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AN EXAMINATION OF THE CHRISTOFFEL-DARBOUX RECURRENCE RELATION

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ABSTRACT

An alternate proof of the Christoffel-Darboux recurrence formula for consecutive orthogonal polynomials is presented. An example is used to demonstrate its application to the normalization of a sequence of orthogonal polynomials.

KEY WORDS: Orthogonal polynomials, Weight function, Recurrence relation.

INTRODUCTION

By orthogonalizing the set of linearly independent non-negative powers of x : 1, x, x^2, ..., x^i, .. in the sense explained in Draper and Smith (1981, p.275), a set of polynomials : $P_o(x)$, $P_1(x)$, $P_2(x)$, ..., $P_i(x)$, ... is obtained, uniquely determined by the following conditions:

(i) $P_i(x)$ is a polynomial of precise degree i in which the coefficient of x^i is positive;

(ii) the system $\{P_i(x)\}$ is orthonomal, that 's, the polynomials $P_i(x)$ are the normalised orthogonal polynomials associated with the weight function w(x).

Related algorithms for constructing orthogonal polynomials have been derived by Snee (1973) and Farebrother (1974).

It has been shown by Christoffel-Darboux (1975) that for any three consecutive orthogonal polynomials; $P_i(x)$, $P_{i-1}(x)$ and $P_{i-2}(x)$, there exist a recurrence relation,

$$P_i(x) = (A_i x + B_i)\, P_{i-1}(x) - C_i P_{i-2}(x) \qquad (1)$$

i = 2, 3, 4, ..., where A_i, B_i and C_i are constants. However, their proof of this recurrence relation does not seem to be quite explicit, hence the need for an alternative prove. A theoretical base has been developed and some computational experience reported in Okolo and Bamiduro (2001), for the determination of orthogonal coefficients, when data points are unequally spaced and/or unequally replicated, in the context of this recurrence relation.

This paper present explicitly another independent proof of the Christoffel-Darboux recurrence relation for any three consecutive orthogonal polynomials. It is shown that this recurrence relation can be used to recognize a normalized sequence of orthogonal polynomials.

ALTERNATIVE PROOF FOR CHRISTOFFEL-DARBOUX RECURRENCE RELATION

Let w(x) be a weight function for the interval [a , b], and $_{i\,k}$ a sequence of coefficients. The

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product $xP_{i-1}(x)$ as a polynomial of the ith degree, is expressible in the form

$$xP_{i-1}(x) = {}_{i,0}P_0(x) + {}_{i,1}P_1(x) + \dots\dots + {}_{i,i-2}P_{i-2}(x) + {}_{i,i-1}P_{i-1}(x) + {}_{i,i}P_i(x), \qquad (2)$$

where $P_0, P_1, \dots\dots, P_i$ are normal orthogonal functions in the sense that

$$\sum_x w(x) P_i(x) P_k(x) = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases} \qquad (3a)$$

for $x = x_1, x_2, \dots, x_n$;

$$\int_x w(x) P_i(x) P_k(x)\, dx = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases} \qquad (3b)$$

for continuous x.

Multiplication of equation (2) by $w(x) P_k(x)$, followed by summation and integration from a to b gives

$$\alpha_{ik} = \sum_x w(x)\, x P_{i-1}(x) P_k(x) \qquad (4a)$$

for $x = x_1, x_2, \dots, x_n$;

$$\alpha_{ik} = \int_a^b w(x)\, x P_{i-1}(x) P_k(x)\, dx \qquad (4b)$$

for continuous x and $0 \; k \; i$.

One may next evaluate $\sum_x w(x) P_t(x) xP_{i-1}(x)$ from equation (2), which certainly vanishes, by equation (3a) when $t > i$ because the expansion of $xP_{i-1}(x)$ contains no orthogonal polynomials of degree higher than i. Since the expression is, furthermore, quite symmetric in t and i-1, one may apply the same argument, interchanging t and i-1, with the conclusion that the expression vanishes whenever $i-1 > t+1$. It has thus a finite value only for $t \; i$ and $i-1 \; t+1$ (i.e. $t \; i-2$).

It follows that only the last three terms in the expansion of $xP_{i-1}(x)$ have coefficients not identically zero, so that, equation (2) reduces to

$$xP_{i-1}(x) = \alpha_{i,i-2} P_{i-2}(x) + \alpha_{i,i-1} P_{i-1}(x) + \alpha_{i,i} P_i(x) \qquad (5a)$$

or, after some rearrangement,

$$\alpha_{i,i} P_i(x) = (x - \alpha_{i,i-1}) P_{i-1}(x) - \alpha_{i,i-2} P_{i-2}(x). \qquad (5b)$$

Let K_j denote the coefficient of x^j in $P_i(x)$ for each value of

$i,\ j = 0, 1, 2, \ldots, i.$

Multiply equation (5b) through by $w(x)\, P_i(x)$ and sum. This gives according to equation (3a),

$$\alpha_{i,i} = \sum_x w(x)\, P_i(x)\ x\, P_{i-1}(x)$$

$$= \sum_x w(x)\, P_i(x)\, [K_{i-1} x^i + K_{i-2} x^{i-1} + \ldots\ldots]$$

$$= \sum_x w(x)\, P_i(x)\ K_{i-1}\ x^i$$

$$= \frac{K_{i-1}}{K_i} \sum_x w(x) P_i(x) K_i x^i$$

$$= \frac{K_{i-1}}{K_i} \sum_x w(x) P_i^2(x) = \frac{K_{i-1}}{K_i} \qquad (6)$$

Similarly, multiply equation (5b) through by $w(x)\, P_{i-2}(x)$ and sum. By virtue of equation (3a), this gives,

$$0 = \sum_x w(x)\, P_{i-1}(x)\ x\, P_{i-2}(x) - \alpha_{i,\, i-2}$$

that is,

$$\alpha_{i,\, i-2} = \sum_x w(x)\, P_{i-1}(x)\, x\, P_{i-2}(x)$$

$$= \sum_x w(x)\, P_{i-1}(x)\, [K_{i-2}\, x^{i-1} + K_{i-3}\, x^{i-2} + \ldots\ldots]$$

$$= \sum_x w(x)\, P_{i-1}(x)\ K_{i-2}\ x^{i-1}$$

$$= \frac{K_{i-2}}{K_{i-1}} \sum_x (x) P_{i-1}(x) K_{i-1} x^{i-1}$$

$$= \frac{K_{i-2}}{K_{i-1}} \sum_x w(x) P_{i-1}^2(x) = \frac{K_{i-2}}{K_{i-1}} \qquad (7)$$

Substitute equations (6) and (7) into (5a). This gives

$$xP_{i-1}(x) = \frac{K_{i-2}}{K_{i-1}} P_{i-2}(x) + \alpha_{i,i-1} P_{i-1}(x) + \frac{K_{i-1}}{K_i} P_i(x) \qquad (8)$$

Now, consider the left hand side of (8). This can be expressed as

$$xP_{i-1}(x) = K_{i-1}x^{i} + K_{i-2}x^{i-1} + K_{i-3}x^{i-2} + \ldots.. \quad (9a)$$

On the other hand, the right hand side of (8) can be expanded as

$$\frac{K_{i-2}}{K_{i-1}}P_{i-2}(x) + \alpha_{i,i-1}P_{i-1}(x) + \frac{K_{i-1}}{K_i}P_i(x)$$

$$= \frac{K_{i-2}}{K_{i-1}}P_{i-2}(x) + \alpha_{i,i-1}[K_{i-1}x^{i-1} + K_{i-2}x^{i-2} + \ldots + K_0x^0]$$

$$+ \frac{K_{i-1}}{K_i}[K_ix^i + K_{i-1}x^{i-1} + \ldots + K_0x^0] \quad (9b)$$

Comparison of the coefficients of x^{i-1} in (9a) and (9b) gives

$$K_{i-2} = \alpha_{i,i-1}K_{i-1} + \frac{K_{i-1}}{K_i}K_{i-1}$$

Solving for $_{i,\,i-1}$, this gives

$$\alpha_{i,i-1} = \frac{K_{i-2}}{K_{i-1}} - \frac{K_{i-1}}{K_i} \quad (10)$$

Finally, substitution of equations (6), (7) and (10) into (5b) gives

$$\frac{K_{i-1}}{K_i}P_i(x) = \left(x - \left(\frac{K_{i-2}}{K_{i-1}} - \frac{K_{i-1}}{K_i}\right)\right)P_{i-1}(x) - \frac{K_{i-2}}{K_{i-1}}P_{i-2}(x)$$

which can be rearranged to give

$$P_i(x) = \left(\frac{K_i}{K_{i-1}}x + \left(1 - \frac{K_iK_{i-2}}{K_{i-1}^2}\right)\right)P_{i-1}(x) - \frac{K_iK_{i-2}}{K_{i-1}^2}P_{i-2}(x) \quad (11)$$

Set the constants A_i, B_i and C_i to be

$$A_i = \frac{K_i}{K_{i-1}}, \quad B_i = 1 - \frac{K_iK_{i-2}}{K_{i-1}^2}, \text{ and } C_i = \frac{K_iK_{i-2}}{K_{i-1}^2} \quad (12)$$

It follows that

$$P_i(x) = (A_i x + B_i)\, P_{i-1}(x) - C_i P_{i-2}(x);$$

$$i = 2, 3, 4, \ldots\ldots$$

This recurrence relation can be written in the form

$$P_i(x) = \left(\frac{K_i}{K_{i-1}}x + B_i\right)P_{i-1}(x) - \frac{K_iK_{i-2}}{K_{i-1}^2}P_{i-2}(x) \quad (13)$$

or, after some rearrangement,

$$xP_{i-1}(x) = \frac{K_{i-1}}{K_i} P_i(x) + \frac{K_{i-2}}{K_{i-1}} P_{i-2}(x) + \beta_{i-1} P_{i-1}(x) \tag{14}$$

where

$$\beta_{i-1} = \frac{-(k_{i-1})B_i}{K_i}.$$

Inspection of (14) shows how one can recognize a normalized sequence by its recurrence relation, for if i is replaced by i-1 in the coefficient of $P_i(x)$, the coefficient of $P_{i-2}(x)$ must result (Wilf, 1962, p. 54).

ILLUSTRATIVE EXAMPLE

Suppose a certain sequence of orthogonal polynomials satisfies

$$P_{i+1}(x) = [(i + 1)\, x + 1]\, P_i(x) - 3\,(i + 1)\, P_{i-1}(x). \tag{15}$$

This sequence, according to Wilf (1962, P.54), is not normalized, but it is intended to find the normalization constants.

Solving equation (15) for $xP_i(x)$ gives,

$$xP_i(x) = \frac{1}{i+1} P_{i+1}(x) + 3P_{i-1}(x) - \frac{1}{i+1} P_i(x) \tag{16}$$

Let $\Psi_i(x) = \lambda_i P_i(x)$ denote the normalized sequence, where $\lambda_0, \lambda_1, \ldots$ are to be found.

Substituting in equation (16),

$$x\frac{1}{\lambda_i}\Psi_i(x) = \left(\frac{1}{i+1}\right)\left(\frac{1}{\lambda_{i+1}}\right)\Psi_{i+1}(x) + 3\left[\frac{1}{\lambda_{i-1}}\right]\Psi_{i-1}(x) - \left(\frac{1}{i+1}\right)\left(\frac{1}{\lambda_i}\right)\Psi_i(x)$$

so that

$$x\Psi_i(x = \left[\frac{\lambda_i}{(i+1)(\lambda_{i+1})}\right]\Psi_{i+1}(x) + 3\left[\frac{\lambda_i}{\lambda_{i-1}}\right]\Psi_{i-1}(x) - \frac{1}{i+1}\Psi_i(x) \tag{17}$$

The condition for normalization is

$$\frac{\lambda_{i-1}}{i\lambda_i} = \frac{3\lambda_i}{\lambda_{i-1}} \quad \text{or} \quad \lambda_i = \frac{1}{\sqrt{3i}}\lambda_{i-1}.$$

Hence

$$\lambda_i = 3^{-i/2}\,(i!)^{½}\,\lambda_0$$

are the required normalization constants.

CONCLUSION

An explicit proof of the Christoffel-Darboux recurrence relation for any three consecutive orthogonal polynomials have been described. The relation is quite useful whenever it is desired to find the normalization constants for a sequence of orthogonal polynomials.

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