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An Optimal Production Model for Non-Instantaneous Decaying Goods with Two-Stage Production Period, Power Demand Pattern, Linear Holding Cost and Partial Backorder

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Abstract

Many existing economic production quantity (EPQ) models are based on the assumption of an exponential demand rate. However, this assumption does not accurately reflect the demand pattern of most real-life items, as product demand rarely changes at an exponential rate. In practice, demand often varies throughout the inventory cycle; it may start slow, increase gradually, and decline toward the end, making the power demand rate a more realistic representation. This research proposes an optimal production model for non-instantaneous processes of decaying goods with a two-stage production period, power demand pattern, linear holding cost, and partial backorder. Shortages are allowed and partially backlogged. The necessary and sufficient conditions for the existence of an optimal solution are established. The model determines the optimal time with a positive inventory, duration of shortages, total production cost, and economic production quantity. We demonstrate the models validity through a numerical example and then conduct a comparative analysis with an existing model. Results show that the proposed model produces a lower average total production cost, indicating better performance. Furthermore, a sensitivity analysis is conducted on selected parameters to evaluate their impact on total cost and EPQ, with recommendations provided to help manufacturers minimize costs.

Keywords: Optimal Production Model, Non-Instantaneous Decaying Goods, Two-Stage Production Period, Power Demand Pattern, Partial Backorder.

INTRODUCTION

Efficient inventory control is essential for achieving operational efficiency in today's business environment and competitive manufacturing and supply chain environments. The EPQ model is a traditional decision-making tool that is used to determine the optimal production lot size to reduce total inventory cost. However, real world production and inventory systems often exhibit complexities such as deterioration of items, variable demand, shortages, and nonlinear production processes that standard EPQ models fail to capture. In many sectors such as food processing, pharmaceuticals, fashion, chemicals, and electronics possess a finite shelf life and

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start degrade after a specified duration. For instance, fresh fruits remain sellable for a few days before spoilage sets in, and vaccines retain full potency for a defined grace period after manufacture. This behavior is referred to as delayed deterioration and is crucial to inventory systems, where early deterioration assumptions lead to over estimated costs and suboptimal decisions. Modelling this delay allows businesses to optimize production scheduling and minimize avoidable spoilage costs. Baraya and Sani (2012) formulated an economic production model for items with delayed deterioration, characterized by a stock-dependent demand rate and a time-dependent deterioration rate. The model was developing in the form a one product exhibits delayed deterioration, characterized by a consistent production rate, a demand rate that is linearly dependent on inventory levels both before and after the production process has a linearly increasing degradation rate. Malumfashi et al. (2021a) formulate an EPQ model for items with delayed deterioration, variable production rates, dual-phase demand rates, and shortages. The model specifically tackles the intricacies of inventory management for products that do not begin to deteriorate immediately after production and where demand fluctuates over two distinct phases. It also acknowledges the existence of shortages, permitting the backordering of unfulfilled demand. Malumfashi et al. (2021b) formulate an EPQ model for items with delayed deterioration, incorporating a two-phase production period, variable demand rate, and linear holding cost. The model accounts for a delay in the deterioration of items, indicating that deterioration does not commence immediately upon stocking. The model includes a two-phase production period, wherein the production rate may vary between phases, alongside a fluctuating demand rate that evolves. The holding cost is presumed to escalate linearly over time.

A model with instantaneous deterioration assumes that items begin to deteriorate immediately after being stocked or produced, without any grace period. Instantaneous deterioration refers to a condition in inventory systems where items begin to lose their usability, quality, or value immediately after production or procurement, without any delay. In such cases, the deterioration process starts from time zero, and the inventory level continuously decreases over time due to decay, spoilage, or obsolescence regardless of whether the item is being sold or not. Sivashankari (2016) developed an EPQ model for items that deteriorate instantaneously, incorporating constant, linear, and quadratic holding costs, and conducted a comparative analysis of these cost structures. The research sought to ascertain the ideal production quantity and cycle duration by equilibrating production expenses, ordering expenses, holding expenses, and deterioration expenses.

A non-instantaneous deterioration model assumes that items do not begin to deteriorate immediately upon arrival or production. Instead, they remain unaffected for a preservation period, after which deterioration starts Musa and Sani (2012) formulated ordering policies for non-instantaneous deteriorating items with allowable payment delays. The demand rates prior to and subsequent to the onset of deterioration are distinct, with both assumed to be constant. The model assumes a constant deterioration for inventory items, with demand defined as a quadratic function of time and salvage value applied to deteriorated unit. Shortages are not permitted within the system. A mathematical framework is developed based on the assumption of supplier-provided trade credit and analyzed in two scenarios: (i) when the credit period is shorter than the inventory cycle and (ii) when the credit period exceeds the cycle length. Choudhury et al. (2013) formulated an inventory model for non-instantaneously deteriorating items characterized by a stock-dependent demand rate, time-variable holding costs, and shortages that are entirely backlogged. This model integrates a demand rate contingent upon stock levels, indicating that the demand for items is affected by the existing inventory quantity. Furthermore, it accounts for a time-dependent holding cost that fluctuates over time, as well as shortages that are entirely backlogged, indicating that all unmet demand will be satisfied subsequently when inventory becomes available. The model seeks to ascertain optimal order quantities and cycle durations to reduce costs or enhance profits within this intricate inventory system. Babangida and Baraya (2019) developed an inventory model for non-instantaneously deteriorating products characterized by two-phase demand and shortages. Complete backlogs and shortages are permissible. To minimize total variable costs, optimal periods of positive inventory cycles the duration as well as the order quantity.

A number of studies have focused on inventory models where demand varies with time. Demand that rises or reduces gradually can be described using a power or exponential functions. Sicilia et al. (2012) developed a deterministic inventory model characterized by a power demand pattern. This model offers a comprehensive analysis of inventory systems characterized by deterministic demand that fluctuates over time within every order cycle follow power pattern, and Shortages are permitted. The decision variables comprise the initial inventory level and the scheduling duration. Holding costs, shortage costs, and replenishment costs are taken into account. A multitude of scenarios were examined. The analysis commences with the inventory system devoid of shortages, subsequently examining systems that experience shortages. The systems of full backordering and sales that are completely lost s are analyzed, and the optimal policy for each is clearly established.

Traditional EPQ models often assume a uniform production rate. However, in many manufacturing environments, production occurs in multiple phases, often with different rates or capabilities. Power demand patterns have received special attention due to their ability to model a wide range of real-world demand behaviors. Two-phase production models have gained importance in textile, pharmaceutical, and automotive industries, where initial production is followed by secondary processing or quality checks. Sivashankari and Sivashankari and Panayappan (2014a) formulated a production inventory model for dual production levels, addressing defective items and integrating a multi-delivery policy.

The model integrates a policy for multiple deliveries within an inventory of production framework that accounts for items that are defective, considering two distinct production rates. It allows for an initial production rate to be altered after a certain period, which is advantageous, as commencing at a lower production rate mitigates the accumulation of excessive inventory at the outset, thereby minimizing holding costs. Sivashankari and Panayappan (2015) examined a production inventory model involving two tiers of production, deteriorating items, and shortages. The paper presents a model for inventory management of items that deteriorate over a time incorporates a pair of distinct production rates, allowing for a transition from one rate to another after a certain duration. This situation is advantageous as it prevents the accumulation of a substantial inventory of manufactured goods at the outset by commencing production at a low rate, thereby decreasing holding costs. Variable production rate satisfies customers and generate profit. Two models first, the production inventory model without shortages, then the model that accommodates shortages.

Most EPQ models assume holding cost to be constant per unit per time, but some adopt linear cost structures, especially when holding costs depend on quantity or time. Linear holding cost structures remain relevant in industries such as frozen food, chemicals, and bulk retailing where the cost per unit increases steadily with quantity held. Sivashankari (2016) examined an EPQ model for items that deteriorate instantaneously, incorporating constant, linear, and quadratic holding costs, and conducted a comparative analysis of these cost structures. Three models have been developed. Three models are used to find the optimal time and total cost. The holding cost can be classified 1) constant, 2) linear, or 3) quadratic. A mathematical model is developing for each scenario, the optimal production lot size that minimizes total costs is determined. Deo et al. (2022) formulated an inventory model that incorporates selling prices, time-varying demand, fluctuating holding costs, and dual storage facilities. This model is intended for perishable goods and incorporates elements such as sustainable technologies and a carbon cap-and-trade system, embodying a sustainable strategy for inventory management. The model incorporates learning effects during holding periods, transportation expenses, and sustainability expenditures. Babangida and Baraya (2022) enhanced the EOQ model for non-instantaneously deteriorating items by incorporating two-phase demand rates, time-dependent linear holding costs, and managing shortages with partial backlogging within a trade credit framework. This model seeks to ascertain the ideal order quantity and cycle duration to reduce total variable costs.

Backlogging refers to satisfying unmet demand later when inventory becomes available. Complete backlogging assumes all customers wait, while partial backlogging acknowledges that some customers cancel or substitute after a certain delay. Several studies emphasize that partial backlogging is more realistic, especially in competitive markets where customer loyalty is time-sensitive. Sarkar and Sakar (2013) introduced an inventory model featuring linear time-dependent partial backlogging and time-varying deterioration. This model offers a dynamic backlogging rate and a dynamic deterioration rate. This model determines the best cycle length for each product to reduce total cost, including holding cost, ordering, deterioration, and opportunity cost. Additionally, the necessary and sufficient conditions show the optimal solutions existence and uniqueness Baraya and Sani (2013) formulated an Economic Order Quantity (EOQ) model for non-instantaneously deteriorating items characterized by an inventory-level-dependent demand rate and constant partially backlogged shortages.

The model integrates a demand rate that is contingent upon inventory levels, whereby demand escalates with increasing inventory, and there exist constant, partially backlogged shortages, indicating that some customers are amenable to waiting for backorders while others pursue alternatives. The model seeks to ascertain the optimal order quantity that reduces total inventory expenses, taking into account these particular conditions. Sunshil and Rajput (2014) proposed an inventory model for deteriorating items featuring time-dependent partial backlogging. The aim of the research is to reduce the overall variable inventory cost, accompanied by a numerical example to demonstrate the proposed model. A sensitivity analysis is regarded in the constructed model. Jaggi et al. (2016) examined the impact of inflation and the time value of money on an inventory system characterized by deteriorating items and exponential partial backlogged shortages. This model examined the impact of financial factors on optimal inventory decisions, particularly regarding order quantity and acceptable shortage levels within a finite replenishment cycle. The research emphasized the various impacts of inflation and the time value of money become more significant as the disparity between the inflation rate and the time value of money widens. Krishnamoorthi and Sivashankari (2016) formulated production inventory models for perishable items incorporating three tiers of production and shortages. These models encompass both the production and post-production phases, factoring in deterioration and steady demand throughout both stages. The study concentrates on enhancing cycle time and lot size to reduce total expenses. Bello and Baraya (2018) formulated an inventory model for a non-instantaneously deteriorating item characterized by dual-phase demand rates and a constant partial backlogging rate. This model specifically addresses situations characterized by a two-phase demand rate fluctuation and a persistent partial backlog of shortages.

The model seeks to ascertain the ideal order quantity and cycle duration to minimize total expenses or maximize profit. Babangida and Baraya (2021) formulated an EOQ model for non-instantaneously deteriorating items characterized by a time-dependent quadratic deterioration rate, linear holding costs, and a partial backlogging rate, all within the framework of a trade credit policy. This model seeks to ascertain the ideal the number of items ordered and cycle duration to reduce total variable costs, considering the intricacies of perishable goods and trade credit. Nath and Sen (2022) formulated a partially backlogged inventory model for perishable items incorporating penalty costs and time-variable holding costs. The model seeks to ascertain the ideal order quantity and stock out threshold to reduce the total average cost. It evaluates the influence of time-variable holding costs and the penalty costs linked to shortages, rendering it pertinent for situations where these elements are substantial.

Malumfashi et al. (2022) developed an EPQ model for delayed deteriorating items with two-phase production period, exponential demand rate and linear holding cost. However, the assumption of exponential demand rate is not applicable to most items because the demand rate of almost every product may not change with the higher rate of change as exponential, similarly the demand rate at the beginning of the inventory cycle may not be same with at the middle or at the end of the cycle and this situation can best be represented by power demand Buhari *et al*. (2025) proposed an EPQ model for delayed deteriorating items with two-phase production period, power demand rate, linear holding cost and partially backlogged shortages. The model determines the best time interval and order quantity to reduce inventory costs. However, when scarcity occur, it cannot be assumed that all customers will be willing to wait for a backorder due to their impatient and the dynamic nature of individuals. During times of scarcity, certain customers whose requirements are not passing that moment might experience an impact, some may choose to wait for backorders to be completed, while others might decide to purchase from different sellers. In this research, an optimal production model for non-instantaneous decaying goods with two-stage production period, power demand pattern, linear holding cost and partial backorder is proposed. The model determines the optimal timing with positive inventory, duration of shortages, total production cost, and economic production quantity. The model's validity is demonstrated through a numerical example, followed by a comparative analysis with an existing model. Results show that the proposed model produces a lower average total production cost, indicating better performance. Furthermore, a sensitivity analysis is conducted on selected parameters to evaluate their impact on total cost and EPQ, with recommendations provided to help manufacturers minimize costs.

Notations and Assumptions

Notations

The following notation will be used in developing the proposed models.

b The demand rate during accumulation of inventory is measured in units per unit time.

b_1 The demand rate occurs after production and before deterioration begins within a specified unit of time

b_2 The demand rate after deterioration begins in unit per unit time.

a Quantity of production per unit of time.

I_1 Inventory close at time t_1.

I_2 Inventory close at time t_2.

I_3 Inventory near at time t_3.

ca Unit cost of Production.

ω Deterioration rate

H_C Cost of holding per unit of time.

F The cost of setting up each production cycle.

T The duration of an inventory cycle.

t_i Unit of time in periods, $i = 1,2,3,4$ and 5

Q_d Total deteriorated items

B_z Cost of a shortage per unit of time

C_π The price of lost sales per unit of time.

B Backorder level during a shortage.

C Unit cost

δ Backlogging parameter

Assumption

The following assumptions are adopted from Malumfashi et al. (2022) in addition assumptions 4 and 9 which is not considered in their work.

1- Production that occurs instantaneously

2- Change in production rate that occurs instantaneously

3- Supply of capital is not constrained.

4- Shortage are permitted and partially backlogged at the rate $\delta(0 < \delta < 1)$

5- Following inspection, all defective items are discarded

6- In-stock production exceeds demand.

7- The inventory accumulation is presume to occur in two phases with varying production rates, defined as a function. $f(x) = \begin{cases} a; 0 \leq t \leq t_1 \\ ca; t_1 \leq t \leq t_2 \end{cases}$

Where $ca[(0, 1) \text{ U } (1, 2)]$

8- It assumed that demand rate b and b_2 remain constant during the inventory build-up and prriod of deterioration, respectively.

9- The demand rate b_1 is assumed to be a demand function of time define as $b_1 = t^{\frac{1-n}{n}}$ at the point production stopped and $0 < \beta <$ .This situation occurs before deterioration begins.

10- If the holding cost per unit item is assumed to be a linear function of time, it is defined as $H_C = h_1 + h_2 t$

11- The inventory system has limited time boundaries. That is $t_1 < t_2 < t_3 < t_4 < t_5 T$

12- The continuous deterioration over a period of up to 1 year is deemed unreasonable; therefore it is assumed that $T - t_5 < 1$.

The Models Mathematical Formulation

As illustrated in Figure 1, the level of inventory changes over time under the influence of two production phases, power demand, item deterioration, and eventual shortages. Production begins $t = 0$ at with a constant rate a and continues until time t_1. During this stage, the demand rate b stays constant, allowing the inventory to grow and reach level I_1 at t_2. Afterward, from t_1 to t_2, the production rate shifts to ca, where the parameter c lies within the interval $(0,1) \cup (1,2)$, indicating a decrease or increase in output. The demand remains unchanged, and inventory increases until it hits a maximum level I_2 at t_2, when production stops. In the period t_2 to t_3 , inventory decays only due to demand, which is now modeled as a power demand rate ($b_1 = T^{\frac{1-n}{n}}$), leading to inventory level I_2 at t_3. At this point, deterioration sets in, and between t_3 and t_4, the inventory continues to decrease because both constant demand and item decay, eventually reaching zero at t_4. From onward, shortages accumulate, hit the highest point at level s by time t_5, when the next production run begins. Shortages are permitted and partially backlogged at $t = t_5$.

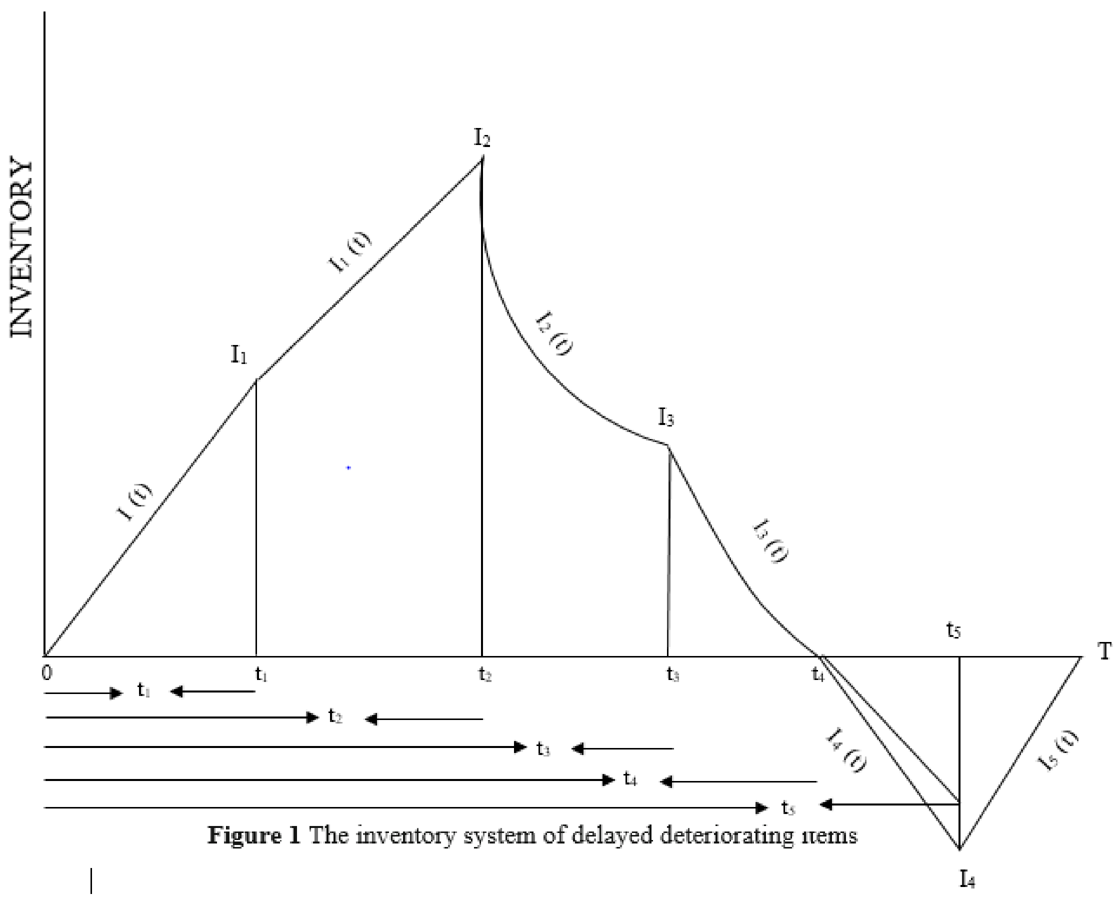

Figure 1 The inventory system of delayed deteriorating items

The model behavior is described mathematically using the following differential equations:

$$\frac{d\mathrm{I}(t)}{dt} = a - b, \qquad 0 \le t \le t_1 \tag{1}$$

with the boundary condition $I(t) = 0$ at $t = 0$ and $I(t_1) = I_1$

$$\frac{dI_1(t)}{dt} = ca - b, \qquad t_1 \le t \le t_2 \tag{2}$$

With the boundary condition $\mathrm{I}_1(t) = \mathrm{I}_1$ at $t = t_1$ and $\mathrm{I}_1(t) = \mathrm{I}_2$ at $t = t_2$

$$\frac{dI_2(t)}{dt} = -b_1 = -\frac{rt^{\frac{1-n}{n}}}{nT^{\frac{1}{n}}} t_2 \le t \le t_3 \tag{3}$$

with the boundary condition $\mathrm{I}_2(\mathrm{t}) = \mathrm{I}_3$ at $\mathrm{t} = t_2$ and $\mathrm{I}_2(\mathrm{t}) = \mathrm{I}_3$ at $\mathrm{t} = t_3$

$$\frac{dI_3(t)}{dt} + \omega I_3(t) = -b_2 t_3 \le t \le t_4 \tag{4}$$

With the boundary condition $\mathrm{I}_3(\mathrm{t}) = \mathrm{I}_3$ at $\mathrm{t} = t_3$ and $\mathrm{I}_3(\mathrm{t}) = 0$ at $\mathrm{t} = t_4$

$$\frac{dI_4(t)}{dt} = -b_2\delta \qquad t_4 \le t \le t_5 \tag{5}$$

With the boundary condition $\mathrm{I}_4(\mathrm{t}) = 0$ at $\mathrm{t} = t_4$

$$\frac{d\mathrm{I}(t)}{dt} = a - b, \qquad t_5 \le t \le T \tag{6}$$

With the boundary condition $I(t) = 0$ at $t = T$ and $I(\mathrm{t}) = B_m$ at $\mathrm{t} = \mathrm{T}$

The solutions of equation $(1-6)$ are as follows:

$$\mathrm{I}(t) = (a-b)t_1 \tag{7}$$

$$I_1(t) = (ca-b)(t-t_1) + (a-b)t_1 \qquad t_1 \le t \le t_2 \tag{8}$$

$$\mathrm{I}_2(\mathrm{t}) = \frac{r}{T^{\frac{1-n}{n}}}\left(t^{\frac{1}{n}} - t_3^{\frac{1}{n}}\right) + \frac{b_2}{\omega}\left(\mathrm{e}^{\omega(t_4 - t_3)} - 1\right) \tag{9}$$

$$\mathrm{I}_3(t) = \frac{b_2}{\omega}\left(\mathrm{e}^{\omega(t_4 - t_3)} - 1\right) \tag{10}$$

$$I_4(t) = b_2\delta(t_4 - t) \tag{11}$$

$$I(t) = (a-b)(t-T) \tag{12}$$

The maximum backorder l_4 is obtained at $t = t_5$, and then from equation (12), it follows that

$$\mathrm{I}_4 = -I(t_5) = (a-b)(t_5 - t_4)$$

The inventory holding cost

$$C_H = h_1 i\left[\int_0^{t_1} \mathrm{I}(t)dt + \int_{t_1}^{t_2} \mathrm{I}_1(t)dt + \int_{t_2}^{t_3} \mathrm{I}_2(t)dt + \int_{t_3}^{T} \mathrm{I}_3(t)dt\right] + h_2 i\left[\int_0^{t_1} \mathrm{I}(t)tdt + \int_{t_1}^{t_2} \mathrm{I}_1(t)tdt + \int_{t_2}^{t_3} \mathrm{I}_2(t)tdt + \int_{t_3}^{t_4} \mathrm{I}_3(t)tdt\right]$$

$$C_H = h_1 i\left[(a-b)\frac{{t_1}^2}{2} + (ca-b)\frac{1}{2}(t_2-t_1)^2 + (a-b)(t_2-t_1)t_1 + \frac{r}{T^{\frac{1-n}{n}}}\left(\frac{n{t_3}^{\frac{n+1}{n}}}{n+1} - {t_3}^{\frac{1+n}{n}} - \frac{n{t_2}^{\frac{n+1}{n}}}{n+1} + t_3^{\frac{1}{n}}t_2\right) + \frac{b_2(t_3-t_2)}{\omega}\left(\mathrm{e}^{\omega(t_4-t_3)}-1\right) + \frac{b_2}{\omega^2}\left(e^{(t_4-t_3)} - 1 - \omega(t_4-t_3)\right)\right] + h_2 i\left\{(a-b)\frac{t_1^3}{3} + \left[(ca-b)\left(\frac{t_2^3}{3} - \frac{t_1^3}{3} - t_1\frac{t_2^2}{2} + \frac{t_1^3}{2}\right) + (a-b)\left(t_1\frac{t_2^2}{2} - \frac{t_1^3}{2}\right)\right] + \left[\frac{r}{T^{\frac{1-n}{n}}}\left(\frac{n{t_3}^{\frac{2n+1}{n}}}{2n+1} - {t_3}^{\frac{2n+1}{n}} - \frac{n{t_2}^{\frac{2n+1}{n}}}{2n+1} + t_3^{\frac{1}{n}}\frac{t_2^2}{2}\right) + \frac{b_2(t_3^2-t_2^2)}{2\omega}\left(\mathrm{e}^{\omega(t_4-t_3)}-1\right)\right] + \left[\frac{b_2}{\omega^3}\left(-\omega t_4 + \omega t_3\mathrm{e}^{\omega(t_4-t_3)} - 1 + e^{\omega(t_4-t_3)} - \frac{\omega^2(t_4^2-t_3^2)}{2}\right)\right]\right\} \quad (13)$$

The total shortage cost is obtained by multiplying number of unit during shortage period by unit shortage cost and is given by

$$SC = B_z\left[\int_{t_4}^{t_5} -I_4(t)\,dt + \int_{t_5}^{T} -I(t)dt\right] = \frac{B_z}{2}\{b_2\delta(t_5-t_4)^2 + (a-b)(T-t_5)^2\} \quad (14)$$

The cost of lost sales per cycle is given by

$$LC = C_\pi\int_{t_4}^{t_5} b_2(1-\delta)dt = C_\pi b_2(1-\delta)(t_5-t_4) \quad (15)$$

To fine the cost of deterioration, multiply the total number of damaged items by the unit cost, which is given by

$$\mathrm{D}_C = C\left[\mathrm{I}_3 - \int_{t_3}^{t_4} b_2 dt\right] = C\left[\frac{b_2}{\omega}\left(\mathrm{e}^{\omega(t_4-t_3)}-1\right) - b_2(t_4-t_3)\right] \quad (16)$$

The total variable cost per unit time is the sum of setup cost, holding cost, deterioration cost and shortages cost divided by production cycle length and is given by

$$TC(t_5) = \frac{1}{T}[F + C_H + \mathrm{D}_C + SC + LC] \quad (17)$$

Substituting equation (13), (14), (15) and (16) into (17) to obtain

$$TPC(t_4, t_5) = \frac{1}{T}\Bigg[F + h_1 i\Bigg[(a-b)\frac{{t_1}^2}{2} + (ca-b)\frac{1}{2}(t_2-t_1)^2 + (a-b)(t_2-t_1)t_1 + \frac{r}{T^{\frac{1-n}{n}}}\left(\frac{n{t_3}^{\frac{n+1}{n}}}{n+1} - {t_3}^{\frac{1+n}{n}} - \frac{n{t_2}^{\frac{n+1}{n}}}{n+1} + t_3^{\frac{1}{n}}t_2\right) + \frac{b_2(t_3-t_2)}{\omega}\left(\mathrm{e}^{\omega(t_4-t_3)}-1\right) + \frac{b_2}{\omega^2}\Big(e^{\omega(t_4-t_3)} - 1 - \omega(t_4-t_3)\Big)\Bigg] + h_2 i\Bigg\{(a-b)\frac{t_1^3}{3} + \left[(ca-b)\left(\frac{t_2^3}{3} - \frac{t_1^3}{3} - t_1\frac{t_2^2}{2} + \frac{t_1^3}{2}\right) + (a-b)\left(t_1\frac{t_2^2}{2} - \frac{t_1^3}{2}\right)\right] +$$

$$\left[\frac{r}{T^{\frac{1-n}{n}}}\left(\frac{nt_3^{\frac{2n+1}{n}}}{2n+1} - t_3^{\frac{2n+1}{n}} - \frac{nt_2^{\frac{2n+1}{n}}}{2n+1} + t_3^{\frac{1}{n}}\frac{t_2^2}{2}\right) + \frac{b_2(t_3^2-t_2^2)}{2\omega}\left(\mathrm{e}^{\omega(t_4-t_3)}-1\right)\right] + \left[\frac{b_2}{\omega^3}\left(-\omega t_4 + \omega t_3\mathrm{e}^{\omega(t_4-t_3)} - 1 + e^{\omega(t_4-t_3)} - \frac{\omega^2(t_4^2-t_3^2)}{2}\right)\right]\Bigg\} + C\left[\frac{b_2}{\omega}\left(\mathrm{e}^{\omega(t_4-t_3)}-1\right) - b_2(t_4-t_3)\right] + \frac{B_z}{2}\{b_2\delta(t_5-t_4)^2 + (a-b)(T-t_5)^2\} + C_\pi b_2(1-\delta)(t_5-t_4)\Bigg] \quad 18)$$

Optimal Decision

The determination is made regarding to the production policy that is optimal in terms of reducing the total production cost per unit of time through the establishment of conditions that are necessary and sufficient. The essential conditions for the total production cost per unit time $TPC(t_4, t_5)$ to be at a minimum are $\frac{\partial TPC(t_4,t_5)}{\partial t_4} = 0$ and $\frac{\partial TPC(t_4,t_5)}{\partial t_5} = 0$. The value of (t_4, t_5) acquired from $\frac{\partial TPC(t_4,t_5)}{\partial t_4} = 0$ and $\frac{\partial TPC(t_4,t_5)}{\partial t_5} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TPC(t_4,t_5)}{\partial t_4^2}\right)\left(\frac{\partial^2 TPC(t_4,t_5)}{\partial t_5^2}\right) - \left(\frac{\partial^2 TPC(t_4,t_5)}{\partial t_4\partial t_5}\right)^2\right\} > 0$ is satisfied gives a minimum value for the total production cost per unit time $TPC(t_4, t_5)$.

Determine the optimal production policy that reduces total production cost per unit time, making conditions necessary and sufficient. The essential conditions for the total production cost per unit time $TPC(t_4, t_5)$ to be minimum are $\frac{\partial TPC(t_4,t_5)}{\partial t_4} = 0$ and $\frac{\partial TPC(t_4,t_5)}{\partial t_5} = 0$. The value of (t_4, t_5) obtained from $\frac{\partial TPC(t_4,t_5)}{\partial t_4} = 0$ and $\frac{\partial TPC(t_4,t_5)}{\partial t_5} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TPC(t_4,t_5)}{\partial t_4^2}\right)\left(\frac{\partial^2 TPC(t_4,t_5)}{\partial t_5^2}\right) - \left(\frac{\partial^2 TPC(t_4,t_5)}{\partial t_4\partial t_5}\right)^2\right\} > 0$ is satisfied gives a minimum value for the total production cost per unit time $TPC(t_4, t_5)$.

Differentiating (18) with respect to t_4 to obtain

$$\frac{\partial TPC(t_4,t_5)}{\partial t_4} = \frac{1}{T}\Bigg[h_1 i\left[b_2(t_3-t_2)\left(\mathrm{e}^{\omega(t_4-t_3)}\right) + \frac{b_2}{\omega}\left(e^{\omega(t_4-t_3)}-1\right)\right] + h_2\Bigg\{\left[\frac{b_2(t_3^2-t_2^2)}{2}\left(\mathrm{e}^{\omega(t_4-t_3)}\right)\right] + \left[\frac{b_2}{\omega^2}\left(-1+\omega t_3\mathrm{e}^{\omega(t_4-t_3)} + e^{\omega(t_4-t_3)} - \omega t_4\right)\right]\Bigg\} + C\left[b_2\left(\mathrm{e}^{\omega(t_4-t_3)}\right) - b_2\right] - B_z\{b_2\delta(t_5-t_4)\} + C_\pi b_2(\delta-1)\Bigg] \quad (19)$$

Similarly, differentiating (18) with respect to t_5 to obtain

$$\frac{\partial TPC(t_4,t_5)}{\partial t_5} = \frac{1}{T}\left[B_z\{b_2\delta(t_5-t_4) - (a-b)(T-t_5)\} + C_\pi b_2(1-\delta)\right] \quad (20)$$

Setting equation (20) to zero to obtain

$$\frac{\partial TPC(t_4,t_5)}{\partial t_5} = \frac{1}{T}\left[B_z\{b_2\delta(t_5-t_4) - (a-b)(T-t_5)\} + C_\pi b_2(1-\delta)\right] = 0 \quad (21)$$

From equation (21), the value of t_5 can be obtain as follows:

$$t_5 = \frac{B_z b_2\delta t_4 + B_z(a-b)T + C_\pi b_2(\delta-1)}{B_z\{b_2\delta + (a-b)\}} \quad (22)$$

Using the Maclaurin Series to get the approximate value of the exponential terms, i.e $e^{\omega(t_4 - t_3)} \cong 1 + \omega(t_4 - t_3)$, and substituting the value of t_5 from equation (22) into equation (18) to obtain

$$\frac{\partial TPC(t_4, t_5)}{\partial t_4} = \frac{b_2}{T}[Xt_4 - Y] \tag{23}$$

where

$$X = \left\{h_1 i[(t_3 - t_2)\omega + 1] + \frac{ih_2\omega(t_3^2 - t_2^2)}{2} + C\omega + \frac{B_z\delta(a-b)}{\{b_2\delta + (a-b)\}}\right\}$$

$$Y = \left\{h_1 i[t_2 + (t_3 - t_2)\omega t_3] + h_2 i\left[\frac{t_2^2}{2} + \frac{\omega t_3}{2}\right] + C\omega t_3 + \frac{B_z\delta(a-b)T}{\{b_2\delta + (a-b)\}} + \frac{C_\pi b_2\delta(\delta - 1)}{\{b_2\delta + (a-b)\}} + C_\pi(1-\delta)\right\}$$

Setting equation (23) to zero to obtain

$$Xt_4 - Y = 0 \tag{24}$$

From equation (24), let

$$F(t_4) = Xt_4 - Y \quad t_4 \in [t_3, \infty) \tag{25}$$

and

$\Delta = F(t_3) = Xt_3 - Y$

Then, the following result is obtained.

Lemma 1.

(i) If $\Delta \le 0$, then the solution of $t_4 \in [t_3, \infty)$ (say t_4^*) which satisfies equation (24) does not only exists but also unique.

(ii) If $\Delta > 0$, then the solution of $t_4 \in [t_3, \infty)$ which satisfies equation (24) does not exist.

Proof of (i): Taking the first-order derivative of $F(t_4)$ with respect to $t_4 \in [t_3, \infty)$ yields

$$\frac{F(t_4)}{dt_4} = X > 0 \tag{26}$$

Since $X = \left\{h_1 i[(t_3 - t_2)\omega + 1] + \frac{h_2\omega(t_3^2 - t_2^2)}{2} + C\omega + \frac{B_z\delta(a-b)}{\{b_2\delta + (a-b)\}}\right\} > 0$, this implies that $F(t_4)$ is a strictly increasing function of t_4 in the interval $[t_3, \infty)$. Moreover, $\lim_{t_4 \to \infty} F(t_4) = \infty$ and $F(t_3) = \Delta \le 0$. Therefore, by applying the intermediate value theorem, there exists a unique value of t_4 say $t_4^* \in [t_3, \infty)$ such that $F(t_4^*) = 0$. Hence,t_4^* is the unique solution of equation (24). Thus, the value of t_4 (denoted byt_4^*) can be found from equation (24) and is given by

$$t_4^* = \frac{Y}{X} \tag{27}$$

Once the value of t_4^* is obtained, then the value of t_5 (denoted by t_5^*) can be found from equation (21) and is given by

$$t_5^* = \frac{B_z b_2\delta t_4^* + B_z(a-b)T + C_\pi b_2(\delta - 1)}{B_z\{b_2\delta + (a-b)\}} \tag{28}$$

Equations (27) and (28) give the optimal values of t_4^* and t_5^* for the cost function in equation (18)

Proof of (ii): If $\Delta > 0$, then from equation (40),$F(t_4) > 0$. Since $F(t_4)$is a strictly increasing function of $t_4 \in [t_3, \infty)$, $F(t_4) > 0$ for all $t_4 \in [t_3, \infty)$. Thus, a value of $t_4 \in [t_d, \infty)$ such that $F(t_4) = 0$ cannot be found. This completes the proof.

Theorem 4.2.1.

(i) If $\Delta \leq 0$, then the total variable cost $TPC(t_4\ , t_5\ )$ is convex and reaches its global minimum at the point (t_4^*, t_5^*), where (t_4^*, t_5^*) is the point which satisfies equations (24) and (21).

(ii) If $\Delta > 0$, then the total variable cost $TPC(t_4\ , t_5\ )$ has a minimum value at the point (t_4^*, t_5^*), where $t_4^* = t_3$and $t_5^* = \frac{B_z b_2 \delta t_3^* + B_z(a-b)T + C_\pi b_2(\delta - 1)}{B_z\{b_2\delta + (a-b)\}}$.

Proof of (i): When $\Delta \leq 0$, it seen that t_4^* and t_5^* are the unique solutions of equations (24) and (21) from Lemma l(i). Taking the second derivative of $TPC(t_4\ , t_5\ )$ with respect to t_4 and t_5 and then finding the values of these functions at the point (t_4^*, t_5^*) yields

$$\left(\frac{\partial^2 TPC(t_4\ , t_5\ )}{\partial t_4^2}\bigg|_{(t_4^*, t_5^*)}\right)\left(\frac{\partial^2 TPC(t_4\ , t_5\ )}{\partial t_5^2}\bigg|_{(t_4^*, t_5^*)}\right) - \left(\frac{\partial^2 TPC(t_4\ , t_5\ )}{\partial t_4 \partial t_5}\bigg|_{(t_4^*, t_5^*)}\right)^2$$

$$= \frac{1}{T^2}\left\{b_2 h_1 i[(t_3 - t_2)\omega + 1][B_z\{b_2\delta + (a-b)\}] + \frac{b_2 i h_2 \omega(t_3^2 - t_2^2)}{2}[B_z\{b_2\delta + (a-b)\}] + b_2 C\omega[B_z\{b_2\delta + (a-b)\}] + B_z^2 b_2 \delta[(a-b) - b_2\delta]\right\} > 0 \quad (29)$$

It is therefore concluded from equation (29) and Lemma 1 that $TPC(t_4^*, t_5^*)$ is convex and (t_4^*, t_5^*) is the global minimum point of $TPC(t_4, t_5)$. Hence, the values of t_4^* and t_5^* in equations (42) and (4 3) are respectively optimal.

Proof of (ii): When $\Delta > 0$, then that $F(t_4) > 0$ for all $t_4 \in [t_3,\ \infty)$. Thus, $\frac{\partial TPC(t_4, t_5)}{\partial t_5} = \frac{F(t_5)}{t_5^2} > 0$ for all $t_4 \in [t_3, \infty)$, which implies that $TPC(t_4, t_5)$ is a strictly increasing function of t_5. Thus, $TPC(t_4, t_5)$ has a minimum value when t_5 is minimum. Therefore, $TPC(t_4, t_5)$ has a minimum value at the point (t_4^*, t_5^*), where $t_4^* = t_3$ and $t_5^* = \frac{B_z b_2 \delta t_3^* + B_z(a-b)T + C_\pi b_2(\delta - 1)}{B_z\{b_2\delta + (a-b)\}}$. This completes the proof.

EPQ =total demand during inventory build-up period + total demand before deterioration sets in+ total demand after deteriorated items+ total number of deteriorated items + total demand during shortage period.

$$= \int_0^{t_1} b dt + \int_{t_1}^{t_2} b dt + \int_{t_2}^{t_3} \frac{r t^{\frac{1-n}{n}}}{n T^{\frac{1}{n}}} dt + \int_{t_3}^{t_4} b_2 dt + \left[\frac{b_2}{\omega}\left(e^{\omega(t_4 - t_3)} - 1\right) - b_2(t_4 - t_3)\right] + \int_{t_4}^{t_5} \delta b_2 dt + \int_{t_5}^{T} b dt$$

$$= b t_2 + \frac{r}{T^{\frac{1-n}{n}}} t_3^{\frac{1}{n}} - \frac{r}{T^{\frac{1-n}{n}}} t_2^{\frac{1}{n}} + \frac{b_2}{\omega}\left(e^{\omega(t_4 - t_3)} - 1\right) + b_2\delta(t_5 - t_4) + b(t_5 - T) \quad (30)$$

Numerical Experiments

The proposed model is validated by conducting some numerical experiments. The parameter values are adopted from malumfashi et.al (2022), in addition to the parameters $c\pi$ and δ which did not appear in their work, and the values of these added parameters were also obtained optimally. The parameters and their values are summarized in table 1 below:

Table 1 Parameters and their values

Parameter	Value
F	₦3300 *set up cost per production*
a	8000 *unit per production*
c	1.2
b	4500 *unit per unit time*
b_2	2100 *unit per unit*
t_1	0.19178 *years*
t_2	0.82192 *years*
$c\pi$	9
δ	0.4
t_3	0.90411 *years*
ω	0.2
i	0.3
h_1	0.001
h_2	0.0001
C	₦1000 *per unit*
T	1.1
n	4
r	60
B_z	₦20

RESULTS

The results of the proposed model are obtained by substituting the parameter values presented in table 4.1 in equations (27), (28), (18) and (30), after which the values of the optimal time with positive inventory, time during shortage, total production cost and economic production quantity are obtained as presented in Table 2

Table 2 Optimal Results

Decision Variables	Values
t_4	0.9313 *year*
t_5	0.9367 *year*
$TPC(t_4, t_5)$	4157.2871
EPQ	4729.1386 *units*

Comparison

The proposed model is an extension of Malumfashi et.al (2022); hence the result from the model can be compared to demonstrate the optimality of the proposed model (see table 3). The average total production cost in the proposed model ($N0.87908$ *per unit*) is smaller than that of Malumfashi et. al (2022) ($N0.9936$ *per unit*). Since the aim of the the model is to minimize the total production cost per unit time, hence the proposed model is more optimal to that of Malumfashi et. al (2022).

Table 3 Comparison of the models

Model	Average Production Cost Per Unit
Malumfashi et.al (2022)	0.9936
Proposed Model	0.87908

Sensitivity Analysis
The sensitivity analysis of some model parameters is carried out on the obtained numerical solutions to observe the effect of those parameters on the total production cost and economic production quantity, after which suggestions and recommendations toward minimizing the total production cost will be given.

Table 4 Sensitivity Analysis

$Parameters$	% $Change\ in\ parameter$	% $Change\ in\ TPC$	% $Change\ in\ EPQ$
C	-50	-114.8263	4.8263
	-25	-23.6670	1.6796
	0	0	0
	25	11.3385	-1.0445
	50	18.2480	-1.7568
b	-50	58.1920	-92.6424
	-25	45.2511	-33.2901
	0	0	0
	25	374.5467	22.3666
	50	131.7828	40.3868
b_2	-50	45.6979	-10.0463
	-25	37.3567	-4.3751
	0	0	0
	25	-901.8078	3.4249
	50	-193.898	6.1302
ω	-50	-92.0619	14.6062
	-25	-19.5829	4.2278
	0	0	0
	25	9.1118	-2.05882
	50	14.3791	8.7496
B_z	-50	268.1917	-0.9790
	-25	-446.7518	3.0171
	0	0	0
	25	45.1963	-1.7983
	50	62.2145	-2.9468
$c\pi$	-50	46.9949	-7.6470
	-25	39.6840	-3.6861
	0	0	0
	25	1246.8686	3.4389
	50	162.3580	6.6548
δ	-50	239.2775	5.3992
	-25	-243.1719	2.5068
	0	0	0
	25	31.2326	-2.1396
	50	40.8717	-3.9326

1. An increase in the unit cost (C) of an item leads to a higher total production cost (TPC) and a reduction in the economic production quantity (EPQ). Consequently, higher unit costs correlate with higher total costs but lower optimal production volumes. To minimize total production cost and maximize economic production quantity, manufacturers must prioritize strategies that reduce unit costs.

2. An increase in the demand rate during the production phase (b) leads to a decrease in total production cost (TPC) and a rise in the economic production quantity (EPQ), while a decrease in demand results in the opposite effect. Practically, when demand grows during the production period, it typically requires a corresponding

increase in production output. This rise in output contributes to higher efficiency. In addition, a higher production-phase demand tends to lower both holding and deterioration costs, thereby reducing the overall production cost. To encourage higher demand during this phase, manufacturers are advised to adopt suitable marketing approaches such as advertising, offering discounts on price or quantity, and providing deferred payment options.

3. An increase in the demand rate during the production period (b) leads to a reduction in total production cost (TPC) and an increase in the economic production quantity (EPQ), while a decrease in demand has the opposite effect. In real-world scenarios, a higher demand during the production phase typically results in increased production output. This not only enhances operational efficiency but also contributes to lower holding and deterioration costs, thereby decreasing the total cost of production. To encourage higher demand during the production stage, manufacturers are advised to implement effective marketing tactics such as advertising, offering price or quantity-based discounts, and providing flexible payment terms like deferred payments.

4. As the deterioration rate (ω) increases, the total production cost (TPC) also rises, while the economic production quantity (EPQ) decrease, and the reverse occurs when the deterioration rate decreases. A higher deterioration rate leads to a higher number of spoiled items, which in turn raises deterioration costs and contributes to a higher total production cost. To manage this, EPQ is often reduced to limit losses from deterioration, thereby helping to lower the overall cost per unit time. It is therefore advisable for manufacturers to adopt effective measures such as improving storage conditions, offering promotional discounts, or introducing flexible payment options to decrease the deterioration rate and the number of items affected. This approach helps minimize production costs and improve the economic production quantity.

5. An increase in shortage cost (Bz) leads to a rise in both the total production cost(TPC) and the economic production quantity(EPQ). This indicates that as shortage costs become more significant, the overall production expenses also escalate, and when shortage costs decrease, total costs tend to decline. Similarly, a larger number of backordered units generally results in a higher economic production quantity, as company aim to balance inventory levels to avoid costly shortages. To address this, manufacturers often adjust their production schedules or increase safety stock to mitigate shortage risks, which can help optimize production costs and improve service levels. Ultimately, managing shortage costs effectively is crucial for maintaining a cost efficient and responsive production system.

6. When the backlogging parameter increases, the total production cost (TPC) tends to decrease, while the economic production quantity (EPQ) increases. This is due to a higher backlogging parameter implies that a greater proportion of the unmet demand during the shortage period is backordered rather than lost. As a result, the cost associated with lost sales which is often more detrimental to the businesses reduced. Conversely, when decreases, indicating that fewer customers are willing

to wait, the lost sales cost increases, leading to a higher total production cost and a lower EPQ. This suggests that allowing for a higher degree of backlogging can help manufacturers minimize costs and stabilize production planning. Therefore, businesses should adopt strategies to encourage customers to accept backorders, such as offering discounts or faster replenishment, in order to reduce the negative impact of lost sales.

7. The analysis confirms that the cost of lost sales per unit (C_π) plays a critical role in economic performance in economic production quantity (EPQ) system. Higher (C_π) values sharply increase total production cost (TPC), highlighting the importance of demand fulfillment and shortage minimization strategies. Manufacture should pay close attention to this parameter when designing inventory policies in environments with partially backlogged shortages.

CONCLUSION

This study shows a model for Economic Production Quantity (EPQ) model that addresses the limitations of classical assumptions by incorporating delayed deterioration, a two-phase production process, power-type demand, linear time-dependent holding costs, and partially shortages that are backlogged. Traditional EPQ models often assume an exponential demand rate, which might not match up with real-life situations where product demand does not change at such a rapid pace. Additionally, demand may vary throughout the inventory cycle, making power-type demand functions a more realistic representation. Using differential and integral calculus, the proposed model is analytically formulated. The existence of optimal solutions is proven through necessary and sufficient conditions supported by theorems and lemmas. The model identifies the optimal durations of inventory availability and shortages, as well as the economic production quantity and total production cost. Numerical simulations validate the model, followed by a comparative analysis with an existing model. Results indicate that the proposed approach yields a lower average total production cost, demonstrating its improved efficiency. A sensitivity analysis is performed to assess the effects of parameter variations on total cost and production quantity. Based on these findings, manufacturers are advised to minimize unit costs, adopt strategic marketing interventions during production and deterioration periods, and utilize improved storage solutions to reduce deterioration effects and costs. Increasing the proportion of backordered items during shortages is also suggested to reduce overall costs. Future studies may extend this model by extending the model to multi-product systems, incorporating demand uncertainty, or integrating environmental and financial constraints for more robust decision-making.

REFERENCES

Babangida, B. and Baraya, Y. M. (2021). An EOQ model for non-instantaneous deteriorating items with time dependent quadratic rate, linear holding cost and partial backlogging rate under trade credit policy. *Transaction of the Nigerian Association of Mathematical Physics*, 17(4), 131−144.

Baraya, Y. M. and Sani, B. (2012). An Economic Production Quantity (EPQ) Model for delayed deteriorating items with stock-dependent demand rate and time dependent deterioration rate, Journal of the Nigerian Association of Mathematical Physics, 22, 345-354.

Babangida, B., and Baraya, Y. M. (2019). An inventory model for non-instantaneous deteriorating item with time dependent quadratic demand and complete backlogging under trade credit policy. ABACUS: *Journal of the Mathematical Association of Nigeria*, **46**(1), 488−505.

Baraya, Y. M. and Sani, B. (2013). An EOQ model for delayed deteriorating items with inventory level dependent demand rate and partial backlogging. *Journal of the Nigerian Association of Mathematical Physics*, 25 (2), 295–308.

Babangida, B. and Baraya, Y. M. (2022). An EOQ model for non-instantaneous deteriorating items with two-phase demand rates, linear holding cost and time dependent partial backlogging rate under trade credit policy. ABACUS: *Journal of the Mathematical Association of Nigeria*, **49**(2), 91−125.

Bello, Y. and Baraya, Y. M. (2018). An inventory model for non-instantaneous deteriorating item with two-phase demand rate and partial backlogging. *Journal of the Nigerian Association of Mathematical Physics*, **47**(4), 77–86.

Choudhury, K. D., Karmakar, B., Das, M. and Datta, T. K. (2013). An inventory model for deteriorating items with stock dependent demand, time-varying holding cost and shortages. *Journal of the Operational Research Society*, **23**(1), 137–142.

Deo, D. A., Yogendra, K. R., Neha, G., Yashpal, S. R., Rakesh, R., Rahul, B. and Ajay, K. (2022). Selling price, time dependent demand and variable holding cost inventory model with two storage facilities. *Material today:* Proceedings, **56**(1), 245-251.

Jaggi, C., Khanna, A. and Nidhi, N. (2016). Effect of inflation and time value of money on an inventory system with deteriorating item and partial backlogged shortages. *International journal of industrial engineering computations,* 7(2), 267-282.

Krishnamoorthi, C. and Sivashankari, C.K. (2016) Production inventory models for deteriorating items with three levels of production and shortages. Yugoslav journal of operations research, [S.I.], v.27, n.4, p.499-519, ISSN 2334-6043.

Malumfashi, M. L., Ismail, M. T., and Ali, M. K. M. (2021a). An EPQ model for delayed deteriorating items with variable production rate, two-phase demand rates and shortages. In Springer Proceedings in Mathematics & Statistics (pp. 231-245). Springer, Singapore.

Malumfashi, M.L., Ismail, M.T., Bature, B., Sani, D., Ali, M.K.M. (2021b). An EPQ Model for Delayed Deteriorating Items with Two-Phase Production Period, Variable Demand Rate and Linear Holding Cost. In Modelling, Simulation and Applications of Complex Systems. CoSMoS 2019.Penang, Malaysia, April 8–11, 2019 (pp. 404–432). Springer Singapore.

Malumfashi, M.L., Ismail, M.T. and Ali, M.K.M. (2022) An EPQ Model for Delayed Deteriorating Items with Two-Phase Production Period, Exponential Demand Rate and Linear Holding Cost. Bulletin of the Malaysian Mathematical Sciences Society, 45, 395 424.https://doi.org/10.1007/s40840-022-01316-x

Musa, A. and Sani, B. (2012). Inventory ordering policies of delayed deteriorating items under permissible delay in payments. *International Journal of Production Economics*, **136**(1), 75–83.

Nath, B. K., Sen, N.A (2022). Partially Backlogged Inventory Model for Time-Deteriorating items Using Penalty Cost. Operation Research Forum3, **62**. http://doi.org/10.1007/s43069-022-00173-5.

Sivashankari, C. K. (2016). Production inventory model with deteriorating items with constant, linear and quadratic holding cost-a comparative study. *International Journal of Operational Research*, **27**(4), 589–609.

Sivashankari, C.K., and Panayappan, S. (2014a). Production inventory model for two levels of production with defective items and incorporating multi-delivery policy, *International Journal of Operation Research,* Vol. 19, No.3, pp.259-279.

Sivashankari C.K., Panayappan, S. (2015). Production inventory model for two levels of production with deteriorating items and shortages, *International Journal of Advanced Manufacturing Technology, Springer,* 76(9-12), 2003-2014.

Sicilia, J., Febles-Acosta, J. and Gonzalez-De la Rosa, M. (2012). Deterministic inventory Systems with power demand pattern. *Asia-Pacific Journal of Operational Research*, 29(5), 1250025.

Sakar, B. and Sakar, S. (2013). Inventory model with linear time dependent demand and partial backlogging. *Journal of Operational Research Society*, **30,** 924-932.

Sunshil, K. and Rajput, U. S. (2014). An inventory model for deteriorating item with time dependent demand and partial backlogged. *International journal operational research,* **15,** 12-23.