EFFICIENCY OF MODIFIED GENERALIZED IMPUTATION SCHEME FOR
ESTIMATING POPULATION MEAN WITH KNOWN AUXILIARY
INFORMATION

Adejumobi, A.,¹  Audu, A.,² Yunusa, M. A.² and Singh, R. V. K.¹
¹Department of Mathematics, Kebbi State University of Science and Technology, Aliero, Nigeria
²Department of Statistics, Usman Danfodiyo University, Sokoto, Nigeria
*Corresponding Author Email: awwaladejumobi@gmail.com

ABSTRACT
Different authors for estimating population mean have proposed several Imputation
schemes. Recently, some authors have suggested generalized imputation schemes that
their estimators are functions of unknown parameters of the study variable. These
unknown parameters need to be estimated for the estimators to be applicable and this
may require additional resources. This paper considered a class of imputation scheme
that is independent of unknown parameter and the point estimator of the suggested
scheme for estimating population mean was derived. The properties (bias and MSE) of an
efficient estimators presented were derived up to first order approximation and also
conditions for which the estimators of the proposed scheme is more efficient than other
estimators of the existing schemes considered in the study were also examined. The
result of the empirical study revealed that the suggested estimators are more efficient
than the existing ones considered in the study.
Keywords: Imputation, Missing Information, Population mean, auxiliary variable,
Efficiency.

INTRODUCTION
In sample surveys, missing information on sampled units is a relevant and crucial
observation. The demographic surveys, social-economic survey, clinical and agricultural
experiments are the fundamental examples of this. It has been proved by some survey
researchers in their findings that the inferences of unknown population parameters can be
spoiled due to missing information. Therefore, the suitable methodology of estimating
population parameters may be used to handle the statistical datasets in case of missing or
incomplete information. The common technique used to handle situations where data is missing
is Imputation. Missing values can be completed with specific substitutes and data can be
analyzed using standard methods. Information about unit of characteristic of interest observed
and auxiliary variable help improve the accuracy of demographic parameter estimates (Pandey et
al., 2021).
Hansen and Hurwitz (1946) was the first researcher who considered the problem of non-
response. Many researchers also worked on imputation methods to deal with non-response
and missing values among them include; Lee et al. (1994), Singh and Horn (2000), Singh and
Deo (2003), Toutenburg et al. (2008), Singh (2009), Wang and Wang (2006), Kadilar and
Cingi (2008), Diana and Perri (2010), Al-Omari et al. (2013), Singh et al. (2014), Gira (2015),
Singh et al. (2016), Bhushan and Pandey (2016), Prasad (2016), Audu et al. (2020a,b,c),
Audu et al. (2021a,b,c,d), Audu and Singh (2021), Yusuf et al. (2022). However, the
estimators of the scheme proposed by Audu et al. (2020a) and Pandey et al. (2021) are
functions of the unknown parameters of the study variable which makes the schemes and estimators impracticable in real life application unless if the unknown parameters are estimated
using large sample which may required additional resources in the conduct of survey.
Audu et al., (2020) suggested On the Class of Exponential-Type Imputation Estimators of
Population Mean with Known Population Mean of Auxillary Variable.
BAJOPAS Volume 15 Number 1, June, 2022

\[ y_i = \begin{cases} \frac{n}{r} y_i & i \in R \\ \frac{n}{n-r} \bar{y}_r \left( \theta_2 \left( \frac{X}{X_r} \right) + \theta_3 \left( \frac{\kappa_2 (X-\bar{X}_r)}{X + \bar{X}_r} \right) \right) & i \in R' \end{cases} \]  

(1)

where \( \kappa_1, \kappa_2 \in (1, -1) \)

The point estimators of population mean from the proposed schemes in (1) are obtained as

\[ t_p = \bar{y}_r \left( \theta_1 + \theta_2 \left( \frac{X}{X_r} \right) + \theta_3 \exp \left( \frac{\kappa_2 (X-\bar{X}_r)}{X + \bar{X}_r} \right) \right) \]  

(2)

\[ \text{Bias} \left( t_p \right) = \bar{Y} \lambda_{r,n} \left( \theta_2 \frac{\kappa_1 (\kappa_1 + 1)}{2} + \theta_3 \frac{\kappa_2 (\kappa_2 + 2)}{8} \right) C_X^2 - \left( \theta_2 \kappa_1 + \theta_3 \frac{\kappa_2}{2} \right) \rho_{XY} C_X C_Y \]  

(3)

\[ \text{MSE} \left( t_p \right) = \bar{Y}^2 \lambda_{r,n} \left( C_Y^2 + \psi^2 C_X^2 - 2 \psi \rho_{XY} C_X C_Y \right) \]  

(4)

where \( \psi = \rho_{XY} C_Y / C_X \)

The expressions for \( \theta_i, i = 1, 2, 3 \), are as follows

\[ \begin{align*} 
\theta_3 &= 4 \left( 2^{-1} (\kappa_1 + 1) C_X - \rho_{XY} C_Y \right) / \kappa_2 \left( \kappa_1 - \kappa_2 / 2 \right) C_X^2 \\
\theta_2 &= - \left( 2^{-1} (\kappa_2 + 2) C_X - 2 \rho_{XY} C_Y \right) / \kappa_2 \left( \kappa_1 - \kappa_2 / 2 \right) C_X^2 \\
\theta_1 &= 1 + \left( 2 \left( 4^{-1} \kappa_1 (\kappa_2 + 2) - \kappa_1 (\kappa_1 + 1) \right) C_X \right) / \kappa_2 \left( \kappa_1 - \kappa_2 / 2 \right) C_X^2 \\
&- \left( \kappa_2 - 2 \kappa_1 \right) \rho_{XY} C_Y 
\end{align*} \]  

(5)

Audu and Singh (2021) proposed Exponential-type regression compromised imputation class of estimators, the generalized class of imputation scheme given as

\[ y_i = \begin{cases} y_i & i \in \Phi \\ \frac{\mu_0 + \beta x (X-\bar{X}_r)}{\pi_1 \bar{X} + \pi_2} (\pi_1 \bar{X} + \pi_2) \exp \left( \frac{\sigma_i (X-\bar{X}_r)}{\sigma_i (\bar{X} + \bar{X}_r) + 2 \sigma_2} \right) & i \in \Phi' \end{cases} \]  

(6)

where \( \pi_1 \) and \( \pi_2 \) are known functions of auxiliary variables like coefficient of skewness \( \beta_{x(i)} \), kurtosis \( \beta_{k(i)} \), variation \( C_s \), standard deviation \( S_s \) etc.

Note that \( \pi_1 \neq \pi_2 \) and \( \pi_1 \neq 0 \)

The estimator, bias and MSE of the Imputation scheme in (6) are given as in (7), (8) and (9) respectively.

\[ \mu_i^{(\pi)} = \frac{\mu_0 + \beta x (X-\bar{X}_r)}{n \pi_1 \bar{X} + \pi_2} (\pi_1 \bar{X} + \pi_2) \exp \left( \frac{\sigma_i (X-\bar{X}_r)}{\sigma_i (\bar{X} + \bar{X}_r) + 2 \sigma_2} \right) \]  

(7)

\[ \text{Bias} \left( \mu_i^{(\pi)} \right) = \psi_{r,n} \left( 1 - \frac{r}{n} \right) \left( \beta_{x(i)} (\eta_1 + \eta_2) + \bar{Y} (\eta_1 + \eta_2 - 1.5 \eta_2^2) C_{XY} \right) \]  

(8)

\[ \text{MSE} \left( \mu_i^{(\pi)} \right) = \psi_{r,n} \left( S_s^2 + \gamma^2 S_X^2 - 2 \gamma S_{XY} \right) \]  

(9)

106
\[ \eta_1 = \frac{\pi_1 \bar{X}}{\pi_1 \bar{X} + \pi_2}, \quad \eta_2 = \frac{\sigma_1 \bar{X}}{2(\sigma_1 \bar{X} + \sigma_2)} \quad \text{and} \quad \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]

Pandey et al., (2021) proposed improved estimators for mean estimation in presence of missing information.

\[ Y = x_i \phi_i + \xi_i \]

\[ \gamma = \left(1 - \frac{r}{n}\right) \left(R(\eta_1 + \eta_2) + \beta_{rg}\right), \]

\[ \beta_{rg} = \frac{S_x^2}{\sum_{i=1}^{n} x_i^2}. \]
The point estimators of finite population mean under the proposed scheme is obtained as:

\[ T_{fs} = \frac{1}{n} \sum_{i \in \tau} y_i + \sum_{i \in \tau'} \left[ \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r \right] \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] \]

where \( F = \left( (a \bar{x})/(a \bar{x} + b) \right); s = 1, 2, \ldots, 10 \) for different choices of \( a(\neq 0) \) and \( b \).

Remark 2.2: The proposed class of imputation estimators is independent of unknown parameter, hence it is practically applicable.

**Table 1**: Some member of \( T_{fs} (s=1, 2, \ldots, 10) \) for different values of \( a \) and \( b \)

<table>
<thead>
<tr>
<th>( s )</th>
<th>Proposed members of the class</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( T_{f1} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2.</td>
<td>( T_{f2} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.</td>
<td>( T_{f3} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>( \beta_{(1)} )</td>
<td>( \beta_{(1)} )</td>
</tr>
<tr>
<td>4.</td>
<td>( T_{f4} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>( \beta_{(1)} )</td>
<td>( \beta_{(1)} )</td>
</tr>
<tr>
<td>5.</td>
<td>( T_{f5} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>( \beta_{(1)} )</td>
<td>( \beta_{(1)} )</td>
</tr>
<tr>
<td>6.</td>
<td>( T_{f6} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>( S_{(1)} )</td>
<td>1</td>
</tr>
<tr>
<td>7.</td>
<td>( T_{f7} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>( S_{(1)} )</td>
<td>( \beta_{(1)} )</td>
</tr>
<tr>
<td>8.</td>
<td>( T_{f8} = \frac{F}{2} \left( \frac{x_i - \bar{x}}{x_i} \right) + w_i (x_i - \bar{x}) + w_2 \bar{y}_r ) | \exp \left[ \frac{F_i (1 - \frac{x_i}{\bar{x}})}{1 + (a \bar{x} + b)/(a \bar{x} + b)} \right] ]</td>
<td>( S_{(1)} )</td>
<td>( \beta_{(1)} )</td>
</tr>
</tbody>
</table>
Properties of the Suggested Estimator

In this section, the bias and MSE of the suggested estimators in this paper are derived and discussed properly.

\[
\begin{align*}
\frac{\bar{Y}}{\bar{Y}} &= (1 + e_0), \quad \frac{\bar{X}}{X} = (1 + e_1), \quad \frac{\bar{X}}{X} = (1 + e_2), \\
E(e_0) &= E(e_1) = E(e_2) = 0, \quad E(e_0^2) = \theta C_x^2, \quad E(e_1^2) = \theta C_y^2, \\
E(e_2^2) &= E(e_1 e_2) = \theta C_x^2, \quad E(e_0 e_1) = \theta \rho C_x C_y, \quad E(e_0 e_2) = \theta \rho C_y C_x \\
\theta &= \left(\frac{1}{r - \frac{1}{N}}\right), \quad \theta' = \left(\frac{1}{n - \frac{1}{N}}\right), \quad \bar{X} = \gamma, \quad \bar{Y} = \gamma' = F_r, \quad Y = \frac{\bar{X}}{\bar{Y}}
\end{align*}
\]

**Theorem 1:** To $O(n^{-1})$, the bias of the suggested estimator $T_{fs}$ is:

\[
Bias(T_{fs}) = \bar{Y} \left[ \left(1 + 3\theta^2\right) \frac{\lambda C_x^2}{2} - \frac{\lambda C_y^2}{2} - \theta \lambda \rho C_y C_x, C_x \right] + w_1 \theta C_x^2 (\lambda - \lambda_1) + w_2 \left[1 + \frac{3\theta^2\lambda C_x^2}{2} - \theta \lambda \rho C_y C_x \right]
\]

**Proof:** Express (19) in terms of $e'_s$, we have

\[
T_{fs} = \left[ \frac{\bar{Y}(1 + e_0)}{2} \right] \left[ (1 + e_1)^{(1 - e_1)^{-1}} + (1 + e_2)^{(1 + e_2)^{-1}} \right] \exp \left[ \frac{F_s^2 \left(1 - (1 - e_1)\right)}{1 + (a\bar{X} + a\bar{e}_1 b)/(a\bar{X} + b)} \right]
\]

where $F_s = \frac{a\bar{X}}{a\bar{X} + b}$ for different suitable choices of $a$ and $b$, and $|F_s e_1| < 1$ so that the term $(1 + F_s e_1)^{-1}$ is convergent.

where

\[
F_1 = 1, \quad F_2 = \bar{X} \left(\bar{X} + 1\right)^{-1}, \quad F_3 = \bar{X} \left(\bar{X} + \beta_{2(s)}\right)^{-1}, \quad F_4 = \bar{X} \left(\bar{X} + \beta_{1(s)}\right)^{-1}, \quad F_5 = \frac{\bar{X}}{\bar{X} + \rho_{yx}}, \quad F_6 = \frac{s X}{s \bar{X} + 1}, \quad F_7 = \frac{s X}{s \bar{X} + \beta_{1(s)}}, \quad F_8 = \frac{s X}{s \bar{X} + \rho_{yx}}, \quad F_9 = \frac{\beta_{1(s)} \bar{X}}{\beta_{1(s)} \bar{X} + \beta_{2(s)}}, \quad F_10 = \frac{C_s \bar{X}}{C_s \bar{X} + \beta_{2(s)}}
\]

Simplify (23) up to $O(n^{-1})$, we have
\[ T_{\beta} - \bar{Y} = \bar{Y} \left( e_0 - \theta, e_1 + e_1^2 + \frac{3\theta^2 e_1^2}{2} + e_2^2 - e_1 \theta e_0 e_1 \right) + w_1 \bar{X} \left( e_2 - \theta, e_2 - e_1 + \theta, e_1^2 \right) \]
\[ + w_2 \bar{Y} \left( 1 - \theta, e_1 + \frac{3\theta^2 e_1^2}{2} + e_0 - \theta, e_0 e_1 \right) \]  
(25)

Subtract \( \bar{Y} \) from both sides of (25), take expectation and apply the results of (20), Theorem (20) is proved.

**Theorem 2:** To \( O(n^{-1}) \), the MSE of the suggested estimator \( T_{\beta} \) is:

\[ MSE(T_{\beta}) = \bar{Y}^2 \left[ A + w_1^2 B + w_1 C + 2 w_1 D + 2 w_1 E + 2 w_1 w_2 F \right] \]  
(26)

where \( A = \gamma(C_1^2 - 2\theta, \rho C_1, C_x), \quad B = \gamma^2 C_1^2 (\lambda - \lambda_1), \)
\[ C = 1 + \lambda(C_1^2 + 3\theta, \rho C_1, C_x) + \theta, \lambda C_x, \quad D = \gamma(\lambda_1(\rho C_1, C_x - \theta, C_x) - \lambda(\rho C_1, C_x + \theta, C_x)), \]
\[ E = \gamma(\lambda_1(C_1^2 - 3\theta, \rho C_1, C_x + (2.5\theta, \rho C_1, C_x - 0.5) C_x) - 0.5\lambda_1 C_x, F = \gamma(2\theta, C_1^2 (\lambda - \lambda_1) + \rho C_1, C_x (\lambda_1 - \lambda)). \]

Differentiating (27) partially with respect \( w_1 \) and \( w_2 \) equate to zero and solve for \( w_1 \) and \( w_2 \) simultaneously, we obtained \( w_1 = \frac{EF - CD}{BC - F^2} \) and \( w_2 = \frac{DF - BE}{BC - F^2} \). Substituting the results in (27), we obtained the minimum \( MSE(T_{\beta}) \).

\[ MSE(T_{\beta}) = \bar{Y}^2 \left[ A + \frac{CD^2 + BE^2 + 2DEF}{(BC - F^2)} \right] \]  
(27)

**RESULTS AND DISCUSSION**

In this section, In order to elucidate the performance of suggested estimators to deal missing data with respect to some existing related estimators by using two data sets below.

Yadav and Zaman (2021)

**Population 1:** Y = The production (Yield) of peppermint oil in kilogram and X = The area of the field in Bigha (2529.3 Square Meter).

\( N = 150, n = 40, \gamma = 0.018333, \bar{Y} = 79.58, \bar{X} = 6.5833, \rho = 0.9363 \)
\( C_y = 0.781333, C_x = 0.661726, S_y^2 = 3866.165, S_x^2 = 18.97791, \beta_1 = 1.4984 \)
\( \beta_2 = 5.408, \)

Murthy (1967)

**Population 2:** Y = Output for 80 factories in a region and X = Number of workers

\( N = 80, n = 20, \bar{Y} = 51.8264, \bar{X} = 11.2646, \rho = 0.9413 \)
\( C_y = 0.3542, C_x = 0.7505, \beta_1 = 1.0500, \beta_2 = -0.0634, \)
<table>
<thead>
<tr>
<th>ESTIMATORS</th>
<th>MSE</th>
<th>ESTIMATORS</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample mean</td>
<td>167.5338</td>
<td>($\pi_1 = \beta_2 (x), \pi_2 = C_x$)</td>
<td>13967.32</td>
</tr>
<tr>
<td>Lee et al. (1994)</td>
<td>83.57332</td>
<td>($\pi_1 = \beta_2 (x), \pi_2 = \beta_1 (x)$)</td>
<td>13938.67</td>
</tr>
<tr>
<td>Singh and Horn (2000)</td>
<td>82.80116</td>
<td>($\pi_1 = \beta_1 (x), \pi_2 = S_x$)</td>
<td>13850.05</td>
</tr>
<tr>
<td>Singh and Deo (2003)</td>
<td>82.80123</td>
<td>($\pi_1 = S_x, \pi_2 = C_x$)</td>
<td>13961.75</td>
</tr>
<tr>
<td>Singh (2009)</td>
<td>84.34548</td>
<td>($\pi_1 = S_x, \pi_2 = \beta_1 (x)$)</td>
<td>13926.69</td>
</tr>
<tr>
<td>Gira (2015)</td>
<td>82.80123</td>
<td>($\pi_1 = S_x, \pi_2 = \beta_2 (x)$)</td>
<td>13786.15</td>
</tr>
<tr>
<td>Singh et al. (2016)</td>
<td>59.37374</td>
<td>Audu et al. (2020)</td>
<td></td>
</tr>
<tr>
<td>Kadilar and Cingi (2008)</td>
<td>94.86998</td>
<td>($\kappa_1 = 1, \kappa_2 = 1$)</td>
<td>167.5818</td>
</tr>
<tr>
<td></td>
<td>56.87213</td>
<td>($\kappa_1 = -1, \kappa_2 = 1$)</td>
<td>321.5926</td>
</tr>
<tr>
<td>Al-Omari et al. (2013)</td>
<td>98.97785</td>
<td>($\kappa_1 = -1, \kappa_2 = -1$)</td>
<td>518.1431</td>
</tr>
<tr>
<td>Audu &amp; Singh (2021)</td>
<td>86.471</td>
<td>Proposed estimators ($T_{\beta}$)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13875.25</td>
<td>($\pi_1 = 1, \pi_2 = 0$)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13872.86</td>
<td>($\pi_1 = 1, \pi_2 = C_x$)</td>
<td>8.409117</td>
</tr>
<tr>
<td></td>
<td>13751.76</td>
<td>($\pi_1 = 1, \pi_2 = \beta_2 (x)$)</td>
<td>20.0644</td>
</tr>
<tr>
<td></td>
<td>13412.72</td>
<td>($\pi_1 = 1, \pi_2 = \beta_2 (x)$)</td>
<td>45.62799</td>
</tr>
<tr>
<td></td>
<td>13479.75</td>
<td>($\pi_1 = 1, \pi_2 = S_x$)</td>
<td>24.61472</td>
</tr>
<tr>
<td></td>
<td>13661.32</td>
<td>($\pi_1 = C_x, \pi_2 = \beta_2 (x)$)</td>
<td>19.43115</td>
</tr>
<tr>
<td></td>
<td>13282.55</td>
<td>($\pi_1 = C_x, \pi_2 = \beta_2 (x)$)</td>
<td>11.44761</td>
</tr>
<tr>
<td></td>
<td>13350.62</td>
<td>($\pi_1 = C_x, \pi_2 = S_x$)</td>
<td>12.87232</td>
</tr>
<tr>
<td></td>
<td>13909.62</td>
<td>($\pi_1 = \beta_1 (x), \pi_2 = C_x$)</td>
<td>11.26137</td>
</tr>
<tr>
<td></td>
<td>13535.92</td>
<td>($\pi_1 = \beta_1 (x), \pi_2 = \beta_2 (x)$)</td>
<td>38.2639</td>
</tr>
<tr>
<td></td>
<td>13596.86</td>
<td>($\pi_1 = \beta_1 (x), \pi_2 = S_x$)</td>
<td>53.06286</td>
</tr>
</tbody>
</table>
Table 3: MSE of the Proposed and some related existing estimators using population 2

<table>
<thead>
<tr>
<th>ESTIMATORS</th>
<th>MSE</th>
<th>ESTIMATORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample mean</td>
<td>134.2434</td>
<td>$(\pi_1 = \beta_2(x), \pi_2 = C_x)$</td>
</tr>
<tr>
<td>Lee et al. (1994)</td>
<td>92.22131</td>
<td>$(\pi_1 = \beta_2(x), \pi_2 = \beta_1(x))$</td>
</tr>
<tr>
<td>Singh and Horn (2000)</td>
<td>87.16081</td>
<td>$(\pi_1 = \beta_2(x), \pi_2 = S_x)$</td>
</tr>
<tr>
<td>Singh and Deo (2003)</td>
<td>87.16081</td>
<td>$(\pi_1 = S_x, \pi_2 = C_x)$</td>
</tr>
<tr>
<td>Singh (2009)</td>
<td>97.28181</td>
<td>$(\pi_1 = S_x, \pi_2 = \beta_1(x))$</td>
</tr>
<tr>
<td>Gira (2015)</td>
<td>106.8948</td>
<td>$(\pi_1 = S_x, \pi_2 = \beta_2(x))$</td>
</tr>
<tr>
<td>Singh et al. (2016)</td>
<td>55.89374</td>
<td>$(\pi_1 = 1, \pi_2 = 2)$</td>
</tr>
<tr>
<td>Kadilar and Cingi (2008)</td>
<td>121.043</td>
<td>$(\pi_1 = 1, \pi_2 = -1)$</td>
</tr>
<tr>
<td>Al-Omari et al. (2013)</td>
<td>122.1959</td>
<td>$(\pi_1 = -1, \pi_2 = 1)$</td>
</tr>
<tr>
<td>Audu &amp; Singh (2021)</td>
<td>100.6903</td>
<td>$(\pi_1 = 1, \pi_2 = 0)$</td>
</tr>
<tr>
<td>Proposed estimators</td>
<td>46672.08</td>
<td>$(T_{f_1})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_2})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_4})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_5})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_6})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_7})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_8})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_9})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_{f_{10}})$</td>
</tr>
</tbody>
</table>

Table 2 and 3 shows the MSEs of the suggested and other existing estimators considered in the study by using information of two different populations. The results revealed that the suggested estimator have the minimum MSE compared to the conventional estimators from each population. This means that the proposed methods shown a high level of efficiency on others considered in the study, and can produce a better estimate of the average population in the presence of a missing observation on average.

CONCLUSION
From the empirical study, the results showed that the suggested estimators were more efficient than the existing estimators considered in the study. So, therefore its use is recommended to estimate the population average when certain values of the variables of the study are missing in the study.

REFERENCES


