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Formal Examination of Operators and Properties of Generalized Ambiguous Four-Degree Logic

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Abstract
In this paper, we formally introduce the Generalized Ambiguous Quadruple Degree Membership Logic (GAF-DL) as an extension of the foundational principles of fuzzy logic and neutrosophic logic, incorporating Belnap's four-valued logic to provide a binary structure for truth evaluation. This paper explores a discrete intermediary system of Four-Degree Membership Logic. It presents a formal definition of a standard set of logical operators (Negation, Conjunction, Disjunction, Implication, and Equivalence). It analyzes their key properties, including associative laws, the Law of Contradiction, De Morgan's Laws, distributive laws, and commutative laws. Our study demonstrates that GAF-DL offers a computationally efficient and semantically rich approach for modelling phenomena where truth is not merely partial but also context-dependent and subject to multiple interpretations.

Keywords: Ambiguous, membership, four-degree, logic, uncertainty.

INTRODUCTION

This paper proposes formal definitions for a logical framework, the Generalized Ambiguous Four-Degree Membership Logic (GAF-DL). Extending the foundational principles of fuzzy, intuitionistic, and neutrosophic logics, GAF-DL introduces a fourth dimension to capture the inherent ambiguity of complex systems and natural language. The framework assigns to each element a quadruple of values representing its degree of truth, partial truth, partial falsity, and falsity. Also presents the mathematical formalisms for GAF-DL, defines its key operators (Negation, Conjunction, Disjunction, Implication, and Equivalence), and proves their fundamental properties. By formally distinguishing between partially true and partially false statements and indeterminacy, GAF-DL provides a more fundamental tool for modeling phenomena where truth is not merely partial but also context-dependent and subject to multiple interpretations.

The decision to choose these values is based on developing a logical connective with four degrees of truth, which offers a significant advantage in terms of simplicity and interpretability compared to the continuous spectrum of truth values in standard fuzzy (Zadeh, 2013) and neutrosophic logic (A.A.Salama, 2012) at the same time, providing more expressiveness than the binary truth of classical logic. Additionally, GAF-DL enables the explicit modeling of intermediate states of truth in real-world situations involving conditions

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that are neither definitely true nor false. The inclusion of partially true and partially false statements enables a more accurate representation of these nuanced states (Singh, 2023), which can be particularly valuable for modelling human reasoning or systems that operate with discrete but multivalued states. For example, the likelihood of an event that we may naturally think in terms of impossible, unlikely, likely, and inevitable.
Belnap (1977) introduced four-valued logic, which forms a logical lattice under the lattice ordering, with four elements: True (T), False (F), None (N), and Both (B). Discuss some of the main themes in medieval modal logic. It provides a high-level overview of various aspects of modal logic, and modal propositions allow one of the terms in a proposition to be a modal expression (Johnston, 2021). (2001) employed four-valued logic in their paper to prove several theorems from relevance logic related to computer science. They extended it to logical connectives conjunction and disjunction, which do not distribute over each other, thereby formulating mathematical semantics. Jakl (2025) provides logical connections via the representation of a bilattice, allowing the use of a large class of models of linear logics based on the Belnap–Dunn logic.

In fuzzy logic, membership degrees were assigned a precise truth degree, $\mu_p \in [0,1]$, to a proposition. However, a limitation arises when the membership grade $\mu_p(x)$ is inherently vague and cannot be accurately defined by a single, crisp value (Zadeh, 2013). Mares (2006) represents a paradigm shift by introducing the concept of fuzzy sets, which incorporate membership functions and continuous truth values. The operators are typically defined using t-norms (for AND) and t-conorms (for OR). Serrano-Guerrero et al. (2021) and Iancu (2012) introduced fuzzy logic controllers and their extensions that operate on linguistic terms and crisp data inputs, and discussed their applications in online shopping. Fuzzy Implication yields more desirable results across all forms of decision theory; the repercussions are modelled in many different ways, especially the most appropriate can be evaluated either empirically or axiomatically. Liang et al (2000) present the theory and design of interval type-2 fuzzy logic systems, which efficiently simplify the computation of the input and its antecedent operations using a general inference formula. Also deals with the concepts of upper and lower membership functions. Zimmermann (2010) uses three topic-neutral elements to identify logics as the foundations of reasoning: truth values, vocabulary (operators), and reasoning processes (tautologies and syllogisms). Truth values in dual logic can be either "true" (1) or "false" (0). Truth tables A and B represent two sentences or statements that can be true (1) or false (0), and operators are defined using these truth tables. Operators can be used to combine these statements. Considering one tautology, fuzzy reasoning relaxes these presumptions. Giordano (2021) investigates the properties of fuzzy logic by extending the fuzzy logic typicality operator and defines fuzzy multi-preference semantics for multiplayer perception. Pusztaházi et al (2025) examined symmetric difference operators within logical systems generated by nilpotent t-norms and t-conorms, specifically addressing their behaviour, functionality in bounded and Łukasiewicz fuzzy logic systems, and analysing their fundamental properties.

Intuitionistic fuzzy logic, as introduced by Atanassov, (2017) represents facts using two values: a degree of truth and a degree of falsity. These values range from 0 to 1 and must add up to 1 or less. This two-part approach enables the system to manage uncertainty and missing information effectively. Smarandache (2025) The definitions and operators of various forms of Neutrosophic sets, standard and non-standard Neutrosophic logic, which are analyzed using the three-degree membership function, are constructed on the mathematical framework of Neutrosophic N-norms and N-conorms, which are widely used in decision-making and artificial intelligence. Sai et al. (2025) imposed a constraint that combined the truth and falsity membership values to at most one and created a new membership that contradicted indeterminacy, spanning 4 degrees of membership, operators, and properties. Singh (2023) introduces an ambiguous four-degree set, ambiguous norms, and ambiguous conorms that satisfy commutativity, monotonicity, associativity, and boundary conditions.

The Generalized Ambiguous Four-Degree Membership Logic (GAF-DL) was a theoretical framework for developing a logical system capable of capturing ambiguity, which is essential for advancing fields such as natural language understanding, legal reasoning, and advanced computational decision-making. For any given element, we assign a quadruple value: a degree of truth (α_A), a degree of partially true (β_A), a degree of partially false (γ_A), and a degree of falsity (η_A). This framework is a conceptual generalization of existing neutrosophic logic. It incorporates Belnap's four-valued logic, providing a comprehensive model for a more nuanced reality in which a statement's truth or falsity is not only a matter of degree but also a function of multiple possible interpretations. The inclusion of a degree of partially true (βA) and a degree of partially false (γA) enables the explicit formalization of the distinction between having insufficient information to judge a statement with multiple conflicting interpretations of a valid statement.

Methodology
The methodological approach involves defining the GAF-DL set structure, establishing the mapping to the dual lattice space, and deriving the operational laws necessary to maintain the dual-lattice properties. The GAF-DL is proposed based on the axiomatic foundation of ambiguous A-norm and A-conorm, which are the genesis of uncertainty logical operators.

Four-value logic
The four-valued logic Belnap (1977) forms naturally logical lattice ordering into a set of four (true T, false F, none N, and both B) as listed in the table below:

Table 1: Negation of four-valued logic

	N	**F**	**T**	**B**
¬	**B**	**T**	**F**	**N**

Table 2: Disjunction of four-valued logic

∧	**N**	**F**	**T**	**B**
N	N	F	N	F
F	F	F	F	F
T	N	F	T	B
B	F	F	B	B

Table 3: Conjunction of four-valued logic

∨	**N**	**F**	**T**	**B**
N	N	N	T	T
F	N	F	T	B
T	T	T	T	T
B	T	B	T	B

Table 4: Implication of four-valued logic

→	**T**	**B**	**N**	**F**
T	T	B	N	F
B	T	B	T	B
N	T	T	N	N
F	T	T	T	T

Neutrosophic logic

Neutrosophic Logic Smarandache (1999)**:** A logic in which each proposition is estimated to have the percentage of truth in a subset T, the rate of indeterminacy in a subset I, and the percentage of falsity in a subset F, where T, I, and F are called Neutrosophic logic.

Standard Neutrosophic Logic Smarandache (1999)**:** If in the above definition of general neutrosophic logic all neutrosophic components, T, I, F, are standard real subsets, included in or equal to the standard real unit interval, $T, I, F \subseteq [0, 1]$, where
$0 \leq_N \mathrm{infT} + \mathrm{infI} + \mathrm{infF} \leq_N \mathrm{supT} + \mathrm{supI} + \mathrm{supF} \leq 3$
We have a standard neutrosophic logic.

Nonstandard Neutrosophic Logic Smarandache (2025): If in the above definition of general neutrosophic logic at least one of the neutrosophic components T, I, F is a nonstandard real mobinad subset, neutrosophically included in or equal to the nonstandard real mobinad unit interval$]-0,1+[_{MB}$ or $T, I, F, \subseteq_N\ ]-0,1+[_{MB}$.

General Neutrosophic logic Smarandache (2025)**:** is a multivalued logic in which each proposition P has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F), where T, I, F are standard or nonstandard real mobinad subsets of the nonstandard real mobinat unit interval $]-0,1+[_{MB}$ with $T, I, F, \subseteq_N\ ]-0,1+[_{MB}$ where
$-0 \leq_N \mathrm{infT} + \mathrm{infI} + \mathrm{infF} \leq_N \mathrm{supT} + \mathrm{supI} + \mathrm{supF} \leq 3+$.

Definitions Smarandache (2025)**:** the set of formulas (well–informed formulas) let (t_1, i_1, f_1) or (T_1, I_1, F_1) And (t_2, i_2, f_2) or (T_2, I_2, F_2) be the element of Neutrosophic logic where the sum of elements of the triplet logic connectives of Negation, conjunction, disjunction, implication, and equivalence $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$ can be defined in the following ways:

Neutrosophic negation:
$\neg(t_1, i_1, f_1) = (f_1, i_1, t_1)$
$\neg(t_1, i_1, f_1) = (f_1, 1 - i_1, t_1)$
$\neg(T_1, I_1, F_1) = (F_1, 1 - I_1, T_1)$

Neutrosophic conjunction:
$(t_1, i_1, f_1) \wedge (t_2, i_2, f_2) = (t = \min\{t_1, t_2\}, i = 1 - (t + f), f = \max\{f_1 f_2\})$
$(t_1, i_1, f_1) \wedge (t_2, i_2, f_2) = (t = \min\{t_1\ t_2\}, i = \max(i_1 i_2), f = \max\{f_1 f_2\})$
$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2)$

Neutrosophic disjunction:
$(t_1, i_1, f_1) \vee (t_2, i_2, f_2) = (t = \max\{t_1\ t_2\}, i = 1 - (t + f), f = \min\{f_1 f_2\})$
$(t_1, i_1, f_1) \vee (t_2, i_2, f_2) = (t = \max\{t_1\ t_2\}, i = \max(i_1 i_2), f = \{\min(f_1 f_2)\}$
$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2)$

Neutrosophic implication:
$(T_1, I_1, F_1) \longrightarrow_N (T_2, I_2, F_2) = (F_1, 1 - I_1, T_1) \vee_N (T_2, I_2, F_2)$
$= (F_1, 1 - I_1, T_1) \vee_N (T_2, i_2, f_2)$
$A_1(T_1, I_1, F_1) \longrightarrow_N A_2(T_2, I_2, F_2) = \neg_N A_1(T_1, I_1, F_1) \vee_N A_2(T_2, I_2, F_2)$

Neutrosophic equivalence:
$A_1(T_1, I_1, F_1) \leftrightarrow_N A_2(T_2, I_2, F_2)$ is neutrosophically is equivalent with $A_1(T_1, I_1, F_1) \rightarrow_N A_2(T_2, I_2, F_2)$ and $A_1(T_2, I_2, F_2) \rightarrow_N A_2(T_1, I_1, F_1)$

N – norms Smarandache (2001)**:** let $N_n: (\,]-0,1+[\times]-0,1+[\times]-0,1+[)^2 \rightarrow\,]-0,1+[\times]-0,1+[\times]-0,1+[$ and $N_n\big(x(T_1, I_1, F_1), y(T_2, I_2, F_2)\big) = \langle N_n T(x,y) N_n I(x,y) N_n F(x,y)\rangle$
Where $N_n T(\cdot,\cdot)$, $N_n I(\cdot,\cdot)$, $N_n F(\cdot,\cdot)$ are the truth membership, indeterminacy, and falsity membership, components respectively, N_n have satisfied for any x, y, z in the neutrosophic set/logic set M of the universe of discourse $\mathcal{U}$ the following axioms:
i. Boundary condition: $N_n(x, 0) = 0$, $N_n(x, 1) = x$
ii. Commutativity: $N_n(x, y) = N_n(y, x)$
iii. Monotonicity: if $x \le y$ then $N_n(x, y) \le N_n(y, x)$ and $N_n(x, z) \le N_n(y, z)$
iv. Associativity: $N_n(N_n(x, y)z) = N_n\big(x(N_n(y, z)\big)$.
N_n represents the AND operator in neutrosophic logic, and respectively the intersection operator in neutrosophic set theory.

N – conorms Smarandache (2001): $(\,]-0,1+[\times]-0,1+[\times]-0,1+[\,)^2 \rightarrow\,]-0,1+[\times]-0,1+[\times]-0,1+[$ and
$N_c\big(x(T_1, I_1, F_1), y(T_2, I_2, F_2)\big) = \langle N_c T(x,y) N_c I(x,y) N_c F(x,y)\rangle$
Where $N_n T(\cdot,\cdot)$, $N_n I(\cdot,\cdot)$,$N_n F(\cdot,\cdot)$ are the truth membership, indeterminacy, and respective falsehood membership components. N_c have satisfied, for any x, y, z in neutrosophic logic/set M of the universe of discourse $\mathcal{U}$, the following axioms:
i. Boundary condition: $N_c(x, 1) = 1$, $N_c(x, 0) = x$
ii. Commutativity: $N_c(x, y) = N_c(y, x)$
iii. Monotonicity: if $x \le y$ then $N_c(x, y) \le N_c(y, x)$ and $N_c(x, z) \le N_c(y, z)$
iv. Associativity: $N_c(N_c(x, y)z) = N_c\big(x, N_c(y, z)\big)$

Ambiguous Norms and Ambiguous Conorm
Ambiguous conorms (A-conorms) Singh (2023b)**:** is an operation $p: (X\star)^2$ on the interval [0.1], a function $A: ([0,1] \times [0,1] \times [0,1], [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1] \times [0,1]$ if $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle$
$A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2}\rangle$ $A_3 = \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3}\rangle$such that
$A(A_1, A_2) = \langle \min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2), \max(\eta_1, \eta_2)\rangle$ for $\forall\, A_1, A_2, A_3 \in A$ A- Conorms satisfied the following conditions:
i. Commutativity: $A(A_1, A_2) = A(A_2, A_1)$
ii. Associativity: $A\big(A_1 A(A_2, A_3)\big) = A\big(A_2, A(A_1, A_3)\big)$
iii. Monotonicity: $A(A_1, A_2) \le A(A_1, A_3)$ if and only if $A_1 \le A_3$
iv. Boundary Condition :$A(A_1, X\star\min) = A_1$

Ambiguous-norms (A-norms) Singh (2023b)**:** is an operation $A:: (X\star)^2 \rightarrow [0,1]$ on the interval [0.1], a function $A: ([0,1] \times [0,1] \times [0,1], [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1] \times [0,1]$ if $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle$
$A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2}\rangle$ $A_3 = \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3}\rangle$such that
$A(A_1, A_2) = \langle \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2), \min(\gamma_1, \gamma_2), \min(\eta_1, \eta_2)\rangle$ for $\forall\ A_1, A_2, A_3 \in A$ A- norms satisfied the following conditions:
i. Commutativity :$A(A_1, A_2) = A(A_2, A_1)$
ii. Associativity: $A\big(A_1 A(A_2, A_3)\big) = A\big(A_2, A(A_1, A_3)\big)$
iii. Monotonicity : $A(A_1, A_2) \le A(A_1, A_3)$ if and only if $A_1 \le A_3$
iv. Boundary Condition: $A(A_1, X\star\max) = A_1$

The Quadruplets membership degree function: with closed unit interval $[0,1]$ let a,b be the sub interval of $A \subseteq [0,1]$ Let U be a universe of discourse. An Ambiguous set A be defined by a membership function A: U→4 such that $A \subseteq [0.1]$, the membership is represented by the four-tuple: $A = \langle \alpha_A, \beta_A, \gamma_A, \eta_{A,} \rangle$ with the truth degree of membership function (α_A), partially true degree of membership function (β_A), partially false degree of membership function (γ_A), and falsity degree of membership function(η_A).

Mapping GAF-DL to the Continuous Bilattice Space: The algebraic structure of Belnap's four-value logic (Jakl, 2025) is based on the twist-product defined over a fundamental lattice $L = (min, max)$. To map the four degrees of GAF-DL into this structure, it is necessary to identify the core positive and negative evidence measures, A+ and A−. (Belnap Components A+ and A−) For a GAF-DL membership vector: $A = \langle \alpha_A, \beta_A, \gamma_A, \eta_A \rangle$, the corresponding positive (A+) and negative (A−) degrees of evidence are defined as: $A+= \alpha_A + \beta_A$ and $A-= \eta_A + \gamma_A$ The inclusion of partially true membership and partially false membership in both A+ and A− is justified by the Jakl, (2025) interpretation. The value both represents maximal information about the proposition, precisely because the conflicting evidence for truth and falsity is maximal. Therefore, partial membership contributes to both the positive and negative evidence base. Conversely, (indeterminacy/none) represents the absence of evidence for both α_A and η_A, and thus if we set $A+$ and $A-$ as (a, b) respectively, it acts as a residual measure of overall incompleteness, distinct from β_A and γ_A. Where:

$\max(a, b) = (\max(\inf a, \inf b), \max(\sup a, \sup b))$ and

$\min(a, b) = (\min(\inf a, \inf b), \min(\sup a, \sup b))$

Defining the Continuous Bilattice Orders in GAF-DL Space: In the GAF-DL, the space is four-structured by two distinct partial orders, which generalize the discrete orders of Belnap's four-value logic. Let $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$ and $A_2 = \langle x: \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$

Definition Logical Order ($\leq$): The logical order defines the flow of truth inference. A_1 is logically less than or equal to A_2, ($A_1 \leq A_2$) if A_2 supports truth at least as strongly as A_1, and supports falsity no more strongly than A_1. $A_1 \leq A_2 \Leftrightarrow (\alpha_{A1} \leq \alpha_{A2}$ and $\eta_{A1} \geq \eta_{A2})$ then $[a, b] = [(\inf a, \inf b)(\sup a, \sup b)]$ and $a \leq b = \inf a \leq \inf b \leq \sup a \leq \sup b$, where which form the logical order, which determines the degree of certainty in the proposition's veracity, is defined solely by the definitive components α_A and η_A. The ambiguous components, partially true (β_A) and partially false (γ_A), do not impact the core logical inference defined by $\leq$. The structure of this order is derived from the requirement that A_2 must be a state of greater overall information than A_1. Information increases when definitive evidence (α_A and η_A) increases, or when the explicit recognition of partially true (β_A) or partially false γ_A increases. Conversely, Information interpreted as acquiring more facts about the contradictory nature of the subject aligns with Belnap's four-value logic, in which both states are informationally maximal.

Operational Laws for GAF-DL: The operational laws must satisfy the dual lattice axioms. We define the standard logical connectives (negation, conjunction, and disjunction) and the informational connectives (meet and join) using generalized A-norms and A-conorms (Singh, 2023a) for algebraic simplicity. For this purpose, the minimum (min) and maximum (max) operators are used as the standard A-norm and A-conorm, respectively.

Generalized Ambiguous Four–Degree Logic

Definition Ambiguous Four-Degree Logic (GAF-DL): is a four-degree membership function which has a true degree membership α_A, partially true degree membership function β_A, partially false membership degree function γ_A, and a falsity degree membership function η_A, with standard unit interval $[0,1]$ where $\alpha, \beta, \gamma,$ and $\eta \in [0,1]$ and

$0^{-} \leq \inf\alpha_A + \inf\beta_A + \inf\gamma_A + \inf\eta_A \leq \sup\alpha_A + \sup\beta_A + \sup\gamma_A + \sup\eta_A \leq 1^{+}$.

Generalized Ambiguous Prepositional Calculus

In the case of generalised ambiguous logic, we set a prepositional with letters denoted by A_1, A_2, and A_3 and with the following connectives: conjunction, disjunction, negation, implication, and equivalence denoted by $\vee, \wedge, \neg, \longrightarrow$ and $\leftrightarrow$ respectively.

Sematic of Generalized ambiguous of preposition calculus we have $\langle \alpha_A, \beta_A, \gamma_A, \eta_A \rangle \subseteq [0,1]$ A_1, A_2 *and* A_3 are subsets of A. The formula for the operators of the Generalised ambiguous prepositional logic is:

i. If A_1 is a formula, we have $(\neg A_1)$

ii. $(A_1 \wedge A_2)$

iii. $(A_1 \vee A_2)$

IV.$(A_1 \longrightarrow A_2)$

V.$(A_1 \leftrightarrow A_2)$

Operation of Generalized Ambiguous Four Degree Logic (GAF-DL)

Negation of GAF-DL ($\neg A_1$): If the negation of A_1 denoted by $(\neg A_1)$ and $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$ then $\neg A_1$ is defined as:

$\neg A = \langle x, \eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1} \rangle$.

Example: consider the generalized ambiguous logics:

$A_1 = \langle 0.6, 0.2, 0.3, 0.7 \rangle$

$\neg A_1 = \langle \eta_A, \gamma_A, \beta_A, \alpha_A \rangle$

$\neg A_1 = \langle x, 0.7, 0.3, 0.2, 0.6 \rangle$

Definition of Conjunction of GAF-DL ($\wedge$): The Generalized ambiguous conjunction logic defined as: $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}(x), \eta_{A1} \rangle$ and $A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$, then

$$A_1 \wedge A_2 = \langle, \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \wedge \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$$
$$= \langle \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}), \max(\gamma_{A1}, \gamma_{A2}), \max(\eta_{A1}, \eta_{A2}) \rangle$$

Example: consider the two generalized ambiguous logics:

$A_1 = \langle 0.7, 0.3, 0.2, 0.8 \rangle$ and $A_2 = \langle 0.9, 0.5, 0.6, 0.1 \rangle$

$$A_1 \wedge A_2 = \left\langle \begin{matrix} \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}) \\ \max(\gamma_{A1}, \gamma_{A2}), \max(\eta_{A1}, \eta_{A2}) \end{matrix} \right\rangle$$

$A_1 \wedge A_2 = \langle 0.7, 0.3, 0.6, 0.8 \rangle$

Definition Disjunction GAF-DL ($\vee$): The ambiguous conjunction of logic A_1 and A_2 is defined as:

$A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$ and

$A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$ then

$\therefore A_1 \vee A_2 = \langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}) \min(\gamma_{A1}, \gamma_{A2}) \min(\eta_{A1}, \eta_{A2}) \rangle$

Example: consider the two generalized ambiguous logics defined over:

$A_1 = \langle \langle 0.7, 0.3, 0.2, 0.8 \rangle \rangle$ and $A_1 = \langle 0.9, 0.5, 0.6, 0.1 \rangle$

$$A_1 \vee A_2 = \left\langle \begin{matrix} \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}) \\ \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2}) \end{matrix} \right\rangle$$

$A_1 \vee A_2 = A_1 = \langle x, 0.9, 0.5, 0.2, 0.1 \rangle$

Definition Implication of GAF-DL ($\longrightarrow$): The generalized ambiguous implication of set A_1 and A_2 is defined as: if $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1,} \rangle$ and

$A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$, then;

$$A_1 \longrightarrow A_2 = \neg A_1 \vee A_2$$
$$= \langle \eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1} \rangle \vee \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$$

$= \langle \max(\eta_{A1}, \alpha_{A2}), \max(\gamma_{A1}, \beta_{A2}), \min(\beta_{A1}, \gamma_{A2}), \min(\alpha_{A1}, \eta_{A2}) \rangle.$

Example: if $A_1 = \langle\langle 0.7, 0.3, 0.2, 0.8 \rangle\rangle$ and $A_1 = \langle 0.9, 0.5, 0.6, 0.1 \rangle$

$$A_1 \longrightarrow A_2 = \left\langle \begin{matrix} \max(\eta_{A1}, \alpha_{A2}), \max(\gamma_{A1}, \beta_{A2}) \\ \min(\beta_{A1}, \gamma_{A2}), \min(\alpha_{A1}, \eta_{A2}) \end{matrix} \right\rangle$$

$A_1 \longrightarrow A_2 = A_1 = \langle 0.9, 0.5, 0.3, 0.1 \rangle$

Definition Equivalence of GAF-DL ($\leftrightarrow$): A_1 and A_2 are said to be generalized ambiguously equivalent if and only if $A_1 = A_2$, that is, if $A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1,} \rangle$ and $A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$, then

$A_1 \leftrightarrow A_2 = \langle \eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1} \rangle \longrightarrow \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle.$

$= \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle \longrightarrow \langle \eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1} \rangle$

Example:

$A_1 = \langle\langle 0.7, 0.3, 0.2, 0.8 \rangle\rangle$ and $A_2 = \langle 0.9, 0.5, 0.6, 0.1 \rangle$

$$A_1 \longrightarrow A_2 = \left\langle \begin{matrix} \max(\eta_{A1}, \alpha_{A2}), \max(\gamma_{A1}, \beta_{A2}) \\ \min(\beta_{A1}, \gamma_{A2}), \min(\alpha_{A1}, \eta_{A2}) \end{matrix} \right\rangle$$

$A_1 \longrightarrow A_2 = A_1 = \langle 0.9, 0.5, 0.3, 0.1 \rangle$

$$A_2 \longrightarrow A_1 = \left\langle \begin{matrix} \max(\eta_{A2}, \alpha_{A1}), \max(\gamma_{A2}, \beta_{A1}) \\ \min(\beta_{A2}, \gamma_{A1}), \min(\alpha_{A2}, \eta_{A1}) \end{matrix} \right\rangle$$

$A_1 \longrightarrow A_2 = A_1 = \langle 0.9, 0.5, 0.3, 0.1 \rangle$

Since $A_1 \longrightarrow A_2 = A_2 \longrightarrow A_1 = \langle 0.9, 0.5, 0.3, 0.1 \rangle$

$\therefore A_1 \leftrightarrow A_2$

Definition: logical Truth of generalizing ambiguous logic as the function membership value $A(\alpha_A) = \langle 1,0,0,0 \rangle$ and falsity logical value as $A(\eta_A) = \langle 0,0,0,1 \rangle$ such that $A \in [0,1]$

Theorem 1: $A_1 = (\alpha_{A1} + 0 + 0 + 1 - \alpha_{A1}) = 1$ and α_{A1} is any variable in generalised ambiguous logic, and $F: \alpha_{A1} \longrightarrow (\alpha_{A1} + 0 + 0 + 1 - \alpha_{A1})$ is a homomorphism of sub subset of generalised ambiguous logic.

Proof: if α_{A1} and α_{A2} are variables in Generalised Ambiguous Logic, the sum of $\alpha_{A1} + 0 + 0 + 1 - \alpha_{A1} = 1$

Let the mapping $f: (\alpha_{A1}) \longrightarrow (\alpha_{A1} + 0 + 0 + 1 - \alpha_{A1})$ and

$f: \alpha_{A2} \longrightarrow (\alpha_{A2} + 0 + 0 + 1 - \alpha_{A2})$

$f: (\neg\alpha_{A1}) = f(1 - \alpha_{A1}) = (1 - \alpha_{A1} + 0 + 0 + \alpha_{A1})$ and

$f: (\neg\alpha_{A1}) = \neg(\alpha_{A2} + 0 + 0 + 1 - \alpha_{A2})$

$= (1 - \alpha_{A1} + 0 + 0 + \alpha_{A1})$

$f: (\neg\alpha_{A1}) = \neg f(\alpha_{A1})$

Also

$f(\alpha_{A1} \wedge \alpha_{A2}) = \langle \mathrm{Min}(\alpha_{A1}, \alpha_{A2}), \mathrm{Min}(0,0), \mathrm{Max}(0,0), 1 - \mathrm{Min}(\alpha_{A1}, \alpha_{A2}) \rangle$

$$f(\alpha_{A1} \wedge \alpha_{A2}) = \left\langle \mathrm{Min}\big(\alpha_{A1}, \alpha_{A2}(x)\big), 1 - \mathrm{Max}\left(\alpha_{A1,}(x), \alpha_{A2}(x)\right) \right\rangle \quad (1)$$

$f(\alpha_{A1}) \wedge f(\alpha_{A2}) = \langle \alpha_{A1}, 0, 0, 1 - \alpha_{A1}(x) \rangle \wedge \langle \alpha_{A2}, 0, 0, 1 - \alpha_{A2} \rangle$

$= \langle \mathrm{Min}(\alpha_{A1}, \alpha_{A2}), \mathrm{Min}(0,0), \mathrm{Max}(0,0), \mathrm{Max}(1 - \alpha_{A1}, 1 - \alpha_{A2}) \rangle$

$$f(\alpha_{A1}) \wedge f(\alpha_{A2}) = \langle \mathrm{Min}(\alpha_{A1}, \alpha_{A2}), \mathrm{Max}(1 - \alpha_{A1}, 1 - \alpha_{A2}) \rangle \quad (2)$$

$\therefore$ From equations 1 and 2 we have

$\mathrm{Min}(\alpha_{A1}, \alpha_{A2}) + \big(1 - \mathrm{Min}(\alpha_{A1}, \alpha_{A2})\big) = 1$

$\mathrm{Min}(\alpha_{A1}, \alpha_{A2}) + \mathrm{Max}(1 - \alpha_{A1}, 1 - \alpha_{A2}) = 1$

$\therefore \mathrm{Max}(1 - \alpha_{A1}, 1 - \alpha_{A2}) = \big(1 - \mathrm{Min}(\alpha_{A1}, \alpha_{A2})\big)$

$\Rightarrow f(\alpha_{A1}) \wedge f(\alpha_{A2}) = f(\alpha_{A1}) \wedge f(\alpha_{A2})$

Similarly

$f(\alpha_{A1} \vee \alpha_{A2}) = \langle \mathrm{Max}(\alpha_{A1}, \alpha_{A2}), \mathrm{Max}(0,0), \mathrm{Min}(0,0), 1 - \mathrm{Max}(\alpha_{A1}, \alpha_{A2}) \rangle$

$= \langle \mathrm{Max}(\alpha_{A1}, \alpha_{A2}), 1 - \mathrm{Max}(\alpha_{A1}, \alpha_{A2}) \rangle$

$\therefore 1 = \mathrm{Max}(\alpha_{A1}, \alpha_{A2}) + \left(1 - \mathrm{Max}(\alpha_{A1}, \alpha_{A2})\right)$
$= \mathrm{Max}(\alpha_{A1}, \alpha_{A2}) + \mathrm{Min}(\alpha_{A1}, \alpha_{A2})$
$\Rightarrow 1 - \mathrm{Max}(\alpha_{A1}, \alpha_{A2}) = \mathrm{Min}(\alpha_{A1}, \alpha_{A2})$
$\Rightarrow f(\alpha_{A1}) \vee f(\alpha_{A2}) = f(\alpha_{A1}) \vee f(\alpha_{A2})$
The mapping in f is a homomorphism since $\langle \alpha_{A1}, 0,0, 1 - \alpha_{A1} \rangle = \langle \alpha_{A2}, 0,0, 1 - \alpha_{A2} \rangle$
$\Longrightarrow \alpha_{A1} = \alpha_{A2}$; then f is a one − to − one mapping.
Proposition 1: If $A_1 =, A_2 = \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2,} \rangle$ and $\alpha, \beta, \gamma, \gamma$and η are elements in generalized ambiguous logic where $A \in [1,0]$ then
i. $\neg A_1 = \langle \eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1} \rangle$
ii. $A_1 \wedge A_2 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \wedge \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$
iii. $A_1 \vee A_2 = \langle \alpha_{A1}, \beta_{A1,} \gamma_{A1}, \eta_{A1} \rangle \vee \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$
Are they also elements in generalized ambiguous logic and closed under the logical operations
Proof:
i. From Theorem 1, we have:
$\alpha_{A1} + \beta_{A1} + \gamma_{A1} + \eta_{A1} = 1$
if $\alpha_{A1} = \alpha_{A1}, \quad \beta_{A1} = 0, \; \gamma_{A1} = 0, \quad \eta_{A1} = 1 - \alpha_{A1}$
$\Rightarrow \langle \alpha_{A1}, 0,0, 1 - \alpha_{A1} \rangle = 1$
$A_1 = \langle \alpha_{A1}, 0,0,1 - \alpha_{A1} \rangle$
$\therefore 1 = \alpha_{A1} + 1 - \alpha_{A1}$
$\Longrightarrow 1 = 1$ (1)
Also $\neg A_1 = \langle 1 - \alpha_{A1}, 0,0, \alpha_{A1} \rangle$
$\therefore 1 - \alpha_{A1} + \alpha_{A1} = 1$ (2)
From equations 1 and 2, the elements are closed.
ii. $A_1 \wedge A_2 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \wedge \langle \alpha_{A2}, \beta_{A2,} \gamma_{A2}, \eta_{A2} \rangle$
$= \langle \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}), \max(\gamma_{A1}, \gamma_{A2}), \max(\eta_{A1}, \eta_{A2}) \rangle$
If $\min(\beta_{A1}, \beta_{A2}) = 0$ and $\max(\gamma_{A1}, \gamma_{A2}) = 0$
$\langle \min(\alpha_{A1}, \alpha_{A2}), \min(0,0), \max(0,0), \max(\eta_{A1}, \eta_{A2}) \rangle$
We prove that $\langle \min(\alpha_{A1}, \alpha_{A2}) + \max(\eta_{A1}, \eta_{A2}) \rangle \leq 1$
if we assume that, $\langle \min(\alpha_{A1}, \alpha_{A2}) + \max(\eta_{A1}, \eta_{A2}) \rangle > 1$
when $\min(\alpha_{A1}, \alpha_{A2}) = \alpha_{A1}$ and $\max(\eta_{A1}, \eta_{A2}) = \alpha_{A2}$
since $\alpha_{A1} + \alpha_{A2} > 1$ it follows that that $\alpha_{A2} < \alpha_{A1}$ this is contradiction from the assumption that $\alpha_{A1} \leq \alpha_{A2}$
$\therefore \langle \min(\alpha_{A1}, \alpha_{A2}) + \max(\eta_{A1}, \eta_{A2}) \rangle > 1$ is correct
$\therefore \langle \min(\alpha_{A1}, \alpha_{A2}) + \max(\eta_{A1}, \eta_{A2}) \rangle \leq 1$
This means the conjunction is closed.
iii. The proof of $A_1 \vee A_2 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$ is obviously similar to (ii) above.

Properties of Generalised Ambiguous Logic
We observed some of the basic properties of generalised ambiguous logic
$A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$
$A_2 = \langle x\colon \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle$
$A_3 = \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle$ with $A \in [0,1]$
Theorem 2: If A_1, $A_{2,}$ and A_3 are variables under Generalised Ambiguous logic, then they satisfy the following operations :
i. Commutative law
a. $A_1 \wedge A_2 = A_2 \wedge A_1$
b. $A_1 \vee A_2 = A_2 \vee A_1$
Proof:
$A_1 \wedge A_2 = A_2 \wedge A_1$

$$A_1 \wedge A_2 = \{\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \cup \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle\}$$
$$= \langle \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}), \max(\gamma_{A1}, \gamma_{A2}), \max(\eta_{A1}, \eta_{A2}) \rangle$$
$$= \langle \min(\alpha_{A2}, \alpha_{A1}), \min(\beta_{A2}, \beta_{A1}), \max(\gamma_{A2}, \gamma_{A1}), \max(\eta_{A2}, \eta_{A1}) \rangle$$
$$= A_2 \wedge A_1$$
$$\Rightarrow A_1 \wedge A_2 \subset A_2 \wedge A_1 \qquad (1)$$

Also, if

$$A_2 \wedge A_1 = \{\langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle \wedge \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle\}$$
$$= \langle \min(\alpha_{A2}, \alpha_{A1}), \min(\beta_{A2}, \beta_{A1}), \max(\gamma_{A2}, \gamma_{A1}), \max(\eta_{A2}, \eta_{A1}) \rangle$$
$$= \langle \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}), \max(\gamma_{A1}, \gamma_{A2}), \max(\eta_{A1}, \eta_{A2}) \rangle$$
$$= A_1 \wedge A_2$$
$$\Rightarrow A_2 \wedge A_1 \subset A_1 \wedge A_2 \qquad (2)$$

∴ From equations 1 and 2 we proved that $A_1 \wedge A_2 = A_2 \wedge A_1$.

b. We prove that $A_1 \vee A_2 = A_2 \vee A_1$ similar to 'a' above.

ii. Associative law

a. $A_1 \vee (A_2 \vee A_3) = (A_1 \vee A_1) \vee A_3$

If

$$A_1 \vee (A_2 \vee A_3) = \left\langle \begin{array}{c} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \cup \\ \{\langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle \cup \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle\} \end{array} \right\rangle$$
$$= \left\langle \begin{array}{c} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \cup \\ \{\langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}) \min(\gamma_{A1}, \gamma_{A2}) \min(\eta_{A1}, \eta_{A2}) \rangle\} \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \max(\alpha_{A1}, \alpha_{A2}, \alpha_{A3}) \max(\beta_{A1}, \beta_{A2}, \beta_{A3}) \\ \min(\gamma_{A1}, \gamma_{A2}, \gamma_{A3}), \min(\eta_{A1}, \eta_{A2}, \eta_{A3}) \end{array} \right\rangle$$
$$= \left\langle \begin{array}{c} \langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}) \min(\gamma_{A1}, \gamma_{A2}) \min(\eta_{A1}, \eta_{A2}) \rangle \\ \cup \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3}(x) \rangle \end{array} \right\rangle$$
$$= (A_1 \vee A_2) \vee A_3$$
$$\Rightarrow [A_1 \vee (A_2 \vee A_3) \subset (A_1 \vee A_2) \vee A_3] \qquad (1)$$

Similarly;

$$\text{If } (A_1 \cup A_2) \cup A_3 = \left\langle \begin{array}{c} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \cup \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle \\ \cup \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle \end{array} \right\rangle$$
$$= \left\langle \begin{array}{c} \langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}) \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2}) \rangle \\ \cup \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \max(\alpha_{A1}, \alpha_{A2}, \alpha_{A3}), \max(\beta_{A1}, \beta_{A2}, \beta_{A3}) \\ min(\gamma_{A1}, \gamma_{A2}, \gamma_{A3}), \min(\eta_{A1}, \eta_{A2}, \eta_{A3}) \end{array} \right\rangle$$
$$= \left\langle \begin{array}{c} \langle \{\max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}) \min(\gamma_{A1}, \gamma_{A2}) \min(\eta_{A1}, \eta_{A2}) \rangle \\ \cup \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle \end{array} \right\rangle$$
$$= (A_1 \vee A_2) \vee A_3$$
$$\Rightarrow [A_1 \vee (A_2 \vee A_3) \subset (A_1 \vee A_2) \vee A_3] \qquad (2)$$

∴ from equations 1 and 2

$A_1 \vee (A_2 \vee A_3) = (A_1 \vee A_1) \vee A_3$.

b. we prove that $A_1 \wedge (A_2 \wedge A_3) = (A_1 \wedge A_1) \wedge A_3$ similar to 'a' above.

Theorem 3: If A_1, A_2, and A_3 are variables under generalised Ambiguous logic, the following properties are satisfied under the operations:

i. Distributive law

a. $A_1 \vee (A_2 \wedge A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$

b. $A_1 \wedge (A_2 \vee A_3) = (A_1 \wedge A_2) \vee (A_1 \wedge A_3)$

ii. Absorption law

a. $A_1 \vee (A_1 \wedge A_2) = A_1$

b. $A_1 \wedge (A_1 \vee A_2) = A_1$

proof:

a. $A_1 \vee (A_2 \wedge A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$. We consider the left-hand side of the equation:

$$A_1 \vee (A_2 \wedge A_3) = \left\langle \begin{matrix} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \\ \{\langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle \wedge \langle \alpha_{A3}, \beta_{A3}, \gamma_{A1}, \eta_{A3} \rangle\} \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \\ \{\min(\alpha_{A2}, \alpha_{A3}), \min(\beta_{A2}, \beta_{A3}) \\ \max(\gamma_{A2}, \gamma_{A3}) \max(\eta_{A2}, \eta_{A3}\} \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \max(\alpha_{A1}, \min(\alpha_{A2}, \alpha_{A3})), \max(\beta_{A1}, \min(\beta_{A2}, \beta_{A3})) \\ \min(\gamma_{A1}, \max(\gamma_{A2}, \gamma_{A3})), \min(\eta_{A1}, \max(\eta_{A2}, \eta_{A3})) \end{matrix} \right\rangle \quad (1)$$

Also, we consider the right-hand side of the equation:

$$(A_1 \vee A_2) \wedge (A_1 \vee A_3) = \left\langle \begin{matrix} \{\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle\} \\ \wedge \\ \{\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle\} \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \langle \max(\alpha_{A1}, \alpha_{A2}) \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2}) \rangle \\ \wedge \\ \langle \max(\alpha_{A1}, \alpha_{A3}) \max(\beta_{A1}, \beta_{A3}), \min(\gamma_{A1}, \gamma_{A3}), \min(\eta_{A1}, \eta_{A3}) \rangle \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \min[\max(\alpha_{A1}, \alpha_{A2}), \max(\alpha_{A1}, \alpha_{A3})], \\ \min[\max(\beta_{A1}, \beta_{A2}), \max(\beta_{A1}, \beta_{A3})] \\ \max[\min(\gamma_{A1}, \gamma_{A2}), \min(\gamma_{A1}, \gamma_{A3})] \\ \max[\min(\eta_{A1}, \eta_{A2}), \min(\eta_{A1}, \eta_{A3})] \end{matrix} \right\rangle \quad (2)$$

From equations 1 and 2, we consider the following cases:

Case i: consider $\alpha_{A1} \geq \alpha_{A2}$ and α_{A3}, $\beta_{A1} \geq \beta_{A2}$ and $\beta_{A3,}$ $\gamma_{A1} \leq \gamma_{A2}$ and $\gamma_{A3,}$ $\eta_{A1} \leq \eta_{A2}$ and $\eta_{A3,}$

∴ Equation 1 becomes:

$$\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (3)$$

Also, equation 2 becomes;

$$\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (4)$$

Case ii: consider $\alpha_{A1} \geq \alpha_{A2} \geq \alpha_{A3}$, $\beta_{A1} \geq \beta_{A2} \geq \beta_{A3,}$ $\gamma_{A1} \leq \gamma_{A2} \leq \gamma_{A3,}$ $\eta_{A1} \leq \eta_{A2} \leq \eta_{A3,}$

∴ Equation 1 becomes:

$$= \langle \max(\alpha_{A1}, \alpha_{A3}), \max(\beta_{A1}, \beta_{A3}), \min(\gamma_{A1}, \gamma_{A3}), \min(\eta_{A1}, \eta_{A3}) \rangle$$
$$= \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (5)$$

Also, equation 2 becomes;

$$= \langle \min(\alpha_{A1}, \alpha_{A1}), \max(\beta_{A1}, \beta_{A1}), \min(\gamma_{A1}, \gamma_{A1}), \min(\eta_{A1}, \eta_{A1}) \rangle$$
$$= \langle x: \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (6)$$

Case iii: consider $\alpha_{A1} \geq \alpha_{A2} \leq \alpha_{A3}$, $\beta_{A1} \geq \beta_{A2} \leq \beta_{A3}$ $\gamma_{A1} \leq \gamma_{A2} \geq \gamma_{A3,}$ $\eta_{A1} \leq \eta_{A2} \geq \eta_{A3}$

∴ Equation 1 becomes:

$$= \langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A1}) \rangle$$
$$= \langle x: \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (7)$$

Also, equation 2 becomes;

$$= \langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2}) \rangle$$
$$= \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (8)$$

Case iv: consider $\alpha_{A2} \geq \alpha_{A1} \geq \alpha_{A3}$, $\beta_{A2} \geq \beta_{A1} \geq \beta_{A3,}$ $\gamma_{A2} \leq \gamma_{A1} \leq \gamma_{A3,}$ $\eta_{A2} \leq \eta_{A1} \leq \eta_{A3,}$

∴ Equation 1 becomes:

$$= \langle \max(\alpha_{A1}, \alpha_{A3}), \max(\beta_{A1}, \beta_{A3}), \min(\gamma_{A1}, \gamma_{A3}), \min(\eta_{A1}, \eta_{A3}) \rangle$$
$$= \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (9)$$

Also, equation 2 becomes;

$$= \langle \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}), \max(\gamma_{A1}, \gamma_{A3}), \max(\eta_{A1}, \eta_{A3}) \rangle$$
$$= \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \quad (10)$$

Case v: consider $\alpha_{A3} \geq \alpha_{A1} \geq \alpha_{A2}$, $\beta_{A3} \geq \beta_{A1} \geq \beta_{A2,}$ $\gamma_{A3} \leq \gamma_{A1} \leq \gamma_{A2,}$ $\eta_{A3} \leq \eta_{A1} \leq \eta_{A2,}$

∴ Equation 1 becomes:

$$= \langle \max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2}) \rangle$$

$= \langle x: \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$ (11)
Also, equation 2 becomes;
$= \langle \min(\alpha_{A1}, \alpha_{A3}), \min(\beta_{A1}, \beta_{A3}), \max(\gamma_{A1}, \gamma_{A3}), \max(\eta_{A1}, \eta_{A3}) \rangle$
$= \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$ (12)
Case vi: consider $\alpha_{A2} \geq \alpha_{A3} \geq \alpha_{A1}$, $\beta_{A2} \geq \beta_{A3} \geq \beta_{A1}$, $\gamma_{A2} \leq \gamma_{A3} \leq \gamma_{A1}$, $\eta_{A2} \leq \eta_{A3} \leq \eta_{A1}$,
$\therefore$ Equation 1 becomes:
$= \langle \max(\alpha_{A1}, \alpha_{A3}), \max(\beta_{A1}, \beta_{A3}), \min(\gamma_{A1}, \gamma_{A3}), \min(\eta_{A1}, \eta_{A3}) \rangle$
$= \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle$ (13)
Also, equation 2 becomes;
$= \langle \min(\alpha_{A2}, \alpha_{A3}), \min(\beta_{A2}, \beta_{A3}), \max(\gamma_{A2}, \gamma_{A3}), \max(\eta_{A2}, \eta_{A3}) \rangle$
$= \langle \alpha_{A3}, \beta_{A3}, \gamma_{A3}, \eta_{A3} \rangle$ (14)
In all the above cases, they have duplicate entries of both the right-hand side (equation 1) and the left-hand side (equation 2); therefore, the distribution law holds for union over intersection, that is
$A_1 \vee (A_2 \wedge A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$.
b. Similarly we can prove that $A_1 \wedge (A_2 \vee A_3) = (A_1 \wedge A_2) \vee (A_1 \wedge A_3)$ as in (a) above.
a. $A_1 \vee (A_1 \wedge A_2) = A_1$
If $A_1 \vee (A_1 \wedge A_2)$

$$A_1 \vee (A_1 \wedge A_2) = \left\langle \begin{matrix} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \\ \{\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \wedge \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle\} \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \\ \langle \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A1}), \max(\gamma_{A1}, \gamma_{A1}), \max(\eta_{A1}, \eta_{A1}) \rangle \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} \max(\alpha_{A1}, \min(\alpha_{A1}, \alpha_{A2})), \max(\beta_{A1}, \min(\beta_{A1}, \beta_{A2})) \\ \max(\gamma_{A1}, \min(\gamma_{A1}, \alpha_{A2})), \max(\eta_{A1}, \min(\eta_{A1}, \eta_{A2})) \end{matrix} \right\rangle$$
$$= \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$$
$$= A_1$$

$A_1 \subset A_1 \vee (A_1 \wedge A_2)$ (1)
If
$A_1 = \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle$
in each case if $\alpha_{A1} \leq \alpha_{A2}, \beta_{A1} \leq \beta_{A2}, \gamma_{A1} \leq \gamma_{A2}, \eta_{A1} \leq \eta_{A1}$ or
$\alpha_{A2} \leq \alpha_{A1}, \beta_{A2} \leq \beta_{A1}, \gamma_{A2} \leq \gamma_{A1}, \eta_{A2} \leq \eta_{A2}$ then

$$A_1 = \left\langle \begin{matrix} \max(\alpha_{A1}, \min(\alpha_{A1}, \alpha_{A2})), \max(\beta_{A1}, \min((\beta_{A1}, \beta_{A2})) \\ \min(\eta_{A1}, \max(\eta_{A1}, \eta_{A2})), \min(\gamma_{A1}, \max(\gamma_{A1}, \gamma_{A2})) \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \\ \left\langle \begin{matrix} \min(\alpha_{A1}, \alpha_{A2}), \min(\beta_{A1}, \beta_{A2}) \\ \max(\gamma_{A1}, \gamma_{A2}), \max(\eta_{A1}, \eta_{A2}) \end{matrix} \right\rangle \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} \langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \vee \\ \{\langle \alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1} \rangle \wedge \langle \alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2} \rangle\} \end{matrix} \right\rangle$$
$$= A_1 \vee (A_1 \wedge A_2)$$

$\Rightarrow A_1 \subset A_1 \vee (A_1 \wedge A_2)$ (2)
From equations 1 and 2, we conclude that $A_1 \vee (A_1 \wedge A_2) = A_1$.
b. $A_1 \wedge (A_1 \vee A_2) = A_1$, we can prove it similarly to 'a above.
Theorem 5: De Morgan's law
The following properties of generalised Ambiguous logic are satisfied under the operations if A_1 and A_2 are variables, then:
a. $\neg(A_1 \vee A_2) = \neg A_1 \wedge \neg A_2$

b. $\neg(A_1 \wedge A_2) = \neg A_1 \vee \neg A_2$

Proof:

a.If $\neg(A_1 \vee A_2) = \neg(\langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle \vee \langle\alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2}\rangle)$

$= \neg(\langle\max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2})\rangle)$

$= \langle\min(\eta_{A1}, \eta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \max(\beta_{A1}, \beta_{A2}), \max(\alpha_{A1}, \alpha_{A2})\rangle$

$= (\langle\eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1}\rangle \wedge \langle\eta_{A2}, \gamma_{A2}, \beta_{A2}, \alpha_{A2}\rangle)$

$= \neg A_1 \wedge \neg A_2$

$= \neg A_1 \vee \neg A_2 \subseteq \neg(A_1 \wedge A_2)$ (1)

$\neg A_1 \wedge \neg A_2 = (\langle\eta_{A1}, \gamma_{A1}, \beta_{A1}\alpha_{A1}\rangle \wedge \langle\eta_{A2}, \gamma_{A2}, \beta_{A2}, \alpha_{A2}\rangle)$

$= \langle\min(\eta_{A1}, \eta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \max(\beta_{A1}, \beta_{A2}), \max(\alpha_{A1}, \alpha_{A2})\rangle$

$= \neg\langle\max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2})\rangle$

$= \neg(\langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle \vee \langle\alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2}\rangle)$

$= \neg(A_1 \vee A_2)$

$\Rightarrow \neg(A_1 \vee A_2) \subseteq \neg A_1 \wedge A_2$ (2)

From equations 1 and 2, it has been clearly shown that $(A_1 \vee A_2)^c = A_1^c \wedge A_2^c$ holds.
Similarly, we can prove that $(A_1 \wedge A_2)^c = A_1^c \vee A_2^c$ as in (a) above. a.$(A_1 \vee A_2)^c = A_1^c \wedge A_2^c$
If

$\neg(A_1 \vee A_2) = \neg(\langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle \vee \langle\alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2}\rangle)$

$= \neg(\langle\max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2})\rangle)$

$= \langle\min(\eta_{A1}, \eta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \max(\beta_{A1}, \beta_{A2}), \max(\alpha_{A1}, \alpha_{A2})\rangle$

$= (\langle\eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1}\rangle \wedge \langle\eta_{A2}, \gamma_{A2}, \beta_{A2}, \alpha_{A2}\rangle)$

$= \neg A_1 \wedge \neg A_2$

$\Rightarrow \neg A_1 \wedge \neg A_2 \subseteq \neg(A_1 \vee A_2)$ (1)

If $\neg A_1 \cap \neg A_2 = \neg\langle\eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1}\rangle \wedge \neg\langle\eta_{A2}, \gamma_{A2}, \beta_{A2}, \alpha_{A2}\rangle$

$= \langle\min(\eta_{A1}, \eta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \max(\beta_{A1}, \beta_{A2}), \max(\alpha_{A1}, \alpha_{A2})\rangle$

$= \neg\langle\max(\alpha_{A1}, \alpha_{A2}), \max(\beta_{A1}, \beta_{A2}), \min(\gamma_{A1}, \gamma_{A2}), \min(\eta_{A1}, \eta_{A2})\rangle$

$= \neg(\langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle \vee \langle\alpha_{A2}, \beta_{A2}, \gamma_{A2}, \eta_{A2}\rangle)$

$= \neg(A_1 \vee A_2)$

$\Longrightarrow \neg(A_1 \cup A_2) \subseteq \neg A_1 \cap \neg A_2$ (2)

From Equations 1 and 2, it is clear that $(A_1 \cup A_2)^2 = A_1^c \cap A_2^c$
b. Similarly, we can prove that $(A_1 \cap A_2)^c = A_1^c \cap A_2^c$ as in (a) above.

Theorem 6: Involution Law:

If A_1 is a variable under generalised Ambiguous logic, then:

$\neg(\neg A_1) = A_1$

Proof:

If $A_1 = \langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle$

also $\neg A_1 = \langle\eta_{A1}, \gamma_A, \beta_A, \alpha\rangle$ then

$\neg(\neg A_1) = \neg(\langle\eta_{A1}, \gamma_A, \beta_A, \alpha_{A1}\rangle)$

$= \langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle$

$= A_1$

$\neg(\neg A_1) \subseteq A_1$ (1)

let $A_1 = \langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}\eta_{A1}\rangle$

$A_1 = \neg(\neg\langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}\eta_{A1}\rangle)$

$= \neg(\neg A_1)$

$\therefore A_1 \subseteq \neg(\neg A_1)$ (2)

From equations 1 and 2, we conclude that $\neg(\neg A_1) = A_1$

Theorem 7: In generalised ambiguous positional logic, the following equivalence holds;

i. $\neg\neg A_1 \leftrightarrow A_1$

ii. $\neg(A_1 \wedge A_2) \leftrightarrow \neg A_1 \vee \neg A_2$

iii. $\neg(A_1 \vee A_2) \leftrightarrow \neg A_1 \wedge \neg A_2$

proof: All the proof is clear from the above properties

Discussion

Table 5: Comparison Between the operation of Neutrosophic logic and Generalized Ambiguous logic

S/N	Operators	Neutrosophic Logic	Generalized Ambiguous logic
1	Negation (¬)	$\neg(t_1, i_1, f_1) = (f_1, 1 - i_1, t_1)$	$\neg A_1\langle\alpha_{A1}, \beta_{A1}, \gamma_{A1}, \eta_{A1}\rangle = \langle\eta_{A1}, \gamma_{A1}, \beta_{A1}, \alpha_{A1}\rangle$
2	Conjunction (∧)	$(t_1, i_1, f_1) \wedge (t_2, i_2, f_2) = \left\langle \begin{matrix} \min(t_1\, t_2), \max(i_1 i_2), \\ \max\,(f_1 f_2) \end{matrix} \right\rangle$	$A_1 \wedge A_2 = \left\langle \begin{matrix} \min(\eta_{A1}, \alpha_{A2}), \min(\gamma_{A1}, \beta_{A2}) \\ \max(\beta_{A1}, \gamma_{A2}), \max(\alpha_{A1}, \eta_{A2}) \end{matrix} \right\rangle$
3	Disjunction(∨)	$(t_1, i_1, f_1) \wedge (t_2, i_2, f_2) = \left\langle \begin{matrix} \max(t_1\, t_2), \min(i_1 i_2), \\ \min\,(f_1 f_2) \end{matrix} \right\rangle$	$A_1 \vee A_2 = \left\langle \begin{matrix} \max(\eta_{A1}, \alpha_{A2}), \max(\gamma_{A1}, \beta_{A2}) \\ \min(\beta_{A1}, \gamma_{A2}), \min(\alpha_{A1}, \eta_{A2}) \end{matrix} \right\rangle$
4	Implication (→)	$A_1(T_1, I_1, F_1) \rightarrow_N A_2(T_2, I_2, F_2) = \neg_N A_1(T_1, I_1, F_1) \vee_N A_2(T_2, I_2, F_2)$	$A_1 \rightarrow A_2 = A_1 \rightarrow A_2 = \neg A_1 \vee A_2$

The generalization Neutrosophic Logic (Smarandache, 2025) creates the Introduction of generalized Ambiguous Logic (GAF-DL), which has four sound membership degrees: truth membership degree, partially true membership degree, partially false membership degree, and falsity membership degree which addresses the Problem of inconsistent in indeterminacy membership degree which was extended to (partially true and partially false membership degrees) as shown in table 4 above. Therefore, handle incomplete uncertainty in real-world scenarios and address issues in paraconsistent neutrosophic logic.

Table 6: Comparison between the uncertainty Logic Properties

S/N	Properties	Fuzzy logic	Neutrosophic logic	Generalized Ambiguous Logic
1	Associative laws	yes	yes	yes
2	Distributive Laws	yes	yes	yes
3	Commutative Laws	yes	yes	yes
4	Absorption Laws	yes	yes	yes
5	Involution Laws	yes	yes	yes
6	De-Morgan laws	yes	yes	yes

According to Table 6, the Generalized Ambiguous Logic (GAF-DL) properties with respect to operators are clearly demonstrated. This shows that the proposed uncertainty logic, GAF-DL, represents a comprehensive generalization of the multi-valued logical frameworks. Fuzzy logic can be seen as a special case of a GAF-DL where the degrees of falsity membership, partial falsity, and partial truth are all zero, and the degree of valid membership is the only value considered. An intuitionistic fuzzy logic can also be viewed as a constrained version of a GAF-DL, where the partially true and partially false values are always zero, and the proper degree and falsity degrees are the residual values of the actual degree and falsity degrees, respectively. Neutrosophic logic, a closer conceptual predecessor, decouples the degrees of truth and falsity. Still, the GAF-DL framework extends this by providing a specific, formally defined dimension of ambiguity separate from indeterminacy. This distinction is critical because it prevents conflating a lack of knowledge with a multiplicity of valid interpretations.

CONCLUSION

The proposed Generalized Ambiguous Four-Degree Membership Logic (GAF-DL) has formalized a mathematical system Framework that extends the lineage of four-valued logics, four-degree membership uncertainty logic, true degrees of membership, partially true degrees of membership, partially false degrees of membership, and falsity degrees of membership as a new, distinct dimension for ambiguity. This framework formally distinguishes ambiguity from other forms of uncertainty, a critical conceptual leap for modeling complex human cognition and language. We have also proposed a new set of operators for this framework and demonstrated how it generalizes and subsumes fuzzy, intuitionistic, and neutrosophic logics.

REFERENCES

A.A.Salama, S. A. (2012). Neutrosophic Set and Neutrosophic Topological Spaces. *IOSR Journal of Mathematics*, *3*(4), 31–35. https://doi.org/10.9790/5728-0343135

Atanassov, K. T. (2017). Type-1 Fuzzy Sets and Intuitionistic Fuzzy Sets †. *Algorithms*, *10*(3), 1–12. https://doi.org/10.3390/a10030106

Belnap, N. D. (1977). A Useful Four-Valued Logic. In *Modern Uses of Multiple-Valued Logic* (pp. 5–37). Springer Netherlands. https://doi.org/10.1007/978-94-010-1161-7_2

Giordano, L. (2021). On the KLM Properties of a Fuzzy DL with Typicality. *Lecture Notes in Computer Science (Including Subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, *12897 LNAI*, 557–571. https://doi.org/10.1007/978-3-030-86772-0_40

Iancu, I. (2012). A Mamdani Type Fuzzy Logic Controller. *Fuzzy Logic - Controls, Concepts, Theories and Applications*. https://doi.org/10.5772/36321

Jakl, T. (2025). *Four imprints of Belnap's useful four-valued logic in computer science*. 1–21.

Johnston, S. C. (2021). Modal logic. In *The Routledge Companion to Medieval Philosophy*. https://doi.org/10.4324/9781315709604-6

Liang, Q., & Mendel, J. M. (2000). Interval type-2 fuzzy logic systems: Theory and design. *IEEE Transactions on Fuzzy Systems*, *8*(5), 535–550. https://doi.org/10.1109/91.873577

Pusztaházi, L. S., & Eigner, G. (2025). *A Comprehensive Study on the Different Approaches of the Symmetric Difference in Nilpotent Fuzzy Systems*. 1–23.

Sai, A. P., Arulselvam, A., & Bobin, A. (2025). Intuitionistic Quadri-Partitioned Neutrosophic Sets with Extended Operators. *Neutrosophic Sets and Systems*, *80*, 443–464.

Serrano-Guerrero, J., Romero, F. P., & Olivas, J. A. (2021). Fuzzy logic applied to opinion mining: A review. *Knowledge-Based Systems*, *222*. https://doi.org/10.1016/j.knosys.2021.107018

Singh, P. (2023a). An investigation of ambiguous sets and their application to decision-making from partial order to lattice ambiguous sets. *Decision Analytics Journal*, *8*(July), 100286. https://doi.org/10.1016/j.dajour.2023.100286

Singh, P. (2023b). An investigation of ambiguous sets and their application to decision-making from partial order to lattice ambiguous sets. *Decision Analytics Journal*, *8*(July), 100286. https://doi.org/10.1016/j.dajour.2023.100286

Smarandache F. (1999). *A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic. American Research Press, Rehoboth,.*

Smarandache, F. (2001). *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics (fourth edition)*. *October*. https://doi.org/10.6084/M9.FIGSHARE.1014204

Smarandache, F. (2025). Short Introduction to Standard and NonStandard Neutrosophic Set and Logic (review paper). *Neutrosophic Sets and Systems*, *77*(2024), 395–404.

Zadeh, L. A. (2013). Fuzzy logic. *Computational Complexity: Theory, Techniques, and Applications*, *9781461418*, 1177–1200. https://doi.org/10.1007/978-1-4614-1800-9_73

Zimmermann, H. J. (2010). Fuzzy set theory. *Wiley Interdisciplinary Reviews: Computational Statistics*, *2*(3), 317–332. https://doi.org/10.1002/wics.82