THE COLLOQUIUM -A Multi disciplinary Thematc Policy Journal
www.ccsonlinejournals.com
Volume 10 , Number 1 , 2022 Pages 112 - 121
 Print ISSN : 2971-6624 eISSN: 2971-6632

HIGH ORDER HYBRID BLOCK METHODS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS

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Abstract
A class of high order hybrid block methods (HOHBM) for the solution of initial value problems. This set of schemes is obtained from a continuous approximation via interpolations means. The order and the linear stability properties of the derived schemes are studied which makes it appropriate for approximating stiff systems. Numerical illustrations are presented to confirm the accuracy of the derived schemes.
Keywords: *Block methods, hybrid methods, Interpolations,stiff problems.*

Introduction

It is worthy to note that numerical integration for initial value problems of ordinary differential equations of the form

$$y' = f(x, y), \qquad x \in [x_0, X], \qquad y(x_0) = y_0;\ f: \mathbb{R} \times \mathbb{R}^v \longrightarrow \mathbb{R}^v \qquad (1)$$

is the major problem of concern in numerical analysis. Here, f satisfies a Lipschitz condition. Although backward differentiation formulas are the most widely used schemes for solving (1) (Curtiss & Hirschfelder, 1952), but suffer the order and stability barrier in (Dahlquist, 1963). In this regard, Bickart and Rubin (1974) noted that to have a schemes with good stability properties for solving (1), the traditional linear multistep methods (LMMs) should be changed to a different schemes.Base on this optimistic approach, Akinfenwa (2011), Cash (1981), Ehigie et al., (2014), Enright (1974), Ogunfeyitimi and Ikhile (2020, 2021b), Yakubu (2016) developed special cases of methods in (Obrehkoff, 1940). Hybrid scheme were also considered to circumvent the stability barrier for LMMs, for instance see, Butcher (2003,2005), Gragg and Stetter (1964), Ikhile and kuoughae (2007).

In this paper, the hybrid block scheme proposed is derived through multistep interpolation (Gladwell & Sayers, 1976; Onumanyi et al., 1994). The newly derived scheme is implemented without the use of a predictor, that is, we adopt the boundary value technique during implementation, see (Axelson & Verwer, 1985; Brugnano & Trigiante 1998; Ogunfeyitimi & Ikhile, 2021a). This allows the hybrid block schemes to obtain a block of numerical solutions simultaneously on the entire interval for (1). The increase in accuracy of the new scheme is obtained by adding hybrid points while retaining the grid-size constant. This flexibility is an added advantage over conventional Runge-Kutta schemes. Moreso, the boundary value implementation procedure has an attribute of overcoming the issues concerning the step-by-step algorithms (Lambert, 1991) and also improves the stability properties of the schemes (Cash, 2000). We note that our hybrid block methods are developed to overcome the Daniel moore conjecture for second derivative methods (Daniel & Moore, 1974; Hairer & Wanner 1996).

Cite this article as
Meninwa, C. A., & Ogunfeyitimi, S. E. (2022). High order Hybrid block methods for the Solution of initial value problems.*THE COLLOQUIUM*, 10(1), 112 - 121

The paper is organized as follows: In Section 2, we state the procedure for deriving block methods. The analysis and the implementation of the new schemes are reported in section 3 and 4. While some numerical experiments are considered in section 5.

2.Development of the block scheme

The continuous extended hybrid method for solving (1) is of the form

$$g_{n+w_i} = \frac{1}{h^2}\left(\sum_{j=0}^{2k} \alpha_{i,j}\, y_{n+w_j} - h \sum_{j=i-1}^{i} \beta_{i,j}\, f_{n+w_j} - h^2 \gamma_{i,i-1} g_{n+w_{i-1}}\right), \quad w_i = \frac{i}{2}, i = 1(1)2k \qquad (2)$$

on an off-step points $t_0, t_1, \dots, t_k$, where the coefficients are determined by depending on the off-step points through interpolation and collocation means. To obtain this scheme, $y(x)$ is approximated by a monomial basis function of the form

$$y(x) \approx Y(x) = \sum_{j=0}^{2k+3} c_j\, x^j \qquad (3)$$

$$y'(x) \approx Y'(x) = \frac{1}{h}\sum_{j=0}^{2k+3} jc_j\, x^{j-1} \qquad (4)$$

$$y''(x) \approx Y''(x) = \frac{1}{h^2}\sum_{j=0}^{2k+3} j(j-1)c_j\, x^{j-2} \qquad (5)$$

where the coefficients c_j are unknown that lie in the $[x_n, x_{n+k}]$ block. Since the normalization of the coefficients in (2) occur in the second derivative part, the basis polynomial in (5) shall be used to derived the block hybrid schemes. The equations (3), (4) and (5) generate the set of k+4 equations

$$Y(x_{n+w_i}) = y_{n+w_i}, i = 0,1,\dots,2k \qquad (6)$$

$$Y'(x_{n+w_i}) = f_{n+w_i}, i = i-1, i \qquad (7)$$

$$Y''(x_{n+w_i}) = g_{n+w_i}, i = i-1 \qquad (8)$$

The compact form of equation (6), (7) and (8) is given as

$$UQ = S \qquad (9)$$

where,

$$Q = (c_0 \quad c_1 \quad c_2 \quad c_3 \quad \cdots \quad c_{2k} \quad c_{2k+1} \quad c_{2k+2} \quad c_{2k+3})^T$$

$$S = (y_n \quad y_{n+w_1} \quad y_{n+w_2} \quad y_{n+w_3} \quad \cdots \quad y_{n+w_{2k}} \quad f_{n+w_{i-1}} \quad f_{n+w_i} \quad g_{n+w_{i-1}})^T$$

THE COLLOQUIUM -A Multi disciplinary Thematc Policy Journal
www.ccsonlinejournals.com
Volume 10 , Number 1 , 2022 Pages 112 - 121
CC-BY-NC-SA 4.0 International Print ISSN : 2971-6624 eISSN: 2971-6632

$$U=\begin{pmatrix}1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ 1 & w_1 & w_1^2 & w_1^3 & \dots & w_1^{2k} & w_1^{2k+1} & w_1^{2k+2} & w_1^{2k+3}\\ 1 & w_2 & w_2^2 & w_2^3 & \dots & w_2^{2k} & w_2^{2k+1} & w_2^{2k+2} & w_2^{2k+3}\\ 1 & w_3 & w_3^2 & w_3^3 & \dots & w_3^{2k} & w_3^{2k+1} & w_3^{2k+2} & w_3^{2k+3}\\ \vdots & & & & & & & & \vdots\\ 1 & w_{2k} & w_{2k}^2 & w_{2k}^3 & \dots & w_{2k}^{2k} & w_{2k}^{2k+1} & w_{2k}^{2k+2} & w_{2k}^{2k+3}\\ 0 & 1 & 2w_{i-1} & 3w_{i-1}^2 & \dots & 2kw_{i-1}^{2k-1} & (2k+1)w_{i-1}^{2k} & (2k+2)w_{i-1}^{2k+1} & (2k+3)w_{i-1}^{2k+2}\\ 0 & 1 & 2w_i & 3w_i^2 & \dots & 2kw_i^{2k-1} & (2k+1)w_i^{2k} & (2k+2)w_i^{2k+1} & (2k+3)w_i^{2k+2}\\ 0 & 1 & 2 & 6w_{i-1} & \dots & 2k(2k-1)w_{i-1}^{2k-2} & (2k+1)2kw_{i-1}^{2k-1} & (2k+2)(2k+1)w_{i-1}^{2k} & (2k+3)(2k+2)w_{i-1}^{2k+1}\end{pmatrix}$$

This is solved simultaneously to get the constants $c_j, j = 0(1)2k+3$. The continuous schemes is developed by replacing the coefficient values of c_j into (5) and after several algebraic manipulation, the scheme is given as

$$g(x)=\frac{1}{h^2}\left(\sum_{j=0}^{2k}\alpha_{i,j}y_{n+w_j}-h\sum_{j=i-1}^{i}\beta_{i,j}f_{n+w_j}-h^2\gamma_{i,i-1}g_{n+w_i}\right) \qquad (10)$$

Replacing $x = x_{n+w_ih}$ for $1 = 1,2,\dots 2k$ in (10) gives the block schemes

$$AY_{n+1}=A_0Y_n+h(BF_{n+1}+B_0F_n)+h^2(DG_{n+1}+D_0G_n) \qquad (11)$$

Where,

$$Y_{n+1}=\left(y_{n+w_1},y_{n+w_2},\dots,y_{n+w_{2k}}\right)^T, Y_n=\left(y_{n-w_{2k-1}},\dots,y_{n-w_1},y_n\right)^T$$
$$F_{n+1}=\left(f_{n+w_1},f_{n+w_2},\dots,f_{n+w_{2k}}\right)^T, F_n=\left(f_{n-w_{2k-1}},\dots,f_{n-w_1},f_n\right)^T$$
$$G_{n+1}=\left(g_{n+w_1},g_{n+w_2},\dots,g_{n+w_{2k}}\right)^T, G_n=\left(g_{n-w_{2k-1}},\dots,g_{n-w_1},g_n\right)^T$$

and the matrices A, A_0, B, B_0, D, D_0 are defined as follows

$$A=\begin{pmatrix}\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,2k}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,2k}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots & \alpha_{3,2k}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \alpha_{2k,1} & \alpha_{2k,2} & \alpha_{2k,3} & \cdots & \alpha_{2k,2k}\end{pmatrix}, A_0=\left(\begin{array}{c|c} & \alpha_{1,0}\\ & \alpha_{2,0}\\ 0_{2k,2k-1} & \alpha_{3,0}\\ & \vdots\\ & \alpha_{2k,0}\end{array}\right),$$

$$B=\begin{pmatrix}\beta_{1,1} & & & & & \\ \beta_{2,1} & \beta_{2,2} & & & & \\ & \beta_{3,2} & \beta_{3,3} & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \beta_{2k,2k-1} & \beta_{2k,2k}\end{pmatrix}, B_0=\left(\begin{array}{c|c} & \beta_{1,0}\\ & 0\\ 0_{2k,2k-1} & 0\\ & \vdots\\ & 0\end{array}\right)$$

$$D=\begin{pmatrix}1 & & & & & \\ \gamma_{2,1} & 1 & & & & \\ & \gamma_{3,2} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \gamma_{2k,2k-1} & 1\end{pmatrix}, \qquad D_0=\left(\begin{array}{c|c} & \gamma_{1,0}\\ & 0\\ 0_{2k,2k-1} & 0\\ & \vdots\\ & 0\end{array}\right)$$

3.Analysis of the hybrid block schemes

Adopting the approach of Fatunla (1989), let the local truncation error (LTE) corresponding with (2) to be the linear difference operator such that

$$\mathcal{L}[y(x);h] = \sum_{j=0}^{k} \alpha_{i,j} y(x + w_j h) - h \sum_{j=i-1}^{i} \beta_{i,j}\, y'(x + w_j h) - h^2 \left(\gamma_{i,i-1} y''(x + w_{i-1} h) + y''(x + w_i h)\right) \quad (12)$$

Assuming that $y(x)$ is sufficiently differentiable function, the terms in (12) is expanded through Taylor series about the point x to have

$$\mathcal{L}[y(x);h] = C_0 y(x) + C_1 h\, y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^p(x) + \cdots$$

where,

$$C_0 = \sum_{j=0}^{2k} \alpha_{i,j}$$

$$C_1 = \sum_{j=0}^{2k} j\, \alpha_{i,j} - \sum_{j=i-1}^{i} \beta_{i,j}$$

$$C_2 = \frac{1}{2!} \sum_{j=0}^{2k} j^2\, \alpha_{i,j} - \sum_{j=i-1}^{i} j\, \beta_{i,j} - \gamma_{i,j} - 1$$

$$C_3 = \frac{1}{3!} \sum_{j=0}^{2k} j^3\, \alpha_{i,j} - \frac{1}{2!} \sum_{j=i-1}^{i} j^2\, \beta_{i,j} - (i-1)\gamma_{i,j} - i$$

$$C_p = \frac{1}{p!} \sum_{j=0}^{2k} j^p\, \alpha_{i,j} - \frac{1}{(p-1)!} \sum_{j=i-1}^{i} j^{p-1}\, \beta_{i,j} - \frac{1}{(p-2)!}\left((i-1)^{p-2}\gamma_{i,j} - i^{p-2}\right)$$

According to Henrici (1962), we have the following definition,
Definition 1. The method in (2) has order p if,
$C_j = 0, j = 0(1)p$ and $C_{p+1} \neq 0$
Where C_{p+1} is the error constant of the scheme in (2) and the principal LTE at point x is given as $C_{p+1} h^{p+1} y^{p+1}(x) + 0(h^{p+2})$.

3.2 Zero stability

In the spirit of Jator (2010) and Akinfenwa (2011), zero-stability deals with the stability of the scheme (11) in the limit as h approaches to zero. That is, as $h \to 0$, the hybrid block scheme in (11) equivalents to

$$AY_{n+1} = A_0 Y_n \qquad (13)$$

Then the first characteristics polynomial associated with (13) is given as

$$\rho(r) = det(r\,A - A_0) = r^{2k-1}(r-1) \qquad (14)$$

Following Fatunla (1991) the block method (11) is zero-stable for $k \geq 1$, since from (14), $\rho(r) = 0$ possesses only one root on the unit modulus.

3.3 Linear stability

We analyze the stability properties of the hybrid block method in (11) by applying them to the test equations

$$y' = \lambda\, y, \;\; y'' = \lambda^2\, y$$

to give

$$Y_{n+1} = (A - zB - z^2 D)^{-1}\, (A_0 + zB_0 + z^2 D_0)\;\, Y_n \qquad (15)$$

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www.ccsonlinejournals.com
Volume 10 , Number 1 , 2022 Pages 112 - 121
 Print ISSN : 2971-6624 eISSN: 2971-6632

The boundary loci of the hybrid block schemes coincide with the imaginary axis (see Figure 1). In particular, the stability region is the left half complex plane C. The newly derived scheme is A-stable for $k = 1(1)5$ with order $p = k + 3$. This shows that the newly derived scheme overcome the Daniel and Moore (1970) conjecture of second derivative linear multistep methods.

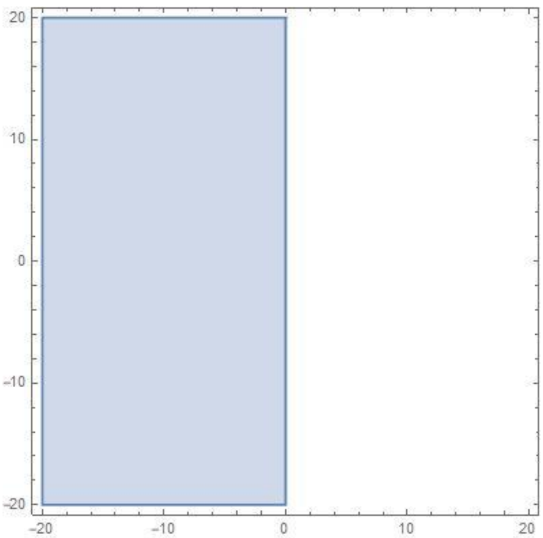

Figure. 1 Regions of absolute stability of hybrid block schemes (11).

4. Implementation of hybrid block methods

The gaussian elimination via pivoting is applied directly to solve linear problem while for a non-linear problems, a modified Newton-Raphson method is considered as details in Jator (2010) (see also, Ogunfeyitimi & Ikhile, 2021a).

Using a particular case of $k = 2$ order $p = 7$ in (11), the coefficient matrices are defined as

$$A = \begin{pmatrix} -1 & \frac{27}{206} & -\frac{1}{103} & \frac{1}{1648} \\ \frac{446}{459} & -1 & \frac{2}{51} & -\frac{1}{918} \\ -\frac{9}{223} & \frac{459}{446} & -1 & \frac{9}{892} \\ \frac{16}{1447} & -\frac{216}{1447} & \frac{1648}{1447} & -1 \end{pmatrix}, A_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{1447}{1648} \\ 0 & 0 & 0 & -\frac{1}{102} \\ 0 & 0 & 0 & \frac{1}{892} \\ 0 & 0 & 0 & -\frac{1}{1447} \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{21}{103} & 0 & 0 & 0 \\ -\frac{2}{9} & -\frac{4}{17} & 0 & 0 \\ 0 & -\frac{54}{223} & -\frac{51}{223} & 0 \\ 0 & 0 & -\frac{336}{1447} & -\frac{294}{1447} \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{147}{824} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Volume 10 , Number 1 , 2022 Pages 112 - 121
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$$D = \begin{pmatrix} \frac{9}{206} & 0 & 0 & 0 \\ -\frac{1}{51} & \frac{1}{34} & 0 & 0 \\ 0 & -\frac{27}{892} & \frac{9}{446} & 0 \\ 0 & 0 & -\frac{72}{1447} & \frac{18}{1447} \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{9}{824} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$Y_{n+1} = \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}\right)^T, Y_n = \left(y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n\right)^T$$

$$F_{n+1} = \left(f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}\right)^T, F_n = \left(f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n{}^T\right)^T$$

$$G_{n+1} = \left(g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{3}{2}}, g_{n+2},\right)^T, G_n = \left(, g_{n-\frac{3}{2}}, g_{n-1}, g_{n-\frac{1}{2}}, g_n\right)^T$$

The seventh order Hybrid methods in (11) is denoted by HOHBM2

5.0 Numerical Experiments

We consider a linear and non-linear standard problem to show the accuracy of the hybrid block schemes. The numerical result were obtained using MATLAB 2010a.

Example 1: Consider the linear system

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{4.1}$$

$$y(0) = \begin{pmatrix} e^{-2x} + e^{-40x}\,(\cos(40x) + \sin(40x)) \\ e^{-2x} - e^{-40x}\,(\cos(40x) + \sin(40x)) \\ 2e^{-40x}\,(\cos(40x) - \sin(40x)) \end{pmatrix}$$

We compare the HOHBM2 of order $p = 7$ with SDAM of order $p = 8$ in Jator and Sahi (2010) and method of order $p = 8$ Amodio and Mazzia (1995). From Table 1, It is observed that the new hybrid block scheme is superior in accuracy than the SDAM of Amodio and Mazzia (1995), and perform slightly better than the method of Jator and Sahi (2010) at lower order $p = 7$.

Table 1: The numerical result of Example 1, Error=$Max|y(x) - y|$

Step	$HOHBM\ k = 2$ $P = 7$	$SDAM\ k = 3$ $P = 8$	Amodio and Mazzia $K = 7\ \ P = 8$
20	$5.4\text{x}10^{-5}$	$7.5\text{x}10^{-4}$	$2.9\text{x}10^{-2}$
40	$2.1\text{x}10^{-6}$	$1.9\text{x}10^{-5}$	$6.8\text{x}10^{-3}$
80	$2.0\text{x}10^{-8}$	$1.4\text{x}10^{-7}$	$7.8\text{x}10^{-5}$
160	$1.6\text{x}10^{-10}$	$6.4\text{x}10^{-10}$	$4.7\text{x}10^{-7}$
320	$1.2\text{x}10^{-12}$	$2.5\text{x}10^{-12}$	$2.3\text{x}10^{-9}$
The error from ODE15s at $x = 1\ is\ 3.660087954199254e\ -5$			

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Volume 10 , Number 1 , 2022 Pages 112 - 121
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Example 2: Vander Pol

$$y'' + \mu(y^2 - 1)y' + y = 0; y(0) = 2, y'(0) = 0, \mu > 0. \quad (4.4)$$

This is solved by transformation into a first-order system of two ODEs given by

$$y_1' = y_2 \quad (4.5)$$

$$y_2' = -y_1 + \mu y_2(1 - y_1^2);\ y_1(0) = 2, y_2(0) = 0. \quad (4.6)$$

The Vander Pol equation is presented to show how robust the hybrid block scheme are in solving stiff system of equations. The Example 2 is solved for μ=10 and 100 with step size h=0.001. The graph of the computed solution is displayed in the Figures 2 and 3. It was observed from Figure 2 and 3 that the numerical solution of the HOHBM2 coincide with ODE15s.

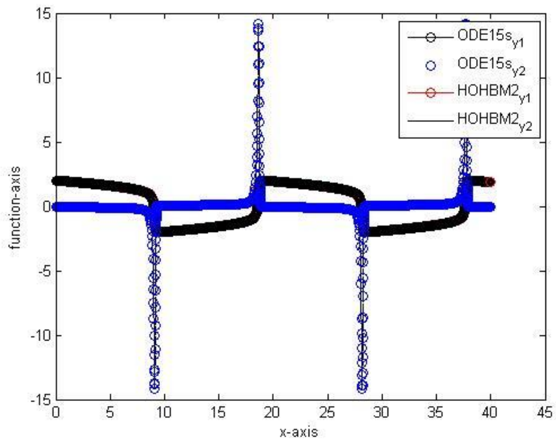

Figure 2: Numerical results for Example 2 using HOHBM2 with $\mu = 10$.

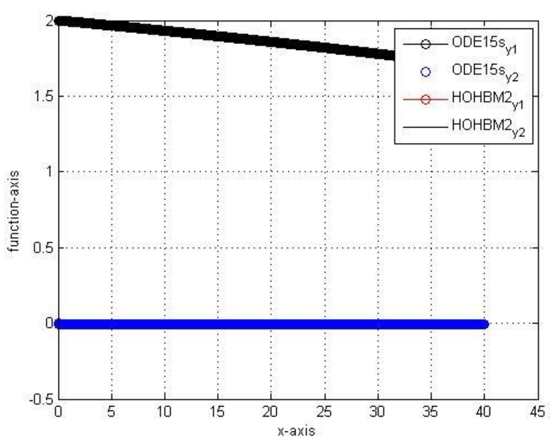

Figure 3: Numerical results for Example 2 using HOHBM2 with $\mu = 100$.

Conclusion

A class of HOHBM (11) for solving IVPs in (1) is considered. The HOHBM (11) is A-stable for $k \leq 5$ with order $p = 2k + 3$. The newly derived scheme in (11) has an advantage of being self-starting and good accuracy. The HOHBM2 has been implemented on some known problems with promising results (see Table 1, Figures 2 and 3).

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Meninwa, C. A., & Ogunfeyitimi, S. E. (2022). High order Hybrid block methods for the Solution of initial value problems.*THE COLLOQUIUM*, 10(1), 112 - 121

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Cite this article as

Meninwa, C. A., & Ogunfeyitimi, S. E. (2022). High order Hybrid block methods for the Solution of initial value problems.*THE COLLOQUIUM*, 10(1), 112 - 121