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A DETERMINATION OF THE GENERALIZED GRAVITY FOR CERTAIN SINGULAR MAPS IN FINITE FULL TRANSFORMATION SEMIGROUPS

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ABSTRACT

For any α in Sing$_n$, it is well known from Saito (1989) that $k(\alpha)=\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ ***or*** $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil + 1$***; and*** $k(\alpha)=\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ ***if*** $g(\alpha)=1 \bmod (d(\alpha))$***. In this paper, using division algorithm, for each α in Sing$_n$ we write*** $g(\alpha)=dm+s$ ***with*** $0 \le s \le d-1$ ***where d is the defect of α. A complete determination of k(α) was obtained for*** $g(\alpha)=dm$ ***(i.e. s = 0) for all d and*** $g(\alpha)=dm+s$ ***for*** $d \le 2$***, where α is assumed to have no cyclic orbits.***

Keywords: Full Transformation Semigroups, Idempotents, Singular Maps, Orbits, Defect, Generalized Gravity.

INTRODUCTION

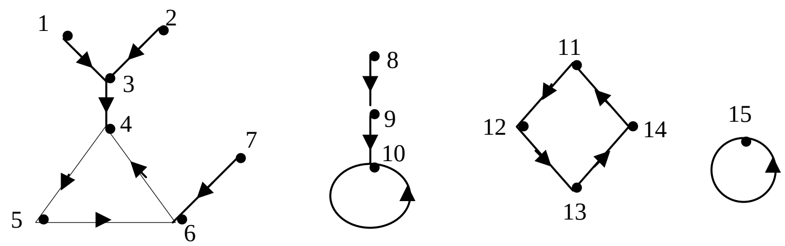
In Howie (1966) started the search for a subsemigroup of a transformation semigroup generated by idempotent elements. He considered the full transformation semigroup T_X that is the semigroup of all maps from a set X to itself. For a finite set X with $|X|=n$, T_X is denoted as T_n and Howie was able to show that Sing$_n$ the set of all singular maps in T_n (i.e all maps $\alpha \in T_n$ such that $|\mathrm{im}\alpha| <n$) is a subsemigroup of T_n ;and moreover E the set of idempotents in T_n generates Sing$_n$ i.e <E>= Sing$_n$. If we let E_1 denote the set of idempotents of defect 1 (i.e all idempotents ε for which $|\mathrm{im}\varepsilon|= n-1$) it was found again that <E_1> = Sing$_n$.

Also, for any α in Sing$_n$ Howie (1980) and Iwahori (1977) where able to obtain the formula for the length of α i.e. the minimum number of idempotents of defect 1 required to generate α. This means, assuming k to be the least number of idempotents in E_1 in the expression of α as a product of idempotents in E_1 i.e. $\alpha = \varepsilon_1\varepsilon_2 \ldots \varepsilon_k$, then k(α)=g(α) the gravity of α, and this number was shown to be g(α)=n+c(α)-f(α), where c(α) is the number of cyclic orbits in the diagraph of α and f(α) is the number of fixed points in α. It should be noted that every map α in T_n is associated with a diagraph with n vertices in which there is an edge i→j if and only if iα=j. Let X={1,2,. . .,n}} then for i,j ∈ X ,we write i ≡ j if and only if there exist r, s ≥ 0 such that $i\alpha^r = j\alpha^s$. This is an equivalence relation and it partitions X into disjoint classes, called orbits. The orbits are the connected components of the associated diagraph and there are four types of components or orbits: standard, acyclic, cyclic and trivial. f(α) the number of fixed points in α is equal to the number of acyclic components plus that of trivial ones.

Example 1: $\alpha \in Sing_{15}$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 14 & 15 & 15 \end{pmatrix}$$

has orbits {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10}, {11, 12, 13, 14}, {15} which are respectively standard, acyclic, cyclic and trivial. $\Gamma(\alpha)$ the associated digraph is



The results of Howie (1980) and Iwahori (1977) were extended to idempotents in E not necessary in E_1 by Saito (1989). That is for every α in $Sing_n$ if d(α is the defect of α (i.e. d(α) = n - |im α |) then k(α) the generalized gravity of α is obtained to be $k(\alpha)=\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ or $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil+1$ and $k(\alpha)=\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ if $g(\alpha)\equiv 1 \bmod (d(\alpha))$

Saito [4] introduced levelling of a set which we will also use in this paper. Let $V_0=\{v_1,v_2,...,v_d\}$ be a set of positive integers ($d \geq 2$) and let $maxV_0 = v_i$ and $minV_0 = v_j$ If $v_i - v_j \geq 3$, let us subtract k_1 from v_i and add $k_1 + 1$ to v_j, where k_1 is some positive integer which satisfies $k_1 \leq v_i - v_j$, and define

$$V_1=\begin{cases}\{v_1,...,v_j-k_1,...v_j+k_1+1,...v_d\} & \text{if } v_i\text{-v}_j \geq 3\\ V_0 & \text{otherwise}\end{cases}$$

by repeating this procedure on V_1, we obtain a new set V_2 if $maxV_1 - minV_1 \geq 3$, $V_2 = V_1$ otherwise.

This process can be continued until no further new sets can be obtained. Then we obtain V_o, V_1,…, V_t, where $maxV_k - minV_k \geq 3$, if $k < t$ and $maxV_t - minV_t \geq 2$. In this case, we have $maxV_0 \geq minV_1 \geq \ldots \geq maxV_t$ and $minV_o \leq minV_1 \leq \ldots \leq minV_t$. Then V_i (i = 0, 1, . . . , t) are called levelled sets of V_o and V_t is a completely levelled set.

We now state the following results from Saito (1989) which we shall use subsequently.

Theorem 1.1 [4, theorem 10]: Let $Sing_n$ be the semigroup of all singular mappings from X in to X, where $X=\{1,2,...,n\}$, and let E be the set of idempotents of $Sing_n$. For α in $Sing_n$, let $k(\alpha)$ be the unique positive integer for which $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)}$ and $g(\alpha)$ the gravity of α and $d(\alpha)$ the defect of α. Then $k(\alpha)=\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ or $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ + 1, and equals $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ if $g(\alpha) \equiv 1 \bmod(d(\alpha))$.

Lemma 1.2 [4, lemma 14]: Let $V_0=\{v_1,v_2,...v_d\}$ be a set of positive integers with $v_1+v_2+...+v_d=g \quad (d \geq 2)$. Then there exists a completely levelled set $V_k \quad (0 \leq k \leq d-1)$ of V_o such that $\left\lceil \frac{g}{d} \right\rceil \leq \max V_k \leq \left\lceil \frac{g}{d} \right\rceil+1$ and $\max V_k=\left\lceil \frac{g}{d} \right\rceil$ if $g \equiv 1 \bmod(d)$.

For basic concepts and definitions on semigroup theory, see for example Howie (195) and for a detailed discussion of idempotents in finite full transformation semigroups Levi and Seif (2002) and Blyth and Santos (2006).

The Generalized Gravity of Certain Singular Maps in Finite Full Transformation Semigroups

In this work we use division algorithm to write $g(\alpha)=dm+s$ for $0 \leq s \leq d-1$, where d is the defect of α, and α is assumed to have no cyclic orbits. Then using the methods of levelling provided in Saito (1989) and Sani (2008) we obtain the following results:

Theorem 2.1: Let α be an element in $Sing_n$ whose defect d(α)=2 and assume α to have no cyclic orbits. Then k(α) $= \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ if and only if g(α) is an odd integer or that g(α) is even and V_o = {m, m} where g(α) = 2m (m ∈ **Z**), where V_0 is the initial levelled set corresponding to any representation of α.

Proof:

i) If g(α) is odd i.e. g(α) = 2m + 1 then g(α) = 1 mod (2) and it follows from Theorem 1.1 that k(α) $= \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$.

ii) If g(α) is even say g(α) = 2m, $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil = \left\lceil \frac{2m}{2} \right\rceil$ = m. Now, suppose V_o = {m, m}, obviously V_o is completely levelled and

k(α) = maxV_o = m $= \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$.

Conversely, if $V_o = \{v_1, v_2\}$ = {2m – s, s}, 1 ≤ s < m.

Take k_1 = 2m – s – m = m – s then

V_1 = {2m – s – (m – s), s + (m – s) + 1} = {m, m + 1} and

k(α) = maxV_1 = m + 1 = $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ + 1. Hence the result.

Theorem 2.2: Let α be an element in $Sing_n$ with defect d(α)=d and assume α to have no cyclic orbits. If g(α) = dm for some positive integer m, then

k(α) $= \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ if and only if V_o = {m, m, . . . , m} that is v_i = m

1 ≤ i ≤ d where V_o is the initial levelled set corresponding to any representation of α.

Proof: Suppose V_o = {m, m, . . . m}. Clearly,

k(α) = maxV_o = m $= \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil = \frac{dm}{d}$

Conversely, suppose $V_o = \{v_1, v_2, \ldots, v_d\}$ with some $v_i \neq m$ 1≤ i ≤ d. Without loss of generality we may assume $V_o = \{v_1, v_2, \ldots, v_d\}$ where $v_1 \leq v_2 \leq \ldots \leq v_d$ and $v_1 < m$. Thus $v_d \geq m + 1$. If $v_d = m + 1$, then V_o is a completely leveled set and k(α) = maxV_o = m + 1 = $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ +1.

On the other hand, if $v_d \geq m + 2$ then maxV_o – minV_o = $v_d - v_1 \geq 3$. In this case V_o is not completely levelled. Now, let V_q be the completely levelled set of V_o (q ≥ 1). Clearly maxV_q > m since every levelling will increase $\sum_1^d v_i$ by 1. And since from Lemma 1.2

m ≤ maxV_q ≤ m +1, it follows that

k(α) = maxV_q = m + 1 = $\left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ + 1. Hence the proof.

Conclusion

In this paper, a complete determination of k(α) was obtained for all α of defect 2 and also for all α with defect d provided g(α) = dm for some positive integer m. However the problem of determining k(α) for the remaining cases remains open.

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