Fixed Point and Common Fixed Point Results in Dislocated Quasi Metric Spaces

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ABSTRACT
In this paper we establish some new fixed point results for self-mappings satisfying certain contractive conditions in the setting of dislocated quasi-metric spaces. Our established results extend and generalize some existing fixed point results in the literature. We also provide appropriate examples for the usability of the established results.

Keywords: Common fixed point; Complete dq-metric space; Contraction mapping; Fixed point

INTRODUCTION
Fixed point theory is one of the most dynamic research subjects in nonlinear analysis and can be used to many discipline branches such as; control theory, convex optimization, differential equation, integral equation, economics etc. In this area, the first important and remarkable result was presented by Banach in 1922 for a contraction mapping in a complete metric space. Dass and Gupta (1975) generalized the Banach contraction principle in a metric space for some rational type contractive conditions. Hitzler and Seda (2000) investigated the useful applications of dislocated topology in the context of logic programming semantics. Furthermore, Zeyada et al. (2005) generalized the results of Hitzler and Seda (2000) and introduced the concept of complete dislocated quasi metric space.

Aage and Salunke (2008) derived some fixed point theorems in dislocated quasi metric spaces. Similarly, Isufati (2010) proved some fixed point results for continuous contractive condition with rational type expression in the context of a dislocated quasi metric space. Kohli et al. (2010) investigated a fixed point theorem which generalized the result of Isufati. In 2012 Zoto (2012) gave some new results in dislocated and dislocated quasi metric spaces. For a continuous self-mapping, a fixed point theorem in dislocated quasi metric spaces was investigated by Shrivastava et al. (2012). In 2013, Patel and Patel constructed some new fixed point results in a dislocated quasi metric space. In 2014, Sarwar et al. and in 2017 Li et al. established fixed point results in dislocated quasi metric spaces.

In the present paper, we establish some fixed point results in the setting of dislocated quasi-metric spaces for single and a pair of continuous self-mappings which extend and generalize the results reported in the cited papers.
MATERIAL AND METHODS

Preliminaries

**Definition 2.1.** Zeyad et al. (2005): Let $X$ be a non-empty set, and let $d : X \times X \to [0, +\infty)$ be a function satisfying the following conditions:

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x) = 0$ implies that $x = y$;
3. $d(x, y) = d(y, x)$ for all $x, y \in X$;
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If $d$ satisfies the conditions from $d_1$ to $d_4$, then it is called a metric on $X$, if $d$ satisfies conditions $d_2$ to $d_4$, then it is called a dislocated metric (d-metric) on $X$, and if $d$ satisfies conditions $d_2$ and $d_4$, only then it is called a dislocated quasi metric (dq-metric) on $X$.

**Definition 2.2.** Zeyad et al. (2005): A sequence $\{x_n\}$ in a dq-metric space $(X, d)$ is called Cauchy sequence if for all $\epsilon > 0$ there exists a positive integer $N$ such that for all $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$.

**Definition 2.3.** Zeyad et al. (2005): A sequence $\{x_n\}$ is dislocated quasi-converges (dq-converges) to $x$ if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.$$  

In this case, $x$ is called a dq-limit of sequence $\{x_n\}$ and we can write $x_n \to x$ as $n \to \infty$.

**Definition 2.4** Zeyad et al. (2005): A dq-metric space $(X, d)$ is called complete if every Cauchy sequence in it is dq-convergent.

**Lemma 2.1** Zeyad et al. (2005): A dq-limits in a dq-metric space are unique.
Definition 2.5. Let \((X, d)\) be a dq-metric space. A self-map \(T : X \rightarrow X\) is called a contraction if there exists \(0 \leq k < 1\) such that
\[
d(Tx, Ty) \leq kd(x, y)
\]
for all \(x, y \in X\).

Theorem 2.1. Zeyad et al. (2005): Let \((X, d)\) be a complete dq-metric space and let \(T : X \rightarrow X\) be a continuous contraction function. Then \(T\) has a unique fixed point in \(X\).

Theorem 2.2. Aage and Salunke (2008): Let \((X, d)\) be a complete dq-metric space and \(T : X \rightarrow X\) be a continuous self-mapping satisfying
\[
d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)]
\]
for all \(x, y \in X\), where \(0 \leq \beta < \frac{1}{2}\). Then \(T\) has a unique fixed point.

Theorem 2.3. Aage and Salunke (2008): Let \((X, d)\) be a complete dq-metric space and \(T : X \rightarrow X\) be a continuous self-mapping satisfying
\[
d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)
\]
for all \(x, y \in X\), where \(a_1, a_2, a_3 \in \mathbb{R}^+\) and \(0 \leq a_1 + a_2 + a_3 < 1\). Then \(T\) has a unique fixed point.

Theorem 2.4. Shirvastava et al. (2012): Let \((X, d)\) be a complete dq-metric space and \(T : X \rightarrow X\) be a continuous self-mapping satisfying the following condition:
\[
d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)}
\]
for all \(x, y \in X\) satisfying \(d(x, y) \neq 0\), where \(a_1, a_2 \in \mathbb{R}^+\) and \(0 \leq a_1 + a_2 < 1\). Then \(T\) has a unique fixed point.

Theorem 2.5. Shirvastava et al. (2012): Let \((X, d)\) be a complete dq-metric space and \(T : X \rightarrow X\) be a continuous self-mapping. If
\[
d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + a_3 [d(x, Tx) + d(y, Ty)]
\]
\[+ a_4 [d(x, Ty) + d(y, Tx)]
\]
for all \(x, y \in X\) satisfying \(d(x, y) \neq 0\), where \(a_1, a_2, a_3, a_4 \in \mathbb{R}^+\) and \(0 \leq a_1 + a_2 + 2a_3 + 2a_4 < 1\), then \(T\) has a unique fixed point.
Theorem 2.6. Zoto et al (2012): Let \((X,d)\) be a complete dq-metric space and \(T : X \to X\) be a continuous self-mapping. If
\[
d(Tx, Ty) \leq a_1d(x, y) + a_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + a_3[d(x, Tx) + d(y, Ty)]
\]
\[
+ a_4[d(x, Ty) + d(y, Tx)] + a_5[d(x, Tx) + d(x, y)]
\]
for all \(x, y \in X\) satisfying \(d(x, y) \neq 0\), where \(a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^+\) and
\(0 \leq a_1 + 2a_3 + 2a_4 + 2a_5 < 1\), then \(T\) has a unique fixed point.

Theorem 2.7. Panthi et al. (2013): Let \((X,d)\) be a complete dq-metric space and \(T : X \to X\) be a continuous self-mapping. If
\[
d(Tx, Ty) \leq a_1d(x, y) + a_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + a_3[d(x, Tx) + d(y, Ty)]
\]
\[
+ a_4[d(x, Ty) + d(y, Tx)] + a_5[d(x, Tx) + d(x, y)]
\]
\[
+ a_6[d(y, Ty) + d(x, y)]
\]
for all \(x, y \in X\) satisfying \(d(x, y) \neq 0\), where \(a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}^+\) and
\(0 \leq a_1 + 2a_3 + 4a_4 + 2a_5 + 2a_6 < 1\), then \(T\) has a unique fixed point.

Theorem 2.8. Li et al. (2017): Let \((X,d)\) be a complete dq-metric space and \(T : X \to X\) be a continuous self-mapping. If
\[
d(Tx, Ty) \leq a_1d(x, y) + a_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + a_3[d(x, Tx) + d(y, Ty)]
\]
\[
+ a_4[d(x, Ty) + d(y, Tx)] + a_5[d(x, Tx) + d(x, y)]
\]
\[
+ a_6[d(y, Ty) + d(x, y)] + a_7[d(x, Ty) + d(x, y)]
\]
for all \(x, y \in X\) satisfying \(d(x, y) \neq 0\), where \(a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{R}^+\) and
\(0 \leq a_1 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1\). Then \(T\) has a unique fixed point.

Theorem 2.9. Patel and Patel (2013): Let \((X,d)\) be a complete dq-metric space and \(T : X \to X\) be a continuous self-mapping satisfying
\[
d(Tx, Ty) \leq a_1d(x, y) + a_2d(x, Ty) + a_3d(y, Tx) + a_4[d(x, Tx) + d(y, Ty)]
\]
\[
+ a_5[d(x, Ty) + d(y, Tx)]
\]
for all \( x, y \in X \), where \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^+ \) and \( 0 \leq a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1 \). Then \( T \) has a unique fixed point.

**Theorem 2.8.** Li et al. (2017): Let \((X,d)\) be a complete dq-metric space and \( T : X \to X \) be a continuous self-mapping. If

\[
d(x, y) \leq a_1d(x, y) + a_2d(x, y) + a_3d(y, y) + a_4d(x, y) + a_5d(x, y) + a_6d(x, y)
\]

for all \( x, y \in X \), where \( a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}^+ \) and \( 0 \leq a_1 + a_2 + a_3 + 2a_4 + 2a_5 + 2a_6 < 1 \). Then \( S \) and \( T \) have a unique common fixed point.

**Theorem 2.12.** Li et al. (2017): Let \((X,d)\) be a complete dq-metric space and \( S, T : X \to X \) be two continuous self-mappings satisfying

\[
d(x, y) \leq a_1d(x, y) + a_2d(x, y) + a_3d(y, y) + a_4d(x, y) + a_5d(x, y) + a_6d(x, y)
\]

for all \( x, y \in X \), where \( a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}^+ \) and \( 0 \leq a_1 + a_2 + a_3 + 2a_4 + 2a_5 + 2a_6 < 1 \). Then \( S \) and \( T \) have a unique common fixed point.

**RESULTS AND DISCUSSION**

**Theorem 3.1.** Let \((X,d)\) be a complete dislocated quasi metric space and \( T : X \to X \) be a continuous self-mapping satisfying the following condition:
\[ d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + a_3 [d(x, Tx) + d(y, Ty)] \]
\[ + a_4 [d(x, Ty) + d(y, Tx)] + a_5 [d(x, Tx) + d(x, y)] \]
\[ + a_6 [d(y, Ty) + d(x, y)] + a_7 [d(x, Ty) + d(x, y)] \]
\[ + a_8 [d(y, Tx) + d(x, y)] \]

(1)

for all \( x, y \in X \) with \( d(x, y) \neq 0 \), where \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \geq 0 \) and
\[ 0 \leq a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 + 3a_8 < 1. \] Then \( T \) has a unique fixed point.

**Proof:** Let \( x_0 \in X \) and define a sequence \( \{x_n\} \) in \( X \) as follows:

\[ Tx_n = x_{n+1} \quad \text{for } n = 0, 1, 2, 3, \ldots, \] where \( d(x_{n-1}, x_n) \neq 0 \). Set \( x = x_{n-1} \) and \( y = x_n \).

By using (1), the triangle inequality and \( d(x_n, x_n) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \) we have

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a_1 d(x_{n-1}, x_n) + a_2 \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + a_3 [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \]
\[ + a_4 [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] + a_5 [d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)] \]
\[ + a_6 [d(x_n, Tx_n) + d(x_{n-1}, x_n)] + a_7 [d(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)] \]
\[ + a_8 [d(x_n, Tx_{n-1}) + d(x_{n-1}, x_n)] \]
\[ = a_1 d(x_{n-1}, x_n) + a_2 \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + a_3 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \]
\[ + a_4 [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + a_5 [d(x_{n-1}, x_n) + d(x_{n-1}, x_n)] \]
\[ + a_6 [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + a_7 [d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)] \]
\[ + a_8 [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \]
\[ \leq (a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7 + 2a_8) d(x_{n-1}, x_n) \]
\[ + (a_2 + a_3 + 2a_4 + a_6 + a_7 + a_8) d(x_n, x_{n+1}). \]

Hence, we have:
Proof: we have

\[ d(x_n, x_{n+1}) \leq \frac{a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7 + 2a_8}{1 - (a_2 + a_3 + 2a_4 + a_6 + 2a_7 + 2a_8)} d(x_{n-1}, x_n). \]

Let

\[ \lambda = \frac{a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7 + 2a_8}{1 - (a_2 + a_3 + 2a_4 + a_6 + 2a_7 + 2a_8)}. \]

Clearly, \( 0 \leq \lambda < 1 \), since \( 0 \leq a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 + 3a_8 < 1 \). So,

\[ d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n). \]

Similarly,

\[ d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1}). \]

Thus

\[ d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1}). \]

Continuing the same procedure, we get

\[ d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \]

Now, for any \( m, n \in \mathbb{N} \) with \( m > n \), using the triangle inequality, we get

\[
\begin{align*}
    d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\
    & \leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \ldots + \lambda^{m-1} d(x_0, x_1) \\
    & = (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m-1}) d(x_0, x_1) \\
    & \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1).
\end{align*}
\]

For any \( \varepsilon > 0 \), we can choose a positive integer \( N \) such that, \( \frac{\lambda^N}{1 - \lambda} d(x_0, x_1) < \varepsilon \).

It follows that for any \( m, n \geq N \), we have

\[
\begin{align*}
    d(x_n, x_m) & \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1) \leq \frac{\lambda^N}{1 - \lambda} d(x_0, x_1) < \varepsilon.
\end{align*}
\]

This shows that \( \{ x_n \} \) is a Cauchy sequence in a complete dislocated quasi metric space \( (X, d) \). So, there exists \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \). Since \( T \) is continuous, so we have

\[
Tu = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = u.
\]

Hence, \( u \) is a fixed point of \( T \).
Suppose that $T$ has two distinct fixed points $u$ and $v$. Condition (1) implies that $d(u,u) = 0$ and $d(v,v) = 0$.

If $d(u,u) > 0$ and $d(v,v) > 0$, then by condition (1) we have

$$d(u,u) = d(Tu,Tu) \leq a_1 d(u,u) + a_2 \frac{d(u,Tu)d(Tu)}{d(u,u)} + a_3 [d(u,Tu) + d(v,v)]$$

$$+ a_4 [d(u,Tu) + d(v,v)] + a_5 [d(u,Tu) + d(u,v)]$$

$$+ a_6 [d(u,Tu) + d(u,v)] + a_7 [d(u,Tu) + d(u,v)]$$

$$+ a_8 [d(v,v) + d(u,v)].$$

Since $a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8 < 1$, which is a contradiction. Hence, $d(u,u) = 0$.

Similarly, $d(v,v) = 0$.

To show $d(u,v) = d(v,u) = 0$ using condition (1) we have

$$d(TuTv) \leq a_1 d(u,v) + a_2 \frac{d(u,Tu)d(vTv)}{d(u,v)} + a_3 [d(u,Tu) + d(vTv)]$$

$$+ a_4 [d(u,Tv) + d(vTv)] + a_5 [d(u,Tu) + d(u,v)]$$

$$+ a_6 [d(vTv) + d(u,v)] + a_7 [d(u,Tv) + d(u,v)]$$

$$+ a_8 [d(vTv) + d(u,v)].$$

Finally, from (2) we get:

$$d(u,v) \leq (a_1 + a_4 + a_5 + a_6 + 2a_7 + a_8) d(u,v) + (a_4 + a_8) d(v,u). \quad (3)$$

Similarly, we have:

$$d(v,u) \leq (a_1 + a_4 + a_5 + a_6 + 2a_7 + a_8) d(v,u) + (a_4 + a_8) d(u,v). \quad (4)$$

Subtracting (4) from (3) we have:

$$|d(u,v) - d(v,u)| \leq (a_1 + a_5 + a_6 + 2a_7) |d(u,v) - d(v,u)|. \quad (5)$$

Since $0 \leq (a_1 + a_5 + a_6 + 2a_7) < 1$, so the inequality (5) is possible if
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\[ d(u,v) - d(v,u) = 0. \] (6)

Taking equations (3), (4) and (6) into account, we have \( d(u,v) = 0 \) and \( d(v,u) = 0 \). Thus by condition \((d_2), u = v\). Hence \( T \) has a unique fixed point in \( X \).

**Example 3.1.** Let \( X = [0,1] \) with a complete dq-metric defined by

\[ d(x,y) = |x|, \text{ for all } x, y \in X \] and define a continuous self-mapping \( T \) by \( Tx = \frac{x}{2} \).

With

\[ a_1 = \frac{1}{8}, a_2 = \frac{1}{12}, a_3 = \frac{1}{14}, a_4 = \frac{1}{32}, a_5 = \frac{1}{20}, a_6 = \frac{1}{16}, a_7 = \frac{1}{18} \text{ and } a_8 = \frac{1}{24}, \]

\( T \) satisfies all the conditions of Theorem 3.1 and \( x = 0 \) is the unique fixed point of \( T \) in \( X \).

**Remarks:** In the Theorem 3.1:

If \( a_8 = 0 \), then we get Theorem 2.8.

If \( a_7 = a_8 = 0 \), then we get Theorem 2.7.

If \( a_6 = a_7 = a_8 = 0 \), then we get Theorem 2.6.

If \( a_5 = a_6 = a_7 = a_8 = 0 \), then we get Theorem 2.5.

If \( a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0 \), then we get Theorem 2.4.

In the following we prove a common fixed point result.

**Theorem 3.2.** Let \((X,d)\) be a complete dq-metric space and \( S, T : X \rightarrow X \) be two continuous self-mappings satisfying

\[ d(Sx,Ty) \leq a_1d(x,y) + a_2d(x,Sx) + a_3d(y,Ty) + a_4[d(x,Sx) + d(y,Ty)] + a_5[d(x,Ty) + d(y,Sx)] + a_6[d(x,Sx) + d(x,y)] + a_7[d(y,Ty) + d(x,y)] \] (2)

for all \( x, y \in X \), where \( a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{R}^+ \) and

\[ 0 \leq a_1 + a_2 + a_3 + 2a_4 + 4a_5 + 2a_6 + 2a_7 < 1. \]

Then \( S \) and \( T \) have a unique common fixed point.

**Proof:** Let \( x_0 \in X \), we define a sequence \( \{x_n\} \) in \( X \) as follows:
\[
Sx_0 = x_1, \quad Sx_1 = x_2, \ldots, \quad Sx_{2n} = x_{2n+1}
\]

and

\[
Tx_1 = x_2, \quad Tx_2 = x_3, \ldots, \quad Tx_{2n-1} = x_{2n}
\]

For all \( n \in \mathbb{N} \) we have

\[
d\left(x_{2n+1}, x_{2n+2}\right) = d\left(Sx_{2n}, Tx_{2n+1}\right).
\]

By using (2), the triangle inequality and \( d\left(x_n, x_{n}\right) \leq d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right) \)

We have,

\[
d\left(x_{2n+1}, x_{2n+2}\right) = d\left(Sx_{2n}, Tx_{2n+1}\right) \\
\leq a_1 d\left(x_{2n}, x_{2n+1}\right) + a_2 d\left(x_{2n}, Sx_{2n}\right) + a_3 d\left(x_{2n+1}, Tx_{2n+1}\right) \\
+ a_6 d\left(x_{2n}, Sx_{2n}\right) + d\left(x_{2n+1}, Tx_{2n+1}\right)] + a_7 d\left(x_{2n+1}, Sx_{2n}\right)
+ a_6 d\left(x_{2n}, Sx_{2n}\right) + d\left(x_{2n+1}, Tx_{2n+1}\right)] + a_7 d\left(x_{2n+1}, Sx_{2n}\right)
\]

\[
= a_1 d\left(x_{2n}, x_{2n+1}\right) + a_2 d\left(x_{2n}, x_{2n+1}\right) + a_3 d\left(x_{2n+1}, x_{2n+2}\right) \\
+ a_6 d\left(x_{2n}, x_{2n+1}\right) + d\left(x_{2n+1}, x_{2n+2}\right)] + a_7 d\left(x_{2n+1}, x_{2n+1}\right)
+ a_6 d\left(x_{2n}, x_{2n+1}\right) + d\left(x_{2n+1}, x_{2n+1}\right)] + a_7 d\left(x_{2n+1}, x_{2n+1}\right)
\]

\[
\leq (a_1 + a_2 + 2a_6 + 2a_7) d\left(x_{2n}, x_{2n+1}\right) + (a_3 + a_4 + 2a_5 + a_7) d\left(x_{2n+1}, x_{2n+2}\right),
\]

which implies that:

\[
d\left(x_{2n+1}, x_{2n+2}\right) \leq \frac{(a_1 + a_2 + 2a_6 + 2a_7 + a_7)}{(1-a_1 + a_2 + 2a_6 + a_7)} d\left(x_{2n}, x_{2n+1}\right).
\]

Let

\[
\lambda = \frac{a_1 + a_2 + a_4 + 2a_5 + 2a_6 + a_7}{1-(a_1 + a_2 + 2a_5 + a_7)}.
\]

Clearly, \( 0 \leq \lambda < 1 \), since \( 0 \leq a_1 + a_2 + a_3 + 2a_4 + 4a_5 + 2a_6 + 2a_7 < 1 \).

So,

\[
d\left(x_{2n+1}, x_{2n+2}\right) \leq \lambda d\left(x_{2n}, x_{2n+1}\right).
\]

Similarly,

\[
d\left(x_{2n}, x_{2n+1}\right) \leq \lambda d\left(x_{2n-1}, x_{2n}\right).
\]

Thus

\[
d\left(x_{2n+1}, x_{2n+2}\right) \leq \lambda^2 d\left(x_{2n-1}, x_{2n}\right).
\]
Continuing the same procedure, we get
\[
d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).
\]
Now, for any \(m, n \in \mathbb{N}\) with \(m > n\), using the triangle inequality, we get
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)
\[
\leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \ldots + \lambda^{m-1} d(x_0, x_1)
\[
= (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m-1}) d(x_0, x_1)
\[
\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).
\]

For any \(\varepsilon > 0\), we can choose a positive integer \(N\) such that \(\frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon\).

For any \(m, n \geq N\), we have
\[
d(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1) \leq \frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon.
\]

This shows that \(\{x_n\}\) is a Cauchy sequence in a complete dislocated quasi metric space \((X, d)\). So, there exists \(u \in X\) such that \(\lim_{n \to \infty} x_n = u\). Also the sub-sequences \(\{x_{2n}\}\) and \(\{x_{2n+1}\}\) converge to \(u\). Since \(T\) is a continuous mapping, therefore
\[
\lim_{n \to \infty} x_{2n+1} = u \Rightarrow \lim_{n \to \infty} T x_{2n+1} = Tu \Rightarrow \lim_{n \to \infty} x_{2n+2} = Tu
\]

Hence,
\[
Tu = u.
\]

Similarly, taking the continuity of \(S\), we can show that \(Su = u\). Hence \(u\) is the common fixed point of \(S\) and \(T\).

**Uniqueness:** Suppose that \(S\) and \(T\) have two distinct common fixed points \(u\) and \(v\).

Consider
\[
d(Su, Tv) \leq a_1 d(u, v) + a_2 d(u, Su) + a_3 d(v, Tv) + a_4 [d(u, Su) + d(v, Tv)] 
\[
+ a_5 [d(u, Tv) + d(v, Su)] + a_6 [d(u, Su) + d(u, v)] + a_7 [d(v, Tv) + d(u, v)]
\]

\[(3)\]

Since \(u\) and \(v\) are common fixed points of \(S\) and \(T\), condition (2) implies that
\[
d(u, u) = 0 \text{ and } d(v, v) = 0.
\]
Finally, from (3) we get
Similarly, we have:

\[ d(v, u) \leq (a_1 + a_5 + a_6 + a_7)d(u, v) + (a_5)d(v, u). \]  

(5)

Subtracting (5) from (4) we have

\[ |d(u, v) - d(v, u)| \leq (a_1 + a_6 + a_7)|d(u, v) - d(v, u)|. \]  

(6)

Since \( 0 \leq a_1 + a_6 + a_7 < 1 \), so the above inequality (6) is possible if

\[ d(u, v) - d(v, u) = 0. \]  

(7)

Taking equations (4), (5) and (7) into account, we have \( d(u, v) = 0 \) and \( d(v, u) = 0 \).

Thus by condition \( d_2 \) we have \( u = v \). Hence \( S \) and \( T \) have a unique common fixed point in \( X \).

**Example 3.2.** Let \( X = [0,1] \) and a complete dq-metric defined by

\[ d(x, y) = |x|, \text{ for all } x, y \in X \]

and define a continuous self-mappings \( S \) and \( T \) by

\[ Sx = 0, Tx = \frac{x}{8}. \]

With \( a_1 = \frac{1}{8}, a_2 = \frac{1}{12}, a_3 = \frac{1}{16}, a_4 = \frac{1}{18}, a_5 = \frac{1}{32}, a_6 = \frac{1}{20} \) and \( a_7 = \frac{1}{24} \),

\( S \) and \( T \) satisfy all the conditions of Theorem 3.2 and \( x = 0 \) is the unique common fixed point of \( S \) and \( T \) in \( X \).

**Remarks:** In the Theorem 3.2:

If \( a_7 = 0 \), then we get Theorem 2.12.

If \( a_6 = a_7 = 0 \), then we get Theorem 2.11.

If \( S = T \) and \( a_6 = a_7 = 0 \), then we get Theorem 2.9.

If \( a_4 = a_5 = a_6 = a_7 = 0 \), then we get Theorem 2.10.

If \( S = T \) and \( a_4 = a_5 = a_6 = a_7 = 0 \), then we get Theorem 2.3.

If \( S = T \) and \( a_1 = a_2 = a_3 = a_5 = a_6 = a_7 = 0 \), then we get Theorem 2.2.
CONCLUSION
The derived results extend and generalize theorems from Theorem 2.2 to Theorem 2.12 in the setting of dislocated quasi-metric spaces.

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REFERENCES


