

As shown in fig 2 the current frame is predicted with reference pictures before and after. This makes it unsuitable for interactive application due to the delay of several frames. It is an example of multi-hypothesis motion-compensated prediction where two motion-compensated predicted signals are superimposed to reduce the bit-rate of a video codec. Therefore, different coding standards: Motion Picture Expert Group: MPEG-1 and MPEG-2 or ITU-T Rec.H.263 utilizes B-pictures for bi-directional motion compensation. Motion estimation problem is solved with the iteration of motion vectors MVs by independent search method which includes minimization of the prediction error by fixing and searching the forward and backward (MVs). So, for joint estimation of forward and backward motion vectors in Bi-directional prediction, a low-complexity iterative algorithm was introduced to minimize the prediction error without considering the rate constraint (Wu and Gersho, 1994).
Overlapped Block Motion Compensation (OBMC)
Overlapped block motion compensation is another example of multi-hypothesis motion compensated prediction similar to bi-directional prediction (Orchard and Sullivan, 1994). This is specifically aimed at reducing the blocking artifacts caused by block-based motion compensation. OBMC does not increase the number of vectors per block unlike the bi-directional prediction, but uses more than one motion vector for predicting the same pixel.

Multi-frame Motion Compensation
As shown in figure 3, this method of motion compensation utilizes multiple frames with additional reference picture parameter chosen to predict the current frame. Therefore, multi-frame block motion compensation makes use of redundancy that exist across multiple frames in typical video conferencing sequences to achieve additional compression over that obtained by single frame block motion compensation.

MATHEMATICAL ANALYSIS OF OPTIMAL BIT ASSIGNMENT
By minimizing the average reconstruction error variance $\sigma_r^2$ [11] as a distortion we allocate bit-rate utilizing the Lagrange multiplier ($\lambda$) method (Li et al., 2009; Flierl et al., 1998) with a Lagrange cost function $j$ given by

$$j = D + \lambda R_c$$  \hspace{1cm} (1)

Where, $D$ and $R_c$ are distortion and constant bit-rate constraints respectively.

Taking into account the constant bit-rate constraints $R_c$ given by

$$R_c = R_i + (Q-1)R_{MHP} - QR_c$$  \hspace{1cm} (2)

Where, $\min j$ is the minimum Lagrange cost function given by

$$\min j = \min \left\{ \frac{1}{Q} \left[ \sigma_{r,i}^2 + (Q-1)\sigma_{MHP}^2 \right] + \lambda \left[ R_i + (Q-1)R_{MHP} - QR_c \right] \right\}$$  \hspace{1cm} (4)

Where, $\min j$ is the minimum Lagrange cost function.
By relating equation (1) and equation (4) the derivative of the function \( j \) is given by
\[
\frac{dj}{dR} = \frac{dD}{dR} + \lambda = 0
\] (5)

Since
\[
j(R_{\text{MHP}}, R, \lambda) = \frac{1}{Q} \left[ \sigma^2_{r,j} + (Q-1)\sigma^2_{\text{MHP}} \right] + \lambda [R_j + (Q-1)R_{\text{MHP}} - QR] \] (Flierl et al., 2002).

Then
\[
\frac{\partial j}{\partial R_{\text{MHP}}} = \frac{1-Q}{2Q} \log_2 \left[ \frac{\sigma^2_{r,j}}{\sigma^2_{\text{MHP}}} \right] + \lambda (Q-1) = 0,
\]
\[
\lambda = \frac{1}{2Q} \log_2 \left[ \frac{\sigma^2_{r,j}}{\sigma^2_{\text{MHP}}} \right],
\] (6)

Since
\[
R_{\text{MHP}} = R + \lambda.
\] (7)

Substituting equation (6) into equation (7) we have
\[
R_{\text{MHP}} = R + \frac{1}{2Q} \log_2 \left[ \frac{\sigma^2_{r,j}}{\sigma^2_{\text{MHP}}} \right]
\] (8)

Hence, where \( \sigma^2 \) and \( R_{\text{MHP}} \) are the quantization performance factor and bit rates for the multi-hypothesis frames respectively and \( R \) is the overall bit rate, we now also obtain that for the cases of P-frame, 2, 3, 4 and 8 hypotheses respectively given
\[
R_{\text{P-frame}} = R + \frac{1}{2Q} \log_2 \left[ 2 - 2\rho^2_{\text{P}} \rho^2_{\text{P}} \right],
\] (9)
\[
R_{\text{2MHP}} = R + \frac{1}{2Q} \log_2 \left[ \frac{3}{2} - 2\rho^2_{\text{P}} \rho^2_{\text{P}} + \frac{1}{2} \rho^2_{\text{2MHP}} \right],
\] (10)
\[
R_{\text{3MHP}} = R + \frac{1}{2Q} \log_2 \left[ \frac{4}{3} - 2\rho^2_{\text{P}} \rho^2_{\text{P}} + \frac{2}{3} \rho^2_{\text{3MHP}} \right],
\] (11)
\[
R_{\text{4MHP}} = R + \frac{1}{2Q} \log_2 \left[ \frac{5}{4} - 2\rho^2_{\text{P}} \rho^2_{\text{P}} + \frac{1}{2} \rho^2_{\text{2MHP}} + \frac{1}{4} \rho^2_{\text{4MHP}} \right],
\] (12)
\[
R_{\text{8MHP}} = R + \frac{1}{2Q} \log_2 \left[ \frac{9}{8} - 2\rho^2_{\text{P}} \rho^2_{\text{P}} + \frac{1}{4} \rho^2_{\text{2MHP}} + \frac{1}{8} \rho^2_{\text{3MHP}} + \frac{1}{4} \rho^2_{\text{4MHP}} + \frac{1}{4} \rho^2_{\text{8MHP}} \right]
\] (13)

MATERIALS AND METHODS
In this work, bit-rates for Multi-hypothesis (equation 9-13) were analysed and optimized with MATLAB in decoding signals of 100 pictures per group of pictures (GOP). Editor’s script and command window of MATLAB were used to run the analysis with the following simulation procedures

- Initially, output functions (bit-rate of multi-hypothesis frames) such as RP, R2MHP, R3MHP, R4MHP, R8MHP and input parameters \( \rho_{\text{P}}, \rho_{\text{2MHP}}, \Delta_{\text{P}}, \Delta_{\text{2MHP}} \) and \( \Delta_{\text{R}} \) were used.
- Also, we generated script for the output functions in the editor’s script environment of the MATLAB by defining a log-function for the input parameters as \( [\text{RP}, \rho_{\text{2MHP}}, \rho_{\text{3MHP}}, \rho_{\text{4MHP}}, \rho_{\text{8MHP}}] = \log (\rho_{\text{P}}, \rho_{\text{2MHP}}, \rho_{\text{3MHP}}, \rho_{\text{4MHP}}, \rho_{\text{8MHP}}) \)
- In the first case, we set \( \rho_{\text{P}}, \rho_{\text{2MHP}}, \rho_{\text{3MHP}}, \rho_{\text{4MHP}}, \rho_{\text{8MHP}} \) and \( Q \) at 0.8, 0.93, 1.0, 0.54, 1.0 and 1-100 respectively.

A script is then used to run a log function defined on the command window for a reasonable optimization.

In the second case, \( \rho_{\text{P}}, \rho_{\text{2MHP}}, \rho_{\text{3MHP}}, \rho_{\text{4MHP}}, \rho_{\text{8MHP}} \) and \( Q \) were set at 0.8, 0.93, 0.3, 0.64, 1.0 and 1-100 respectively. A script is then used to run a log function defined on the command window for a reasonable optimization.

In the third case, we also set \( \rho_{\text{P}}, \rho_{\text{2MHP}}, \rho_{\text{3MHP}}, \rho_{\text{4MHP}}, \rho_{\text{8MHP}} \) and \( Q \) at 0.95, 0.93, 1.0, 0.50, 1.0 and 1-100 respectively. A script is then used to run a log function defined on the command window for a reasonable optimization.

Lastly, setting \( \rho_{\text{P}}, \rho_{\text{2MHP}}, \rho_{\text{3MHP}}, \rho_{\text{4MHP}}, \rho_{\text{8MHP}} \) and \( Q \) at 0.95, 0.93, 0.3, 0.95, 1.0 and 1-100 respectively, a script is then used to run a log function defined on the command window for a reasonable optimization.
The output values of the optimal bit-rates for multi-hypothesis frames were obtained with the given values of the input parameters on the command window due to the generated script in each case.

Hence, we plotted a two dimensional (2-D) graph of bit-rates for multi-hypothesis frames against number of pictures per group for each case using plot command which contains grid, legend of strings and colors with x and y label for Q and the output functions R respectively as:

\[
\text{plot}(Q, \text{RP}, 'm.', Q, \text{R2MHP}, 'b--', Q, \text{R3MHP}, 'k-.', Q, \text{R4MHP}, 'k-', Q, \text{R8MHP}, 'r:');
\]

Finally, the display of the 2-D graphics plot shows the performance limit of bit assignment for multi-hypothesis frames in motion compensated signal compression.

**RESULTS AND DISCUSSION**

Setting the values of parameters as given in Figure 4, bit-rate of 8MHP frames is optimal at 0.948 bpp while that of P frames is optimal at 0.747 bpp. Therefore, an optimal bit-rate of 0.201 bpp is collectively obtained for 8MHP frames over P frames.

**Figure 4:** Bit-rates for multi-hypothesis with \( \rho_r, \rho_v, \Delta_r, \Delta_v, R, Q \) at 0.8, 0.93, 1.0, 0.54, 1.0, 1 – 100 respectively.

Setting the values of parameters as given in Figure 5, bit-rate of 3MHP frames is optimal at 0.964 bpp while that of P frames is optimal at 0.852 bpp. Therefore, an optimal bit-rate of 0.112 bpp is collectively obtained for 3MHP frames over P frames.

**Figure 5:** Bit-rates for multi-hypothesis with \( \rho_r, \rho_v, \Delta_r, \Delta_v, R, Q \) at 0.8, 0.93, 0.3, 0.64, 1.0, 1 – 100 respectively.

Setting the values of parameters as given in Figure 6, bit-rate of 8MHP frames is optimal at 1.638 bpp while that of P frames is optimal at 1.197 bpp. Therefore, an optimal bit-rate of 0.441 bpp is collectively obtained for 8MHP frames over P frames.

**Figure 6:** Bit-rates for multi-hypothesis with \( \rho_r, \rho_v, \Delta_r, \Delta_v, R, Q \) at 0.8, 0.93, 1.0, 0.54, 1.0, 1 – 100 respectively.
**DISCUSSION**

The Figures above (4, 5, 6 and 7) demonstrate the effect of the number of pictures in a group over the optimal bit-rates of multi-hypothesis frames (MHP-frames) using inter-frame video coding with the values of the parameters such as time correlation coefficient ($\rho_t$), radial displacement ($\Delta_r$), time distance between pictures ($\Delta_t$) and spatial correlation coefficient ($\rho_s$) suggested by video sequence (Ohta and Katto, 1995). A MATLAB program is used for the simulation of the parametric equation of optimal bit-rates for the multi-hypothesis frames (9 to 13) over numbers of pictures in a group of pictures (GOP). The optimal bit-rates for multi-hypothesis frames were varied over 100 pictures per group with the radial displacement ranging from 0.3 to 1.0. An optimal bit-rate is obtained with three hypotheses from the Figures (4, 5, 6 and 7) which signify a better prediction and reconstruction of frames using up to eight hypotheses in order to achieve improved compression of video signal transmission. In Figures 4 and 5, an optimal bit-rate of 0.201 bpp and 0.112 bpp is obtained and allocated for 8MHP frames and 3MHP frames respectively over P frames when 100 pictures in a group are considered. Therefore bit-rate decreases for about 0.089 bpp when time correlation coefficient ($\rho_t$) = 0.8 and radial displacement ($\Delta_r$) ranges from 1.0 to 0.3 at a greater time interval ($\Delta_t$) of 0.54 to 0.64.
In Figures 6 and 7, an optimal bit-rate of 0.441 bpp and 0.282 bpp is obtained and allocated for 8MHP frames and 3MHP frames respectively over P frames when 100 pictures in a group are considered. Therefore bit-rate decreases for about 0.159 bpp when time correlation coefficient ($\rho_t$) = 0.95 and radial displacement ($\Delta_r$) ranges from 1.0 to 0.3 at a greater time interval ($\Delta_t$) of 0.5 to 0.95. Also, a greater time correlation coefficient ($\rho_t$) of the range 0.8-0.95 between picture elements with optimal bit assignment of 0.089 bpp-0.159 bpp implies a better video compression for digital storage and transmission.

**CONCLUSION**

Bit-rate for multi-hypothesis frames has been theoretically and computationally analyzed in order to compensate their assignment over number of pictures per group for an achievable video signal compression quality. An optimal bit-rate of the range 0.089 bpp to 0.159 bpp was achieved with up to eight hypotheses. This shows a qualitative signal compression performance analysis for high technology motion compensated digital video codec in communication services and storage processes.

**REFERENCES**


