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Some Generalized Fixed Point Results on Compact Metric Spaces

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Abstract

The goal of this research is to study some generalized fixed point results in compact metric space. It mainly focuses on the existence and unique fixed point of a selfmap on a compact metric space and its generalizations. In this study iterative techniques due to Edelstein, Bhardwaj et al. and Sastry et al. are used to show existence of a unique fixed point for a selfmap satisfying certain generalized contractive conditions involving rational expressions. Examples are also provided in support of our results. Our results generalize that of Edelstein and Fisher.

Key Words : Compact metric space, complete metric space, contractive map, contraction map, iterative sequences, fixed point.

INTRODUCTION

Background of the Study

Fixed point theory is a fascinating subject with an enormous number of applications in various fields of mathematics such as differential equations and numerical analysis. Also the existence of Nash equilibrium in game theory can be formulated as a fixed point problem. It has an important role in Mathematical economics (Nachbar, 2010).

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Definition 1.1. Let (X, d) be a metric space. A mapping $T: X \to X$ is called

i. Contraction if $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall\, x, y \in X$ with $\alpha \in [0,1)$.
ii. Contractive if $d(Tx, Ty) < d(x, y)$, $\forall x, y \in X$ with $x \neq y$.

Fixed point theory in metric spaces perhaps originated from the well known contraction principle of Banach which is stated as "A contraction T on a complete metric space (X, d) has a unique fixed point. If z is the fixed point of the map T then for any $x \in X$, the sequence $\{T^n x\}$ of iterates converges to z".

While considering Lipschitzian mappings, a question arises whether it is possible to weaken contraction assumption in Banach contraction principle and still obtain the existence of a fixed point? In general, the answer to this question is no. In this regard, we observe the following interesting example.

Example 1.1 (Kannan and Sharma, 1990) Let $C\,[0,1]$ denote the complete metric space of real valued continuous functions defined on [0,1] with respect to supermum metric d and consider the closed subspace Z of $C[0,1]$ consisting of those functions $f \in C\,[0,1]$ satisfying $f(1) = 1$. Since Z is closed subspace of $C\,[0,1]$, Z is also complete. Now define mapping $T: Z \to Z$

$$(T(f))(t) = tf(t),$$

for each $t \in (0,1]$.

One can easily verify that

$$d\big(T(f), T(g)\big) < d(f, g),$$

whenever $f \neq g$.

But T has no fixed point as $tf(t) = f(t)$ implies $f\,(t) = 0$ for all $t \epsilon\,[0,1)$ and $f(1) = 1$ which contradicts the continuity of f and so T can not have a fixed point in Z. Here one may note that T is a Contractive mapping on Z.

The question arises whether there is a theorem on the existence of fixed points of contractive mappings or not? The answer is in affirmative, but the class of space to which it applies is much more restrictive. In this direction, Edelstein (1961) established the first ever fixed point theorem for contractive mappings defined on a compact metric space stated as follows.

Theorem 1.1: (Edelstein, 1961) Let (X, d) be a compact metric space with $T: X \to X$ satisfying

$$\mathrm{d}\big(\mathrm{T(x)}, \mathrm{T(y)}\big) < d(\mathrm{x}, \mathrm{y}), \tag{1.1}$$

$\forall \mathrm{x}, \mathrm{y} \in \mathrm{X}$ with $\mathrm{x} \neq \mathrm{y}$, then T has a unique fixed point in X. Moreover, for any $\mathrm{x} \in \mathrm{X}$, the sequence $\{\mathrm{T}^{\mathrm{n}}(\mathrm{x})\}$ converges to the unique fixed point of T.

Also, Fisher (1978) proved the following theorem.

Theorem 1.2: (Fisher, 1978) If T is a continuous mapping of a compact metric space (X, d) into itself satisfying the condition

$$d(Tx, Ty) < \frac{1}{2}[d(y, Tx) + d(x, Ty)] \tag{1.2}$$

for all x, y in X with $x \neq y$, then T has a unique fixed point in X.

The following examples show that Theorem 1.1 and Theorem 1.2 are independent.

Example 1.2. Consider the maps

a) T: [0,1]→ $[0,1]$ defined by $Tx = \frac{1}{x+1}$.

b) T: $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 2\right\} \to \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 2\right\}$ defined by $T\left(\frac{2}{3}\right) = \frac{3}{2}, T\left(\frac{1}{2}\right) = T(2) = T\left(\frac{3}{2}\right) = 2.$

Here we observe that the contractive map T in Example 1.2 (a) has fixed point $\frac{\sqrt{5}-1}{2}$ and does not satisfy Theorem 1.2, for

$$d(T\left(\frac{1}{2}\right), T(1)) = \frac{1}{6} = \frac{1}{2}\left[d(T\left(\frac{1}{2}\right), 1) + d(\frac{1}{2}, T(1))\right].$$

Similarly, the map T in Example 1.2 (b) has fixed point $x = 2$ and satisfies the conditions of Theorem 1.2, but it does not satisfy Theorem 1.1, since

$$d(T\left(\frac{1}{2}\right), T\left(\frac{2}{3}\right)) = \frac{1}{2} > \frac{1}{6} = d(\frac{2}{3}, \frac{1}{2}).$$

Hence, the above two results are independent.

The purpose of this research is to study some generalized fixed point results of selfmaps on compact metric spaces.

Statement of the Problem

Fixed point results on compact metric spaces were obtained as a generalization of Banach contraction mapping principle. For instance, in 1961 Edelstein established the existence of a unique fixed point of a selfmap on a compact metric spaces which is a generalization of Banach contraction mapping principle. In the past few years a number of authors such as Iseki (1977), Kannan (1990) and Fisher et al. (2004) have established a number of interesting results related to fixed points of mappings defined on compact metric spaces as generalizations of the contraction mapping principle using either of the following approach:

i. Weakening the contractive properties of the map and possibly by simultaneously giving the space under consideration a sufficiently rich structure, in order to compensate the relaxation of the contractiveness assumptions;
ii. Extending the structure of the ambient space; or a combination of (i) and (ii).

In this research paper the researchers answer the following leading questions, based on the work of Bhardwaj et al. (2008) and Sastry et al. (2000).

1. What are the sufficient conditions for the existence of unique fixed point of self map of compact metric spaces?
2. What are the generalized fixed point results for discontinuous self maps of compact metric spaces?
3. Among generalized fixed point results in compact metric spaces which one generalizes the other and which of them are independent?

Objective of the Study

General Objective

To study some generalized fixed point result in compact metric spaces.

Specific Objectives

1. To identify sufficient conditions for the existence of fixed points of a selfmap in compact metric spaces.
2. To list generalized fixed point results in compact metric spaces.
3. To compare and contrast the generalized fixed point results in compact metric spaces.

4. To prove theorems related to the existence of unique fixed point of selfmap in compact metric spaces.
5. To identify types of selfmap of compact metric space to which the generalized fixed point results are valid.

Significance of the Study

This study is going to discuss about some generalized fixed point results in compact metric spaces and also the study identifies types of mapping which has a unique fixed point on compact metric spaces.

Delimitation of the Study

This study is delimited to the existence of unique fixed points of contractive type selfmappings of compact metric spaces and its generalized results.

STUDY DESIGN AND METHODOLOGY

Study Site and Period

This study was conducted in Jimma University under Mathematics Department, particularly in the Analysis stream from December 2005 - June 2005 E.C. It is mainly on the existence of fixed points of selfmap of compact metric spaces and its generalized results.

Study Design

This study was a documentary review study on existence of fixed points of selfmap of compact metric spaces and its generalized results. The secondary data were collected from the relevant sources of information to achieve each specific objective of the study.

Source of Information

The available sources of information for the study are related books, journals and similar studies which are collected from the internet, library and experts in the field.

Data Collection Process

To conduct this research necessary information were collected through the following process.

- Related books /documents were collected from the library.
- Necessary parts of books that helped us to conduct this study were copied and used.
- Experts in the related fields were consulted to collect different soft copies and hardcopies of related books and journals.

Study Procedures

In order to achieve the above mentioned objectives the following methods were followed.

(a) Iterative methods due to Edelstein (1965).
(b) Iterative methods by Sastry et al. (2000) and Bhardwaj et al. (2008).

Ethical Issues

The research was done through the support of official letter from the department.

RESULTS AND DISCUSSION

Now we shall state and proff the main result of this research.

Main Result

Theorem 3.1. Let T be a continuous selfmap of a compact metric space (X, d) satisfying the condit ion

$$d(Tx,Ty) < \alpha\frac{d(x,Tx)\big(1+d(Ty,y)\big)}{1+d(Tx,Ty)} + \beta\frac{d(Ty,y)\big(1+d(Tx,x)\big)}{1+d(x,y)}$$
$$+ \frac{\gamma}{2}\,[(d(x,Tx) + d(Ty,y)]$$
$$+ \frac{\mu}{2}\,[d(Tx,y) + d(Ty,x)] + \delta d\,(x,y)$$

for all $x, y \in X$ with $x \neq y$ where $\alpha, \beta, \gamma, \mu$ and δ are nonnegative real numbers such that $\alpha+\beta+\gamma+\mu+\delta = 1$. Then T has a unique fixed point in X.

Proof: Define $F: X \to [0,\infty)$ by $F(x) = d(x,Tx)$ for every $x \in X$.
Suppose $x \neq Tx$. Then

$$F(Tx) = d(Tx,T^2x)$$
$$< \alpha\frac{d(x,Tx)\big(1+d(Tx,T^2x)\big)}{1+d(Tx,T^2x)} + \beta\frac{d\big(Tx,T^2x\big)\big(1+d(x,Tx)\big)}{1+d(x,Tx)}$$
$$+ \frac{\gamma}{2}\,[(d(x,Tx) + d(Ty,y)] + \frac{\mu}{2}[d(x,T^2x) + d(Tx,Tx)] + \delta d(x,Tx)$$
$$= \alpha\frac{F(x)\big(1+F(Tx)\big)}{1+F(Tx)} + \beta\frac{F(Tx)(1+F(x))}{1+F(x)} + \frac{\gamma}{2}[(F(x)+F(Tx)]$$
$$+ \frac{\mu}{2}[(F(x)+F(Tx)] + \delta F(x). \tag{3.1}$$

Now we assume $F(x) \le F(Tx)$ for $x \neq Tx$. Then (3.1) reduced to

$$F(Tx) < (\alpha + \beta + \gamma + \mu + \delta)F(Tx) = F(Tx)$$

which is a contradiction. Hence, we have

$$F(x) > F(Tx) \text{ for } x \neq Tx. \tag{3.2}$$

Since T is continuous, $F$is also continuous on the compact metric space X, and hence it attains its minimum on X say at x_o. Suppose $F(x_o) = d\,(x_o, T(x_o)) > 0$, i.e., $x_o \neq Tx_o$. Then by (3.2), we obtain $F(T(x_o)) < F(x_o)$ which contradicts the minimality of the value of F at x_o. Hence, our assumption $F(x_o) > 0$ is false. Hence,

$$F(x_o) = d(x_o,T(x_o)) = 0.$$

That is, x_o is a fixed point of T.

Uniqueness of the fixed point of *T*.

Suppose if possible $y \neq x_o$ is another fixed point of T .

$$(x_o, y) = d\big(T(x_o), T(y)\big)$$
$$< \alpha\frac{d(x_o,Tx_o)(1+d(Ty,y))}{1+d(Tx_o,Ty)} + \beta\frac{d(Ty,y)(1+d(Tx_o,x_o))}{1+d(x_o,y)}$$
$$+ \frac{\gamma}{2}\,[(d(x_o,Tx_o) + d(Ty,y)] + \frac{\mu}{2}\,[d(Tx_o,y) + d(Ty,x_o)] + \delta d\,(x_o,y)$$
$$= \alpha(0) + \beta(0) + \frac{\gamma}{2}(0) + \mu d(x_o,y) + \delta d(x_o,y)$$
$$= (\mu+\delta)d(x_o,y).$$
$$\Longrightarrow d(x_o,y) < (\mu+\delta)d(x_o,y) \le d(x_o,y),$$

which is a contradiction, since $\mu + \delta \le 1$.

Theorem 3.2: Let T be as in Theorem 3.1. If $\alpha = \beta$, then for every $x \in X$, the sequence $\{T^n x\}$ of iterates converges to the unique fixed point of T.

Proof: By Theorem 3.1, T has a unique fixed point x_o (say) in X. Now for each $n = 0,1,2,\cdots$, define $d_n = d(T^n x, x_o)$ for every $x \in X$ with $x \neq x_o$.
We consider the following two cases.

Case 1. If $d_n = 0$ for some n, then $T^m x = x_o$ for each $m \ge n$ and hence the sequence $\{T^n x\}$ converges to x_o.

Case 2. If $d_n \neq 0$ for each n, then

$$d_{n+1} = d(T^{n+1}x, x_o) = d(T^{n+1}x, T^{n+1}x_o)$$
$$+ \alpha \frac{d(T^{n+1}x_o, T^n x_o)(1+d(T^{n+1}x, T^n x))}{1+d(T^n x, T^n x_o)} + \beta \frac{d(T^{n+1}x, T^n x)(1+d(T^{n+1}x_o, T^n x_o))}{1+d(T^n x, T^n x_o)}$$
$$+ \frac{\gamma}{2}[d(T^{n+1}x_o, T^n x_o) + d(T^{n+1}x, T^n x)]$$
$$+ \frac{\mu}{2}[d(T^{n+1}x_o, T^n x) + d(T^{n+1}x, T^n x_o] + \delta\, d(T^n x_o, T^n x)$$
$$\leq \alpha(0) + \beta(d_n + d_{n+1}) + \frac{\gamma}{2}(d_n + d_{n+1}) + \frac{\mu}{2}(d_n + d_{n+1}) + \delta d_n.$$
$$\Rightarrow d_{n+1} < \left(\beta + \frac{\gamma}{2} + \frac{\mu}{2} + \delta\right) d_n + \left(\beta + \frac{\gamma}{2} + \frac{\mu}{2}\right) d_{n+1}$$
$$\Rightarrow \left(1 - \left(\beta + \frac{\gamma}{2} + \frac{\mu}{2}\right)\right) d_{n+1} < \left(\beta + \frac{r}{2} + \frac{\mu}{2} + \delta\right) d_n$$
$$\Rightarrow \left(\alpha + \frac{\gamma}{2} + \frac{\mu}{2} + \delta\right) d_{n+1} < \left(\beta + \frac{\gamma}{2} + \frac{\mu}{2} + \delta\right) d_n$$
$$\Rightarrow d_{n+1} < d_n, \tag{3.3}$$

since $\alpha = \beta$ and 1-$(\beta + \frac{\gamma}{2} + \frac{\mu}{2})$ = $\alpha + \frac{\gamma}{2} + \frac{\mu}{2} + \delta$.

Hence $\{d_n\}$ is strictly decreasing sequence of positive real numbers and hence converges to a real $r \geq 0$(say), which is the greatest lower bound of the sequence $\{d_n\}$.

By compactness of X, the sequence $\{T^n(x)\}$ has a subsequence $\{T^{n_k}(x)\}$ which converges to $z \in X$ (say). Since T is continuous, as $k \to \infty$,

$$T^{n_k+1}(x) = T(T^{n_k}(x)) \to Tz.$$

By the continuity of the metric d, letting $k \to \infty$,

$$d_{n_k} = d(T^{n_k}(x), x_o) \to d(z, x_o) = r,$$

since the sequence $\{d_{n_k}\}$ is a subsequence of $\{d_n\}$.
Also by the continuity of the metric d, as $k \to \infty$,

$$d_{n_k+1} = d(T^{n_k+1}(x), x_o) \to d(Tz, x_o) = r,$$

since the sequence $\{d_{n_k+1}\}$ is a subsequence of $\{d_n\}$. So,

$$r = d(z, x_o) = d(Tz, x_o). \tag{3.4}$$

Now we claim $r = 0$. Suppose $r \neq 0$. Then $z \neq x_o$. By (3.3), we get

$$d(Tz, x_o) = d(Tz, Tx_o) < d(z, x_o),$$

which contradicts (3.4). Hence, $z = x_o$, which means $r = d(z, x_o) = 0$. This shows $d_n \to 0$ as $n \to \infty$ and hence the conclusion of the theorem follows.

Remarks:

i. Putting $\alpha = \beta = \gamma = \mu = 0, \delta = 1$, we get Edelstein [3] theorem.
ii. Putting $\alpha = \beta = \gamma = \delta = 0, \mu = 1$, we get Fisher [5] theorem.

Theorem 3.3: Let T be a self-map of a compact metric space (X, d) such that for some $n \geq 1$, T^n is continuous and satisfying the condition

$$d(T^n(x), T^n(y)) < \alpha \frac{d(x, T^n x)(1+d(T^n y, y))}{1+d(T^n x, T^n y)} + \beta \frac{d(T^n y, y)(1+d(T^n x, x))}{1+d(x,y)}$$
$$+ \frac{\gamma}{2}[(d(x, T^n x) + d(T^n y, y)] + \frac{\mu}{2}[d(T^n x, y) + d(T^n y, x)]$$
$$+ \delta d(x, y)$$

for every $x, y \in X$ with $x \neq y$, where$\alpha, \beta, \gamma, \mu$ and δ are nonnegative real numbers such that $\alpha + \beta + \gamma + \mu + \delta = 1$. Then T has a unique fixed point in X.

Proof: Define. $F: X \to [0, \infty)$ by $F(x) = d(x, T^n x)$ for every $x \in X$.

Suppose $x \neq T^n x$. Then,

$$F(T^n x) = d\big(T^n x, T^n(T^n x)\big)$$
$$< \alpha \frac{d(x,T^n x)\big(1+d(T^n x,T^n(T^n x))\big)}{1+d\big(T^n x,T^n(T^n x)\big)} + \beta \frac{d(T^n(T^n x),T^n x)(1+d(x,T^n x)}{1+d(x,T^n x)}$$
$$+\frac{\gamma}{2}[d(x,T^n x) + d(T^n x,T^n(T^n x)] + \frac{\mu}{2}\big[d\big(x,T^n(T^n x)\big) +$$
dTnx,Tnx
$$+\delta d(x,T^n x)$$
$$\leq \alpha \frac{F(x)\big(1+F(T^n x)\big)}{1+F(T^n x)} + \beta \frac{F(T^n x)\big(1+F(x)\big)}{1+F(x)} + \frac{\gamma}{2}[(F(x) + F(T^n x)]$$
$$+ \frac{\mu}{2}[(F(x) + F(T^n x)] + \delta F(x). \qquad (3.5)$$

If we assume $F(x) \leq F(T^n x)$, then (3.5) reduced to

$$F(T^n x) < (\alpha + \beta + \gamma + \mu + \delta)F(T^n x) = F(T^n x)$$

which is a contradiction. Hence our assumption $F(x) \leq F(T^n x)$ for $x \neq (T^n x)$ is false. So,

$$F(x) > F(T^n x) \quad \text{for } x \neq T^n x \qquad (3.6)$$

Since T^n is continuous, $F$is continuous on the compact metric space$X$, hence it attains its minimum on X at x_o (say).

Suppose $F(x_o) =$ d $(x_o,T^n(x_o)) > 0$. Then by (3.6), we obtain

$$F(T^n(x_o) < F(x_o),$$

which contradict minimality of the value of F at x_o.

Hence, our assumption $F(x_o) > 0$ is false.

Therefore,

$$F(x_o) = d(x_o,T^n(x_o)) = 0.$$

That is, x_o is a fixed point of T^n.

Uniqueness of the fixed point of T^n.

Suppose if possible $y \neq x_o$ is another fixed point of T^n. Then

$$d(x_o, y) = d(T^n x_o, T^n y)$$
$$< \alpha \frac{d(T^n x_o,x_o)(1+d(T^n y,y))}{1+d\,(T^n x_o,T^n y)} + \beta \frac{d(T^n y,y)(1+d(T^n x_o,x_o))}{1+d(x_o,y)}$$
$$+ \frac{\gamma}{2}[d(T^n x_o, x_o) + d(T^n y, y)] + \frac{\mu}{2}[d(T^n x_o, y) + d(T^n y, x_o)]$$
$$+ \delta d(x_o, y)$$
$$= (\mu + \delta)d(x_o, y).$$
$$\Longrightarrow d(x_o, y) < (\mu + \delta)d(x_o, y) \leq d(x_o, y),$$

which is a contradiction, since $\alpha + \beta + \gamma \geq 0$ and $\mu + \delta \leq 1$.

Now let x_o is a fixed point of T^n and since T^n (T (x_o)) $=T(T^n(x_o))$, then

$$T^n(\mathrm{T}(x_o)) = T(T^n(x_o)) = \mathrm{T}(x_o) = x_o.$$

That is, the uniqueness of the fixed point of T follows from that of T^n and the fact that any fixed point of T is a fixed point of T^n.

Remarks: Theorem 3.1 is obtained by putting $n = 1$ in Theorem 3.3.

Now consider the following example.

Example 3.1: Define $T: [-1,1] \to [-1,1]$ by $T(x) = \begin{cases} 0 & \text{for} \quad -1 \leq x < 0 \\ 1 & \text{for} \quad 0 \leq x \leq 1 \end{cases}$.

In this example the map T is not continuous and hence Theorems 1.1 and Theorem 1.2 are not applicable.

Now since $T^2 x = 1$ for each $x \in [-1,1]$; for all $x, y \in [-1,1]$ with $x \neq$y , we have

$$d(T^2 x, T^2 y) = d(1,1) = 0 < d(x, y),$$

which shows Theorem 3.3 is more general than Theorem 3.1 and hence Theorem 1.1 and Theorem 1.2 if we choose $\alpha = \beta = \gamma = \mu = 0$ and $\delta = 1$.

CONCLUSIONS

Fixed point theory in metric spaces perhaps, originated from the well-known contraction mapping principle of Banach. The generalized result of this principle is opted by weakening the contractive properties of the map and possibly by simultaneously giving the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness assumptions and extending the structure of the ambient space. And also several fixed point theorems have been obtained by combining the two ways or by adding supplementary conditions.

Hence the conclusion of contraction mapping principle is valid if we consider compact spaces instead of using complete spaces and the conclusion of contraction mapping principle is not valid if we consider contractive mapping instead of contraction. In 1961 Edelstein proved fixed point theorem of contractive mapping of compact metric spaces. In the past few years a number of authors such as Bhardwaj et al. (2008), Bhatnagar et al. (2012) and Sastry et al. (2000) have established the generalization of Edelstein result.

This paper proved some general fixed point theorems for self-mapping satisfying a new contractive condition in compact metric space s which generalized the result of Edelstein and Fisher.

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REFERENCES

Banach S. (1922). Sur les operations dans les ensembles abstraits et leurs application aux equations integrales. Fund. Math. 3, 133-181.

Bhardwaj R., Rajput S. S., Choudhary S., and Yadava R. N. (2008). Some Fixed Point Theorems in Compact Metric Space. Int. J. Math. Analysis. 2(12), 551 – 555.

Bhatnagar P, Tenguriya A, Wadkar B. R. and Yadava R. N. (2012). Fixed Point and Common Fixed Point Theorems in Compact Metric Spaces and Pseudo Compact Tichnov Spaces for Self Mappings. Int. J. of Theoretical and Applied Science, 4(1), 8-13.

Browder F.E. (1965). Fixed point theorem for non compact mappings in Hilbert space, Proc. Nat. Acad. Sci. USA. 53,1272-1279.

Edelstein M. (1961). On fixed point and periodic under contracting mappin gs, J. Land, Mathsec. 37, 74-79.

Fisher, B. (1978). A fixed point theorem for compact metric spaces, PubI. Inst. Math. 25, 193-94

Iseki, K. (1977). A simple application of Reich's fixed point theorem. Maths. Sem. Notes Kobe University. 5(1), 75–80.

Kanan S. and Sharma P.L. (1990). Fixed point mapping for compact metric space. The Math. Educ. Vol. XXIV, No. 1

Nachbar J. (2010). Fixed Point Theorem.www.artsci.wustl.edu/~econ467/FixedPoint.pdf

Namdeo R. K. and Fisher B. (2004). Related fixed point theorem for two pairs of set valued mappings on on two complete and compact metric space. Indian J. Math. 46 (2-), 161 – 171.

Sastry K.P.R. and Srinivasa Rao, C. (2000). Fixed point theorems for self map on complete and compact metric spaces, Acta Ciencia Indica. 76 (4), 295-298.