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DEMONSTRATING THE PHYSICS OF SIMPLE PENDULUM VIA COMPUTER SIMULATION

Tsegaye Kassa

P. O. Box 79, Department of Physics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: tsegaye8684@yahoo.com

Abstract: This paper investigates how one can teach the physics of simple pendulum in the classroom by means of computer simulation. Employing the fourth-order Runge-Kutta method carries out the simulation. It is believed that the simulation helps students to understand the physics of simple pendulum easily. The simulation results agree quite well with the analytical solution which is quite often overlooked in the classroom. The numerical method used is stable in simulating the physics of simple pendulum. Moreover, since we developed the simulation by using matlab software graphical user interface (GUI), its interactivness is unquestionable.

Key words: Angular displacement, Angular velocity, Analytical solutions, Numerical solutions, Runge-Kutta method.

Introduction

Many physics textbooks in high school, undergraduate and graduate levels deal with the physics of simple pendulum by employing analytical approach. However most of them have failed to compare theoretical and experimental results. With this approach students will face difficulties in interpreting tabulated data obtained from experiments by comparing with theoretical results. This is because of the fact that the analytical solution of equation of motion of a simple pendulum can only be obtained by approximating for small displacement so that the non-linear differential equation of motion will be changed to liner one (Keller *et. al.* 1993). This approximation results in error in describing the actual situations of a simple pendulum. Such problems can be handled by employing the well-known numerical methods. The purpose of this paper is to describe how one can demonstrate the physics of a simple pendulum by means of computer simulation. The convergence and stability of a numerical method are used as the criterion to check the validity of the simulation. It is believed that simulation can help teachers to demonstrate the physics of simple pendulum to their students. The first section of this paper describes the model for a simple pendulum. The second section presents the analytical solution of the model and the third section illustrates the corresponding numerical method used to solve the problem. Finally results, discussions and conclusions are presented.

Equation of Motion

Let us consider a simple pendulum shown in Figure 1a. The pendulum is oriented in its equilibrium state. There are two external forces: the weight Fg of the bob and a force Fs exerted on the cord at the upper end by a fixed support (Keller *et. al.*, 1993). We choose an axis O at the upper end of the cord and perpendicular to the plane of the figure. For this orientation the torque about O is zero for each external force. If the pendulum is displaced, as seen in Figure 1b, there is a net external torque about axis O due to the weight. For the positive value of angle Θ, as shown in the Figure 1b, this torque tends to cause a clockwise rotation so as to restore the pendulum to its equilibrium orientation. The perpendicular distance from the axis to the line of the action of the weight is $L\sin\theta$. With the z-axis out of the plane of the figure, the torque component is

$$\tau_z = F_g L\sin\theta\,\hat{k}, \quad (1)$$

where $\hat{k}$ is the unit vector along τ_z. Since the mass of the cord can be neglected, the moment of inertia (I) of the simple pendulum about axis O is due to the mass m of the bob at a distance L from the axis. That is

$$I = mL^2 \quad (2)$$

Now applying Newton's second law for the angular motion yields

$$\tau_z = \mathrm{Im}\,\alpha_z \quad (3)$$

Combining equations 1 and 3 and solving for the angular acceleration gives

$$\ddot{\theta} = -\frac{g}{L}\sin\theta\hat{k}, \quad (4)$$

where $\alpha_z = \ddot{\theta}$,

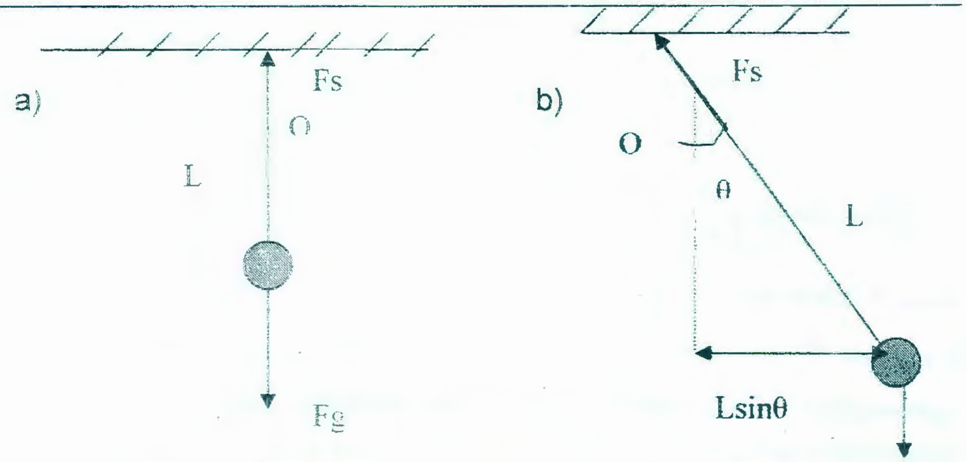

Figure 1: A simple Pendulum at Different States

a) The net torque about axis O is zero when the pendulum is at the equilibrium orientation.
b) The torque about axis O due to the weight tends to restore the pendulum to the equilibrium orientation. The torque about O due to the support Fs is zero.

Analytical Solutions

The angular motion of a simple pendulum for small vibrations is obtained by using equation 4 after we approximate it for small vibration where $\sin\theta = \theta$. Now the equation becomes

$$\ddot{\theta} = -\frac{g}{L}\theta \,. \qquad (5)$$

This ordinary differential equation can be solved analytically by using appropriate techniques. Since the equation is a linear second order homogenous differential equation, we can first write the characteristics equation and then solve its roots. Depending on the roots of the characteristics equation, we can determine the form of the solution of the given differential equation. The characteristics equation of the given differential equation is

$$a^2 + \frac{g}{L} = 0 \,, \qquad (6)$$

which gives a complex number.

Therefore the solution of the given differential equation can be written in such a way that

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right) \tag{7}$$

where A and B are constants to be determined depending on the initial conditions.

If we use $\theta = \theta_o$ and $\dot{\theta} = 0$ as initial conditions at t = 0 and carry out the mathematical manipulations properly, the analytical solution of the angular displacement will be

$$\theta(t) = \theta_o \cos\left(\sqrt{\frac{g}{L}}t\right). \tag{8}$$

The analytical solution of the angular velocity can be easily obtained by differentiating equation 8 with respect to time. The result becomes

$$\omega(t) = -\sqrt{\frac{g}{L}}\sin\left(\sqrt{\frac{g}{L}}t\right). \tag{9}$$

The left and the right parts of Figure 2 display the analytical solutions of angular displacement and velocity respectively.

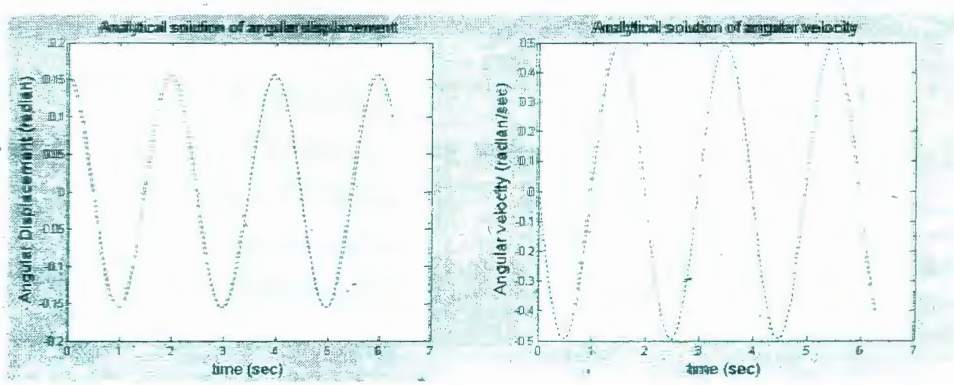

Figure 2: Analytical Solutions

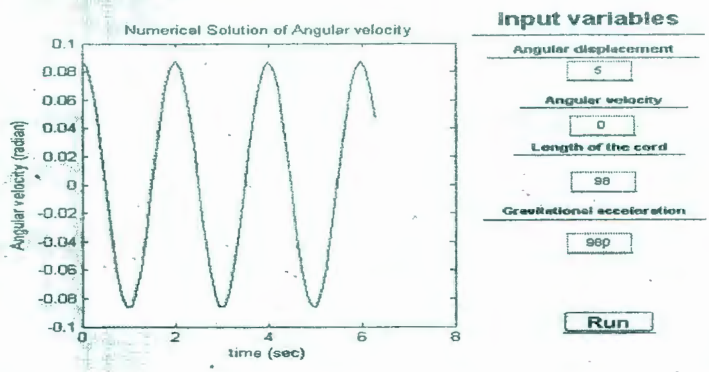

Figure 3: Matlab graphical user interface working window (GUI)

Numerical Solutions

Let us solve the equation of motion of simple pendulum given by equation 4 by using the most popular numerical method called fourth order Runge-Kutta method. There are two versions of Runge-Kutta methods that are the most popular ones. The first version is based on the Simpson's 1/3 rule (Burden and Faires, 1993) and the second is based on the Simpson's 3/8 (Shochiro, 1996). Consider the following ordinary differential equation.

$$\frac{dy}{dt} = f(y,t), \quad y(0) = y_o. \tag{10}$$

The first version of Runge-Kutta method is given by:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{11}$$

where,

$$k_1 = hf(y_n, t_n)$$

$$k_2 = hf\left(y_n + \frac{k_1}{2}, t_{n+\frac{1}{2}}\right)$$

$$k_3 = hf\left(y_n + \frac{k_2}{2}, t_{n+\frac{1}{2}}\right)$$

$$k_4 = hf\left(y_n + \frac{k_3}{2}, t_{n+1}\right)$$

The second versions of fourth-order Runge-kutta method is given by:

$$y_{n+1} = y_n + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4) \qquad (12)$$

where

$$k_1 = hf(y_n, t_n)$$

$$k_2 = hf\left(y_n + \frac{k_1}{3}, t_{n+\frac{1}{3}}\right)$$

$$k_3 = hf\left(y_n + \frac{k_2}{3} + \frac{k_2}{3}, t_{n+\frac{1}{3}}\right)$$

$$k_4 = hf(y_n + k_1 - k_2 + k_3, t_{n+1})$$

Let us consider the first version to solve the solutions of angular displacement and velocity numerically. Since the equation given by 4 is a second order differential equation, let us first convert it into coupled first order differential equations (Chapara and Canale, 1998). If we let $\omega(t) = \dot{\theta}$, we can get the following two coupled first order differential equations:

$$\dot{\theta} = \omega(t) \qquad (13)$$

$$\dot{\omega} = -\frac{g}{L}\sin\theta, \qquad (14)$$

where $\omega(t)$ denotes angular velocity.

We can solve the angular displacement and velocity by employing fourth-order Runge-Kutta method using equations 13 and 14.The steps to be considered when we use fourth-order-Runge-Kutta are given below.

$$k_1 = h(\omega_o) \qquad (15)$$

$$m_1 = -h\frac{g}{L}\sin\theta_o \tag{16}$$

$$k_2 = h\left(\omega_o + \frac{m_1}{2}\right) \tag{17}$$

$$m_2 = -h\frac{g}{L}\left(\sin\theta_o + \frac{k_1}{2}\right) \tag{18}$$

$$k_3 = h\left(\omega_o + \frac{m_2}{2}\right) \tag{19}$$

$$m_3 = -h\frac{g}{L}\left(\sin\theta_o + \frac{k_2}{2}\right) \tag{20}$$

$$k_4 = h(\omega_o + m_3) \tag{21}$$

$$m_4 = -h\frac{g}{L}(\sin\theta_o + k_3) \tag{22}$$

$$\theta = \theta_o + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \tag{23}$$

$$\omega = \omega_o + \frac{1}{6}(m_1 + 2(k_2 + k_3) + k_4), \tag{24}$$

where h is time step for computation. Equations 23 and 24 are numerical solutions for angular displacement and velocity respectively. They can be easily computed by means of a simple computer program which we have developed by using software called matlab. Fig. 4 shows the solutions. The left part of Fig. 4 is the solution for angular displacement where as the right part is for angular velocity.

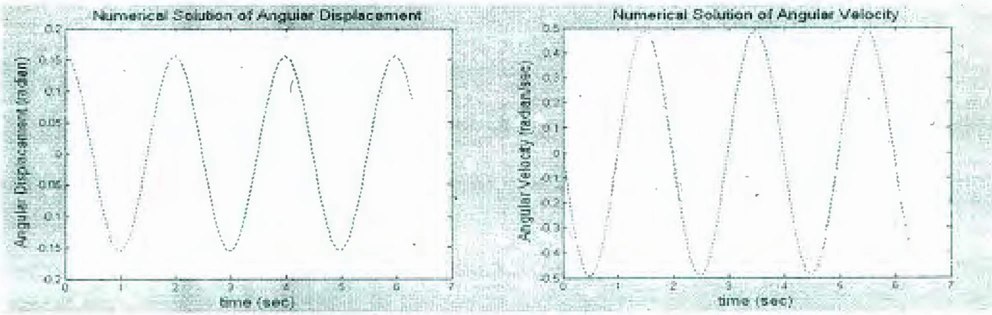

Figure 4: Numerical Solutions

Comparison of analytical and numerical solutions

The analytical and numerical solutions of both angular displacement and velocity have been compared. The comparison is performed for both small and large angles (Fig. 5). The discrepancy (error) between analytical and numerical solutions by looking at the relation between distributions of error and variation in initial angles (Fig. 7) has been carefully examined, besides analyzing the relation between error distribution and the time step (Fig. 7).

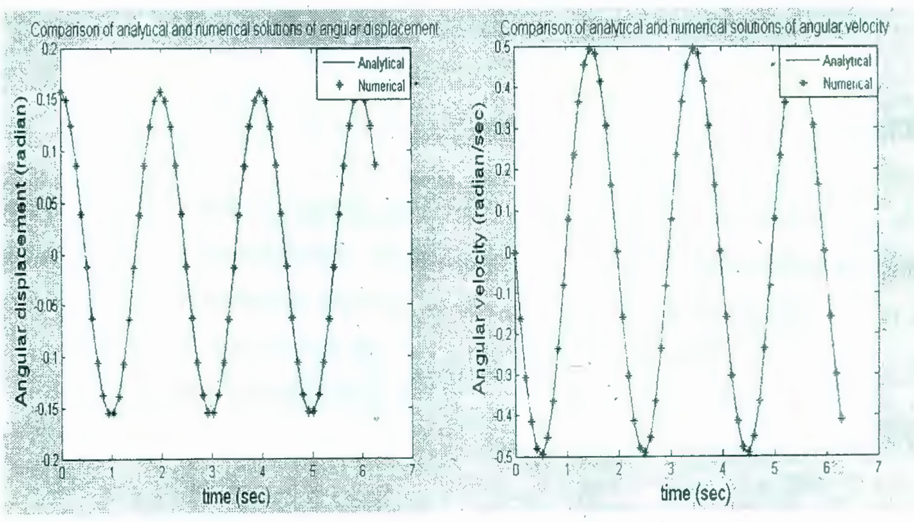

Figure 5: Comparison of Analytical and Numerical Solutions for small angles

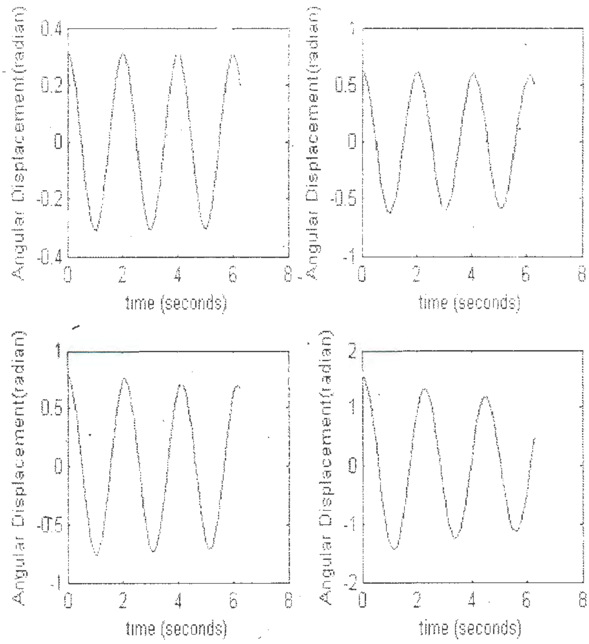

Figure 6: Numerical solutions for large angle (non-linear cases)

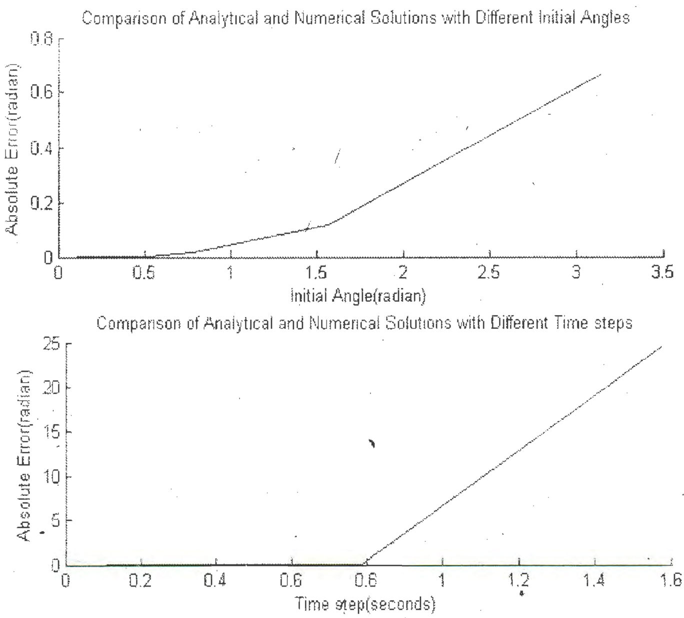

Figure 7: Error Analysis

Results and Discussions

The difference in the initial angle of displaced simple pendulum has its own factor for the increment in discrepancy between analytical and numerical solutions has been shown. That means for small angle θ_o, the analytical and numerical solutions agree (Fig. 4). In particular, it has also been shown that if the initial angle is $\theta_o \leq$ 0.5radian, the numerical simulation can totally describe the expected harmonic motion of a simple pendulum without the requirement of analytical solutions (The top panel of Fig. 5). In addition to this we found out that this simulator describes the behavior of simple pendulum without having a limit in amplitude (see Fig. for large angles). It has been demonstrated in the present study that the method used is conditionally stable since some of the numerical solutions agree with the analytical one for small time step (see the top panel of Fig. 7). The bottom panel of Fig. 7 demonstrates that the numerical result converges to the exact solution as time step is reduced to zero. This provides a power of simulating accurately to our method by achieving consistency.

Conclusions

The analytical method is capable of solving the differential equation of motion of a simple pendulum for small displacements (amplitudes). However, as the amplitude increases, the differential equation will be changed to non-linear form, which is hard to be solved analytically. But our simulator does not face such problems. It can solve both linear and non-linear equations (see Fig. 6). It is known that a simple pendulum attains its periodic motion for small displacements where we can approximate sine $\theta = \theta$. By using this simulator, students and teachers can really see to what extent of displacement the periodic motion is attained. This extent has been demonstrated by the top panel of Fig. 7. While doing experiments, in relation with simple pendulum, students can use this simulator as theoretical base to compare their experimental outcomes. Without performing actual experiments students can study the behavior of simple pendulum by varying parameters used in the simulator such as initial angular displacement, initial angular velocity, length of the cord and gravitational acceleration (see input variables from working window, Fig. 3) that means this simulator can serve as experimental set - up for students while studying the physics of simple pendulum. The present study explains a simulator for motion of a simple pendulum that exhibits the behaviors we expect. The model seems to be accurate since it uses fourth-order Runge-Kutta integration. The graphics of the simulator makes it possible for the user to gain intuition about how a simple pendulum behaves.

Finally this simulator can be implemented in undergraduate physics laboratory as an alternative set-up to teach the physics of simple pendulum.

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