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APPROXIMATION OF COMMON FIXED POINTS OF A FINITE FAMILY OF $\phi-$ DEMICONTRACTIVE MAPPINGS BY AN IMPLICIT ITERATION METHOD.

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ABSTRACT

We prove that the Implicit Iteration process of Xu and Ori (2001) converges strongly to the commonfixed pointsof a finite family of $\phi-$demicontractive mappings in real Hilbertand Banachspaces. Our results extend the results of Osilike (2004a) from strictly pseudocontractive maps to the much more general $\phi-$demicontractive maps;complement and generalize several others in the literature.

KEY WORDS AND PHRASES: $\phi-$Demicontractive Maps, Implicit Iteration Process, Fixed Points,Strong Convergence.

INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space. Let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x)=\{f \in \mathrm{E}^*:\langle x, f\rangle=\|x\|^2;\ \|x\|^2=\|f\|^2\}$,where E^* denotes the dual space of E and $\langle , \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote single-valued duality mapping by j.

Let K be a nonempty subset of E. A mapping $T: K \to K$ is said to be *demicontractive* (see, for example,Hicks and Kubicek, (1979).) if F(T)= $\{x \in K : Tx = x\} \neq \phi$ and for all $x \in K$ and $p \in F(T)$, there exists $j(x-p) \in J\ (x-p)$ and a constant $\lambda > 0$ such that

$$\langle x - Tx, j(x-p)\rangle \geq \lambda\|x - Tx\|^2. \qquad (1)$$

In Hilbert spaces equation(1) is equivalent to:

$$\|Tx-p\|^2 \ \geq\ \|x-p\|^2 + (1-2\lambda)\|x-Tx\|^2 \qquad (2)$$

and we may assume that $1-2\lambda = k > 0$.

Furthermore, T is L-Lipschitzian if there exists a constant $L > 0$ such that $\|Tx-Ty\| \leq L\|x-y\|$ for all $x, y \in K$.

The class of demicontractive maps was first introduced in Hilbert space by Hicks and Kubicek (1979). Maruster (1977) also considered this class of mappings using equation (1) which he referred to as condition (A).

A mapping $T : K \to K$ is called *strictlypseudocontractive* in the terminology of Browder and Petryshyn (1967) if for all $x, y \in K$ and $j\ (x-y) \in J\ (x-y)$there exists $\lambda > 0$ such that $\langle Tx-Ty,\ j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Tx-Ty)\|^2$. A strictly pseudocontraction with a nonempty fixed point set $F(T)$ is demicontractive. An example of a demicontractive map which is not strictly pseudocontractive is given by Hicks and Kubicek (1979).

A mapping $T : K \to K$ is said to be $\phi-$*demicontractive* (Isiogugu, 2005) if $F\ (T) \neq \phi$ and there exists a strictly increasing continuous function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle x-Tx, j(x-p)\rangle \geq \phi(\| x-Tx \|),$$
$$\forall x \in D(T), p \in F(T) and j(x-p) \in J(x-p). \qquad (3)$$

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This class of maps was first studied in arbitrary real Banach space by Isiogugu (2005).

Observe that in real Hilbert spaces, j is the identity and for all $x \in K$ *and* $p \in F(T)$, we obtain from equation(3) that

$||Tx-Tp||^2 = ||x-p-[(I-T)x-(I-T)p]||^2$

$= \|Tx-p\|^2 - 2\langle (I-T)x-(I-T)p, x-p\rangle + \|(I-T)x-(I-T)p\|^2 \leq ||x-p||^2 + ||x-Tx||^2 - 2\phi(||x-Tx||)$. (4)

Hence it follows that in a real Hilbert space, equation (3) is equivalent to equation (4). Every demicontractive map is $\phi-$demicontractive with $\phi: [0,\infty) \to [0,\infty)$ given by $\phi(t) = \lambda t^2$. The following example (Isiogugu, 2005) shows that the class of demicontrative maps is a proper subclass of $\phi-$demicontractive maps:

Let $\Re$ denote the reals with the usual norm and let $K = (-\infty,1)$. Define $T: K \to K$ by

$$Tx = \begin{cases} \dfrac{x}{1-x} & , \quad -\infty < x \leq 0 \\[2ex] \dfrac{x}{x-1} & , \quad 0 \leq x < 1 \end{cases}$$

Then $F(T) = \{0\}$ *and if* $p = 0$, then $|Tx-Tp| = \dfrac{|x|}{1-x} \leq |x-p|$

Thus, $|Tx-Tp|^2 = |x-p|^2 - 2\langle x-Tx, x-p\rangle + |x-Tx|^2 \leq |x-p|^2$

so that

$$\langle x-Tx, x-p\rangle \geq \frac{1}{2}|x-Tx|^2 \geq \frac{|x-Tx|^2}{2+|x-Tx|}, \qquad (5)$$

for all $x \in (-\infty,0]$ *and* $p \in F(T)$.

For every $x \in [0,1)$, we have $Tx = \dfrac{x}{x-1}$ *and* $x - Tx = \dfrac{x(2-x)}{1-x} = |x-Tx|$.

Thus, $2+|x-Tx| = \dfrac{2-x^2}{1-x}$ and $\dfrac{|x-Tx|}{2+|x-Tx|} = \dfrac{x(2-x)}{2-x^2} \leq x = |x|$.

For $p = 0 \in F(T) = \{0\}$, we have

$$\langle x-Tx, x-p\rangle = x(x-Tx) = |x||x-Tx| \geq \frac{|x-Tx|^2}{2+|x-Tx|}. \qquad (6)$$

Equations (5) and (6) now imply that

$$\langle x-Tx, x-p\rangle \geq \frac{|x-Tx|^2}{2+|x-Tx|}, \forall x \in (-\infty,1) \qquad (7)$$

and $\forall\ p \in F(T)$.

It follows that $\langle x-Tx, x-p\rangle \geq \phi(|x-Tx|)$ where $\phi:[0,\infty) \to [0,\infty)$ is given by $\phi(t) = \dfrac{t^2}{2+t}$. Clearly, ϕ is continuous, strictly increasing and $\phi(0)=0$. Hence T is ϕ-demicontractive. Given any $L > 0$, if we choose $x \in \left(1-\dfrac{1}{L},1\right)$,then $|Tx-p| = \dfrac{|x|}{1-x} = \dfrac{1}{1-x}|x-p| > L|x-p|$.

Since every demicontractive mapping $T : D(T) \subseteq E \rightarrow E$ satisfies $||Tx - p|| \leq L||x - p||, \forall x \in D(T), p \in F(T)$ and for some $L > 0$ (see for example Chidume and Nnoli, 2002; Hicks and Kubicek, 1979; Maruster,1977; Osilike and Udomene, 2001), it follows that T is not demicontractive. Isiogugu (2005), proved the convergence of the Mann (1953) iteration scheme to the fixed points of $\phi-$demicontractive maps. Specifically, the author proved the following theorem:

Theorem 1.1 (Isiogugu (2005)) Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $T : K \longrightarrow K$ be Lipschitz $\phi-$demicontractive map with Lipschitzconstant L. Let $\{\alpha_{\bar{n}}\}$ be a real sequence satisfying the conditions:

(i) $0 < \alpha_n < 1$ *(ii)* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *(iii)* $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$

Let $\{\alpha_{\bar{n}}\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \ , \ n \geq 1$$

Then

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ (ii) $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$

(iii) $\{x_n\}$ converges strongly to a fixed point p of T if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to p.

Let K be a convex subset of E, and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansiveself maps of K.Xu and Ori (2001), introducedthe implicit iteration process.For $x_1 \epsilon K$ and $\{\alpha_n\}$ in (0,1),the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{array}{l} x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 \\ x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N \\ x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_{N+1} x_{N+1} \end{array}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ n \geq 1 \qquad (8)$$

where $T_n = T_{n \bmod N}$. Using this iterative process they proved weak convergence theorem for approximation of common fixed points of finite family of nonexpansive maps in Hilbert spaces (i.e. a subclass of asymptotically nonexpansive mappings for which $\|Tx - Ty\| \leq \|x - y\| \quad \forall \ x, y \in K)$.

The iterative scheme (8) has been studied by many authors (see for example Xu and Ori, (2001); Sun, 2003; Osilike, 2004a, 2004b; Su and Li, 2006; Zang and Yao, 2006). Using the iteration process (8), Osilike(2004a) proved the following:

Theorem 1.2 (Osilike, 2004a). Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $x_o \in K$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1$$

where $T_n = T_{n \bmod N}$, converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Lemma 1.1 (Osilike, 2004a)
Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:

(i) $0 < \alpha_n < 1$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ (iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$.

Let $x_1 \epsilon K$ and let $\{x_n\}_{n=1}^{\infty}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \qquad n \geq 1$$

where $T_n = T_{n \bmod N}$.
Then

(i) $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$

(ii) $\lim_{n\to\infty} d(x_n, F)$ exists, where $d(x_n, F) = \inf_{p\in F} \|x_n - p\|$

(iii) $\lim_{n\to\infty} \inf \|x_n - T_n x_n\| = 0$.

Theorem 1.3 (Osilike, 2004a)
Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:

(i) $0 < \alpha_n < 1$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ (iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$

Let $x_0 \in K$ and let $\{x_n\}_{n=1}^{\infty}$ be defined by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1$
where $T_n = T_{n \bmod N}$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$ if and only if $\lim_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F}\|x_n - p\|$.

The above results were established for strictly pseudocontractive maps in realBanach spaces using the implicit iteration scheme (8)of Xu and Ori. It is our purpose in this paper to extend these results from strictly pseudocontractive maps to the much more general $\phi - demicontractive$ maps in real Hilbert and arbitrary real Banach spaces. Besides our results complement the results of Isiogugu(2005),extend and generalize several others in the literature (see for exampleXu and Ori, (2001, Su and Li, 2006; Zang and Yao, 2006).
In the sequel we need the following:

Lemma 1.2 (Osilike et al, 2002). Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1+\delta_n)a_n + b_n$, $n \geq 1$. If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Main Results

Theorem 2.1. Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^{N}$ be N ϕ–demicontractive self-maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$, where $F(T_i) = \{x \in K : T_i x = x\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$

be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$, *where* $\alpha=\frac{1+L}{2+L}$ $(for\ 2\alpha-1>0)$. Let $x_1 \in K$ and let $\{x_n\}_{n=1}^{\infty}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \qquad n \ge 1 \qquad (9)$$

where $T_n = T_{n \bmod N}$ Then,

(i) $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ (ii) $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$

(iii) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of the mappings $\{T_i\}_{i=1}^{N}$ if there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p.

PROOF

We shall use the following well-known result of Reinermann (1969),(see also Ishikawa (1974)):

$$\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \qquad (10)$$

which holds for all $x, y \in H$ and $t \in [0,1]$. Let $p \in F(T)$, then using (9)and (10) we obtain

$$\|x_n - p\|^2 = \|(\alpha_n x_{n-1} + (1-\alpha_n)T_n x_n) - p\|^2$$

$$= \|\alpha_n(x_{n-1}-p) + (1-\alpha_n)(T_n x_n - p)\|^2$$

$$=\alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2.$$

$$\le \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\{\|x_n-p\|^2 + \|x_n - T_n x_n\|^2 - \phi(\|x_n - T_n x_n\|)\}$$
$$- \alpha_n(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2.$$

$$\|x_n - p\|^2 \le \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\left[\|x_n-p\|^2 + \alpha_n^2\|x_{n-1}-T_n x_n\|^2 - \phi(\|x_n - T_n x_n\|)\right]$$
$$- \alpha_n(1-\alpha_n)\|x_{n-1} - T_n x_n\|^2$$

$$= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 + \alpha_n^2(1-\alpha_n)\|x_{n-1}-T_n x_n\|^2$$
$$-(1-\alpha_n)\phi(\|x_n - T_n x_n\|) - \alpha_n(1-\alpha_n)\|x_{n-1}-T_n x_n\|^2$$

$$= \alpha_n\|x_{n-1}-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 - \alpha_n(1-\alpha_n)(1-\alpha_n)\|x_{n-1}-T_n x_n\|^2$$
$$-(1-\alpha_n)\phi(\|x_n - T_n x_n\|).$$

Hence,

$$\|x_n-p\|^2 - (1-\alpha_n)\|x_n-p\|^2 \le \alpha_n\|x_{n-1}-p\|^2 - \alpha_n(1-\alpha_n)^2\|x_{n-1}-T_n x_n\|^2 - (1-\alpha_n)\phi(\|x_n - T_n x_n\|)$$

$$[1-(1-\alpha_n)]\|x_n-p\|^2 \le \alpha_n\|x_{n-1}-p\|^2 - \alpha_n(1-\alpha_n)^2\|x_{n-1}-T_n x_n\|^2$$
$$-(1-\alpha_n)\phi(\|x_n - T_n x_n\|)$$

$$\|x_n-p\|^2 \le \|x_{n-1}-p\|^2 - (1-\alpha_n)^2\|x_{n-1}-T_n x_n\|$$
$$-\frac{(1-\alpha_n)}{\alpha_n}\phi(\|x_n - T_n x_n\|)$$
$$\le \|x_{n-1}-p\|^2 - (1-\alpha_n)^2\|x_{n-1}-T_n x_n\|^2 - (1-\alpha_n)\phi(\|x_n - T_n x_n\|)$$
$$\le \|x_{n-1}-p\|^2 - (1-\alpha_n)\phi(\|x_n - T_n x_n\|) \le \|x_{n-1}-p\|^2. \qquad (11)$$

Hence $\lim_{n\to\infty}\|x_n - p\|$ exists, and this completes the proof of *(i)*

Since $\|x_n - p\| \le M \quad \forall\ n \ge 1,$ for some $M > 0$, we obtain from (11) that,

$\sum_{j=N+1}^{n}(1-\alpha_j)\phi(\|x_j - T_j x_j\|) \le \sum_{N+1}^{n}[\|x_{j-1}-p\|^2 - \|x_j - p\|^2] \le \|x_N - p\|^2 < \infty$

Hence, $\sum_{n=1}^{\infty}(1-\alpha_n)\phi(\|x_n - T_n x_n\|) < \infty$

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Since $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then we must have $\liminf_{n\to\infty} \phi(\|x_n - T_n x_n\|) = 0$.

Since ϕ is strictly increasing and continuous, then $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$.

Thus completing the proof of (ii).

Since $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p and $\lim_{n\to\infty}\|x_n - p\|$ exists, by Lemma 1.2, $\lim_{n\to\infty}\|x_n - p\| = 0$. Thus completing the proof of Theorem 2.1.

Remark: A mapping T with domain $D(T)$ and range $R(T)$ is said to be demiclosed at a point (see for exampleOsilike and Udomene, 2001) if whenever $\{x_n\}$ is a sequencein $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$. If in Theorem 2.1, the iteration parameter $\{\alpha_n\}_{n=1}^{\infty}$ satisfies $0 < \alpha_n \le \alpha < 1$, then $(1-\alpha_n) \ge 1-\alpha$ ($(1-\alpha_n)^2 \ge (1-\alpha)^2$) and it follows from equation(11) that $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. An immediate consequence of this is that if (I – T) is demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the family T_i.

Theorem 2.2

Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be N ϕ-demicontractive self-maps of K such that $\lim_{n\to\infty}\alpha_n = 1$ and the sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfies the conditions:

(i) $0 < \alpha_n < 1$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ (iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$

Let $x_1 \epsilon K$ and let $\{x_n\}_{n=1}^{\infty}$ be defined by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n$, $n \ge 1$

where $T_n = T_{n \bmod N}$ Then,

(i) $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$

(ii) $\lim_{n\to\infty} d(x_n, F)$ exists, where $d(x_n, F) = \inf_{p\in F}\|x_n - p\|$

(iii) $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$

(iv) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of the mappings $\{T_i\}_{i=1}^{N}$ if there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p.

PROOF

It is well known (see for example Chang ,1997) that the inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{12}$$

holds for all $x, y \in E$, and $j(x-y) \in J(x-y)$.

Let $p \in F(T)$, using (9), (12) and (3) respectively, we obtain:

$$\begin{aligned}
\|x_n - p\|^2 &= \|(\alpha_n x_{n-1} + (1-\alpha_n)T_n x_n) - p\|^2 \\
&\le \|\alpha_n(x_{n-1} - p)\|^2 + 2\langle(1-\alpha_n)(T_n x_n - p), j(x_n - p)\rangle \\
&= \alpha_n^2\|x_{n-1} - p\|^2 + 2(1-\alpha_n)\langle T_n x_n - p, j(x_n - p)\rangle \\
&= \alpha_n^2\|x_{n-1} - p\|^2 + 2(1-\alpha_n)[\langle T_n x_n - x_n, j(x_n - p)\rangle + \langle x_n - p, j(x_n - p)\rangle] \\
&\le \alpha_n^2\|x_{n-1} - p\|^2 + 2(1-\alpha_n)\langle T_n x_n - x_n, j(x_n - p)\rangle + 2(1-\alpha_n)\|x_n - p\|^2 \\
&= \alpha_n^2\|x_{n-1} - p\|^2 - 2(1-\alpha_n)\langle x_n - T_n x_n, j(x_n - p)\rangle + 2(1-\alpha_n)\|x_n - p\|^2
\end{aligned}$$

$$\leq \alpha_n^2 \|x_{n-1} - p\|^2 - 2(1-\alpha_n)\phi(\|x_n - T_n x_n\|) + 2(1-\alpha_n)\|x_n - p\|^2 \qquad (13)$$

Since $\lim_{n\to\infty} \alpha_n = 1,$ then there exists a positive integer N_1 such that

$\alpha_n \geq 1 - \frac{(1-\lambda)}{2}, \ \ for\ all\ n \geq N_1,\ \lambda \in (0,1).$ Thus $\alpha_n \geq 1 - 2(1-\alpha_n), \geq \lambda \ \ for\ all\ n \geq N_1.$

Also $\lim_{n\to\infty} [1 - 2(1-\alpha_n)] = 1,$ so that there exists N_2 such that $1 - 2(1-\alpha_n), \leq 1 + \delta \ \ for\ all\ \delta > 0\ \ and\ \ n \geq N_2$. Set $D = 1 + \delta \ \ and\ \ N = \max\{N_1, N_2\}.$

It follows from equation(13) that

$$\|x_n - p\|^2 \leq \frac{\alpha_n^2}{[1 - 2(1-\alpha_n)]}\|x_{n-1} - p\|^2 - \frac{2(1-\alpha_n)}{[1 - 2(1-\alpha_n)]}\phi(\|x_n - T_n x_n\|)$$

$$= \left[\frac{[1 - 2(1-\alpha_n)] + 1 - 2\alpha_n + \alpha_n^2}{[1 - 2(1-\alpha_n)]}\right]\|x_{n-1} - p\|^2 - \left[\frac{2(1-\alpha_n)}{[1 - 2(1-\alpha_n)]}\right]\phi(\|x_n - T_n x_n\|)$$

$$= \left[1 + \frac{(1-\alpha_n)^2}{[1 - 2(1-\alpha_n)]}\right]\|x_{n-1} - p\|^2 - \left[\frac{2(1-\alpha_n)}{[1 - 2(1-\alpha_n)]}\right]\phi(\|x_n - T_n x_n\|)$$

$$\leq \left[1 + \frac{1}{\lambda}(1-\alpha_n)^2\right]\|x_{n-1} - p\|^2 - \frac{2}{D}(1-\alpha_n)\phi(\|x_n - T_n x_n\|)\ \forall n \geq N$$

$$= [1 + \sigma_n]\|x_{n-1} - p\|^2 - \frac{2}{D}(1-\alpha_n)\phi(\|x_n - T_n x_n\|)\ \ \forall\ n \geq N \qquad (14)$$

where $\sigma_n = \frac{1}{\lambda}(1-\alpha_n)^2$.

From condition (iii), $\sum_{n=1}^{\alpha} \sigma_n < \infty,$ sothat it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists.This completes the proof of (i) .

Also, from equation(14) it follows that

$$\|x_n - p\| \leq [1 + \sigma_n]^{\frac{1}{2}}\|x_{n-1} - p\| \leq [1 + \sigma_n]\|x_{n-1} - p\|.$$

Thus, $d(x_n, F(T)) \leq [1 + \sigma_n]d(x_{n-1}, F(T))$. And again it follows from Lemma1.2 that $\lim_{n\to\infty} d(x_n, F(T))$ exists, and this completes the proof of (ii).

From (14), it follows that $\{\|x_{n-1} - p\|\}_{n=1}^{\infty}$ is bounded.Let $\|x_{n-1} - p\| \leq M,\ \ \forall\ \ n \geq 1$ so that,

$$\frac{2}{D}(1-\alpha_j)\phi(\|x_n - T_n x_n\|) \leq [1 + \sigma_n]\|x_{n-1} - p\|^2 - \|x_n - p\|^2$$

$$\leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + \sigma_n\|x_{n-1} - p\|^2$$

Hence, $\frac{2}{D}\sum_{j=N+1}^{n}(1-\alpha_j)\phi(\|x_n - T_j x_j\|) \leq \|x_N - p\|^2 + M\sum_{j=N+1}^{n}\sigma_j$,

and it follows that $\sum_{n=1}^{\infty}(1-\alpha_n)\phi(\|x_n - T_n x_n\|) < \infty$.

Condition (ii) implies that $\liminf_{n\to\infty} \phi(\|x_n - T_n x_n\|) = 0$ .Since ϕ is strictly increasing and continuous ,then $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$, and this completes the proof of (iii).Again as in the proof of Theorem 2.1 (iii), since $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p and $\lim_{n\to\infty}\|x_n - p\|$ exists, by Lemma 1.2, $\lim_{n\to\infty}\|x_n - p\| = 0$.

This completes the proof of Theorem 2.2□

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