A DETERMINATION OF THE GENERALIZED GRAVITY FOR CERTAIN SINGULAR MAPS IN FINITE FULL TRANSFORMATION SEMIGROUPS

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ABSTRACT

For any \(\alpha\) in \(\text{Sing}_n\), it is well known from Saito (1989) that \(k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1\); and if \(g(\alpha) = 1 \mod(d(\alpha))\). In this paper, using division algorithm, for each \(\alpha\) in \(\text{Sing}_n\), we write \(g(\alpha) = dm + s\) with \(0 \leq s \leq d - 1\), where \(d\) is the defect of \(\alpha\). A complete determination of \(k(\alpha)\) was obtained for \(g(\alpha) = dm\) (i.e. \(s = 0\)) for all \(d\) and \(g(\alpha) = dm + s\) for \(d \leq 2\), where \(\alpha\) is assumed to have no cyclic orbits.

Keywords: Full Transformation Semigroups, Idempotents, Singular Maps, Orbits, Defect, Generalized Gravity.

INTRODUCTION

In Howie (1966) started the search for a subsemigroup of a transformation semigroup generated by idempotent elements. He considered the full transformation semigroup \(T_X\) that is the semigroup of all maps from a set \(X\) to itself. For a finite set \(X\) with \(|X| = n\), \(T_X\) is denoted as \(T_n\) and Howie was able to show that \(\text{Sing}_n\), the set of all singular maps in \(T_n\), is a subsemigroup of \(T_n\); and moreover \(E\), the set of idempotents in \(T_n\), generates \(\text{Sing}_n\), i.e \(<E> = \text{Sing}_n\). If we let \(E_1\) denote the set of idempotents of defect 1 (i.e all idempotents \(e\) for which \(|\text{im}e| = n - 1\)) it was found again that \(<E_1> = \text{Sing}_n\).

Also, for any \(\alpha\) in \(\text{Sing}_n\), Howie (1980) and Iwahori (1977) where able to obtain the formula for the length of \(\alpha\), i.e. the minimum number of idempotents of defect 1 required to generate \(\alpha\). This means, assuming \(k\) to be the least number of idempotents in \(E_1\) in the expression of \(\alpha\) as a product of idempotents in \(E_1\), i.e. \(\alpha = e_1 e_2 \ldots e_k\), then \(k(\alpha) = g(\alpha)\) the gravity of \(\alpha\), and this number was shown to be \(g(\alpha) = n + c(\alpha) - f(\alpha)\), where \(c(\alpha)\) is the number of cyclic orbits in the diagraph of \(\alpha\) and \(f(\alpha)\) is the number of fixed points in \(\alpha\). It should be noted that every map \(\alpha\) in \(T_n\) is associated with a diagraph with \(n\) vertices in which there is an edge \(i \rightarrow j\) if and only if \(i \alpha = j\). Let \(X = \{1, 2, \ldots, n\}\) then for \(i, j \in X\), we write \(i \equiv j\) if and only if there exist \(r, s \geq 0\) such that \(i \alpha^r = j \alpha^s\). This is an equivalence relation and it partitions \(X\) into disjoint classes, called orbits. The orbits are the connected components of the associated diagraph and there are four types of components or orbits: standard, acyclic, cyclic and trivial. \(f(\alpha)\) the number of fixed points in \(\alpha\) is equal to the number of acyclic components plus that of trivial ones.

Example 1: \(\alpha \in \text{Sing}_{15}\)

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 14 & 15 & 15
\end{pmatrix}
\]

has orbits \(\{1, 2, 3, 4, 5, 6, 7\}, \{8, 9, 10\}, \{11, 12, 13, 14\}, \{15\}\) which are respectively standard, acyclic, cyclic and trivial. \(\Gamma(\alpha)\) the associated diagraph is
The results of Howie (1980) and Iwahori (1977) were extended to idempotents in E not necessary in $E_1$ by Saito (1989). That is for every $\alpha \in \text{Sing}_n$ if $d(\alpha)$ is the defect of $\alpha$ (i.e., $d(\alpha) = n - \lim |\alpha|$) then $k(\alpha)$ the generalized gravity of $\alpha$ is obtained to be $k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor$ or $\left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1$ and $k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor$ if $g(\alpha) \equiv 1 \mod(d(\alpha))$.

Saito [4] introduced levelling of a set which we will also use in this paper. Let $V_0 = \{v_1, v_2, \ldots, v_d\}$ be a set of positive integers ($d \geq 2$) and let $\max V_0 = v_i$ and $\min V_0 = v_j$ If $v_i - v_j \geq 3$, let us subtract $k_1$ from $v_i$ and add $k_1 + 1$ to $v_j$ where $k_1$ is some positive integer which satisfies $k_1 \leq v_i - v_j$ and define

$$V_1 = \begin{cases} 
\{v_1, \ldots, v_j - k_1, \ldots, v_j + k_1 + 1, \ldots, v_d\} & \text{if } v_i - v_j \geq 3 \\
V_0 & \text{otherwise}
\end{cases}$$

by repeating this procedure on $V_0$, we obtain a new set $V_2$ if $\max V_1 - \min V_1 \geq 3$, $V_2 = V_1$ otherwise. This process can be continued until no further new sets can be obtained. Then we obtain $V_0, V_1, \ldots, V_t$ where $\max V_k - \min V_k \geq 3$, if $k < t$ and $\max V_t - \min V_t \geq 2$. In this case, we have $\max V_0 \geq \min V_1 \geq \ldots \geq \max V_t$ and $\min V_0 \leq \min V_1 \leq \ldots \leq \min V_t$. Then $V_t (i = 0, 1, \ldots)$ are called levelled sets of $V_0$ and $V_t$ is a completely levelled set. We now state the following results from Saito (1989) which we shall use subsequently.

**Theorem 1.1** [4, theorem 10]: Let $\text{Sing}_n$ be the semigroup of all singular mappings from $X$ in to $X$, where $X = \{1,2,\ldots,n\}$, and let $E$ be the set of idempotents of $\text{Sing}_n$. For $\alpha \in \text{Sing}_n$, let $k(\alpha)$ be the unique positive integer for which $\alpha \in Ek(\alpha)$, $\alpha \notin Ek(\alpha)$ and $g(\alpha)$ the gravity of $\alpha$ and $d(\alpha)$ the defect of $\alpha$. Then $k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor$ or $\left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1$ and equals $\left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor$ if $g(\alpha) = 1 \mod(d(\alpha))$.

**Lemma 1.2** [4, lemma 14]: Let $V_0 = \{v_1, v_2, \ldots, v_d\}$ be a set of positive integers with $v_1 + v_2 + \ldots + v_d = g$ ($d \geq 2$). Then there exists a completely levelled set $V_k$ ($0 \leq k \leq d-1$) of $V_0$ such that $\left\lfloor \frac{g}{d} \right\rfloor \leq \max V_k \leq \left\lfloor \frac{g}{d} \right\rfloor + 1$ and $\max V_k = \left\lfloor \frac{g}{d} \right\rfloor$ if $g = 1 \mod(d)$.

For basic concepts and definitions on semigroup theory, see for example Howie (1995) and for a detailed discussion of idempotents in finite full transformation semigroups Levi and Seif (2002) and Blyth and Santos (2006).

**The Generalized Gravity of Certain Singular Maps in Finite Full Transformation Semigroups**

In this work we use division algorithm to write $g(\alpha) = dm + s$ for $0 \leq s \leq d - 1$, where $d$ is the defect of $\alpha$, and $\alpha$ is assumed to have no cyclic orbits. Then using the methods of levelling provided in Saito (1989) and Sani (2008) we obtain the following results:
Theorem 2.1: Let \( \alpha \) be an element in \( \text{Sing}_n \) whose defect \( d(\alpha) = 2 \) and assume \( \alpha \) to have no cyclic orbits. Then

\[
k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor \text{ if and only if } g(\alpha) \text{ is an odd integer or that } g(\alpha) \text{ is even and } V_0 = \{m, m\} \text{ where } g(\alpha) = 2m \ (m \in \mathbb{Z}), \text{ where } V_0 \text{ is the initial levelled set corresponding to any representation of } \alpha.
\]

Proof:

i) If \( g(\alpha) \) is odd i.e. \( g(\alpha) = 2m + 1 \) then \( g(\alpha) = 1 \mod (2) \) and it follows from Theorem 1.1 that \( k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor \).

ii) If \( g(\alpha) \) is even say \( g(\alpha) = 2m \), then

\[
k(\alpha) = \max V_i = m + 1 = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1.
\]

Conversely, if \( V_0 = \{v_1, v_2\} = \{2m - s, s\}, 1 \leq s < m \).

Take \( k_1 = 2m - s - m = m - s \) then \( V_1 = \{2m - s - (m - s), s + (m - s) + 1\} = \{m, m + 1\} \) and

\[
k(\alpha) = \max V_1 = m + 1 = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1.
\]

Hence the result.

Theorem 2.2: Let \( \alpha \) be an element in \( \text{Sing}_n \) with defect \( d(\alpha) = d \) and assume \( \alpha \) to have no cyclic orbits. If \( g(\alpha) = dm \) for some positive integer \( m \), then

\[
k(\alpha) = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor \text{ if and only if } V_0 = \{m, m, \ldots, m\} \text{ that is } v_i = m \text{ and } 1 \leq i \leq d \text{ where } V_0 \text{ is the initial levelled set corresponding to any representation of } \alpha.
\]

Proof: Suppose \( V_0 = \{m, m, \ldots, m\} \). Clearly,

\[
k(\alpha) = \max V_0 = m = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor = \frac{dm}{d}
\]

Conversely, suppose \( V_0 = \{v_1, v_2, \ldots, v_d\} \) with some \( v_i = m \ 1 \leq i \leq d \). Without loss of generality, we may assume \( V_0 = \{v_1, v_2, \ldots, v_d\} \) where \( v_1 \leq v_2 \leq \cdots \leq v_d \) and \( v_1 < m \). Thus \( v_d \geq m + 1 \). If \( v_d = m + 1 \), then \( V_0 \) is a completely leveled set and \( k(\alpha) = \max V_0 = m + 1 = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1 \).

On the other hand, if \( v_d \geq m + 2 \) then \( \max V_0 \geq \min V_0 \text{ and } V_0 \text{ is not completely levelled. Now, let } V_0 \text{ be the completely levelled set of } \sum_i v_i \text{ by } 1. \text{ And since from Lemma 1.2}

\[
m \leq \max V_0 \leq m + 1, \text{ it follows that }
k(\alpha) = \max V_0 = m + 1 = \left\lfloor \frac{g(\alpha)}{d(\alpha)} \right\rfloor + 1.
\]

Hence the proof.

Conclusion

In this paper, a complete determination of \( k(\alpha) \) was obtained for all \( \alpha \) of defect 2 and also for all \( \alpha \) with defect \( d \) provided \( g(\alpha) = dm \) for some positive integer \( m \). However, the problem of determining \( k(\alpha) \) for the remaining cases remains open.
REFERENCES
Sani, S. - On the generalized gravity in finite full transformation semigroups, MSc thesis submitted to the department of Mathematical Sciences, Bayero University, Kano, 2008.