CYCLICAL SUBNORMAL SEPARATED A-GROUPS OF NILPOTENT LENGTH N

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(Received 24 February, 2006; Revision Accepted 13 November, 2006)

ABSTRACT

The existence of Cyclically Subnormally Separated A-groups of nilpotent length n where n is any positive integer has been proved. This shows that there are A-groups of any nilpotent length n in CSn.

KEYWORDS: Cyclic Subnormal group, A-Groups of Nilpotent length n.

1. INTRODUCTION

In Markafi, (2006 and 2005) the existence of CSn, A-groups of nilpotent length four and of nilpotent lengths five and six were presented. The purpose of this communication is to show that there are A-groups of any nilpotent length n in CSn, where n is a positive integer.

A sketch of the case for nilpotent length seven is given before plunging into the general case. The schematic diagram (Figure 1) and the ones given in the two papers sighted above bring out in a simple form the beauty of the subgroup structure in these groups.

2. NILPOTENT LENGTH SEVEN

This case is presented in a way similar to that of nilpotent length six. The diagram (Fig. 1) is to help the mind to imagine what is going on as we go through the various subgroups in the general discussion. The notation is the same as the one adopted in the nilpotent length six and it is easy to see that the group presented is in fact a CSn A-group of nilpotent length seven.

G = P_7 × H nilpotent length seven

P_1 and P_2 cyclic of orders P_1^6 and P_2^6 respectively. P_3, P_4 and P_5 homocyclic of exponents P_3^4, P_4^2 and P_5^2 respectively. P_6 and P_7 elementary abelian P_6 and P_7-groups.

Figure 1

k = <w_1, w_2, w_3, w_4, w_5, w_6, w_7, y_2, x_1^4, x_1^5, w_5^2, w_5^3, w_5^4, w_5^5>

L = <w_3, w_4, w_5, w_6, w_7, w_7^2, w_7^3, w_7^4, w_7^5, w_7^6, w_7^7, y_2^2, x_1^4, x_1^5, w_5^2, w_5^3, w_5^4, w_5^5>

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3. **NILPOTENT LENGTH n**

We start the general construction by recalling some of the facts that will guide us in the various steps we shall take. We know from theorem (2.2) (Markafi, 2006) that an A-group in CSn of whose Sylow p-subgroups have exponent at most $p^{n-1}$ will have nilpotent length at most $n$. We also know that the nilpotent length of any A-group cannot exceed the number of the distinct primes that divide its order, theorem (8.3) of (Taunt, 1949). So we know that an A-group in CSn whose order is divisible by $n$ distinct primes and whose Sylow p-subgroups are of exponent at most $p^{n-1}$ will have nilpotent length at most $n$. This and theorem (2.7) (Markafi 2006) gave us an idea of the number of abelian p-groups we shall need and their exponents.

As we go through the construction we choose our primes in the following way. Suppose we construct the $H_r$ a group of nilpotent length $r$, we choose the next prime $P_{r+1}$ such that the order of $H_r$ divides $P_{r+1}-1$ where

$$H_r = P_r \bigtimes H_{r-1}, \quad 1 \leq r \leq n-1$$

with $H_0 = 1$ and $H_1 = P_1$. We let $P_1 = <x>$ and $P_2 = <y>$ be cyclic of orders $P_1^{n-1}$ and $P_2^{n-1}$ respectively. The next step is to form

$$H_2 = P_2 \bigtimes P_1$$

in a way similar to the $H_2$ in the construction for the nilpotent five case (Markafi, 2005).

For the rest of the prime power order p-groups we get

$$P_r, \quad 3 \leq r \leq n-1$$

in the following way. $P_r$ is a homocyclic $p_r$-groups of exponent $m$ and is the result of inducing up to $H_{r-1}$ a rank one $K_{r-1}$-module, $W_i = <w_i>$, over the ring $k_{r} = \mathbb{Z}[x]$ where

$$m = p_{r-1}^{n-1} \text{ and } K_{r-1} = <W_{r-1}, W_{r-2}^*, W_{r-3}^*, ..., W_3^*, W_2^*, W_1^*>.$$  

Here

$$W_i = (w_i \otimes 1)k_{r}, \quad k_{r} = P_{r-1}^{P_{r-1}}, \quad 3 \leq i \leq r-1; \quad W_2 = y^{P_{r-3}}; \quad W_1 = x^{P_{r-2}}$$

and

$$W_i = <w_i \otimes t > k_{r-1}, \quad 3 \leq i \leq r-1>$$

where $T_{r-1}$ is a transversal to $K_{r-1}$ in $H_{r-1}$ and $T_{r-1} = T_{r-1}(1)$.

So that $W_i$ is $K_{r-1}$ invariant. We then have

$$P_r = <w_0, W_1^*, >, \quad 3 \leq r \leq n-1$$

To see what is going on, we note that if $r = 3$,

$$K_2 = <w_2^*, W_1^*>, \quad <y, x^{P_3}>, $$

while if $r = n-1$

$$K_{n-1} = <W_{n-1}, W_{n-2}^*, ..., W_3^*, W_2^*, y^{P_{n-4}}, x^{P_{n-3}}>, $$

We next get $P_r$, which is elementary abelian and is the result of a similar inducting process as above from

$$K_{r-1} = <w_i^*, W_i^*>, \quad 3 \leq i \leq n-1; \quad y^{P_{r-3}}, x^{P_{r-2}}$$

We construct the group by induction on its nilpotent length $n$, so we assume we have an A-group $H_{r-1}$ in CSn of nilpotent length $r$ and we have

$$L_{r-1} < K_{r-1} \text{ with } K_{r-1}/L_{r-1} \text{ cyclic of order } p_1 p_2 ... p_r, \text{ where } K_{r-1} \text{ is as given above and}$$

$$L_{r-1} = <w_i^{P_{r-1}}, W_i^*, W_j^*, L_{r-1}> \text{ where } 1 \leq l \leq r-1, 3 \leq j \leq r-1>$$

The action of $K_{r-1}/L_{r-1}$ on $W_i$ is lifted to $K_{r-1}$, so that $K_{r-1}$ acts as follows,

$$w_i W_j = \lambda_i w_i, \quad 1 \leq i \leq r-1$$

and $W_i^*$ acts trivially on $W_i$ for $3 \leq j \leq r-1$. Each $\lambda_i$ is an element in $k_i^*$ the group of units of the ring $k_i$ of order $p_i$. Then we get

$$P_r = W_i^{P_{r-1}} \text{ and form } H_r = P_r \bigtimes H_{r-1}.$$
The next step is to define \( K_r \) and \( L_r \), and show that \( L_r < K_r \) and that \( K_r/L_r \) is cyclic of order \( p_1 \ldots p_r \). In order to complete the induction process we also have to show that \( H_r \) is a CS-\( n \) group. Now we know that \( P_i \) is a homocyclic \( p_i \)-group of exponent \( p_i^{q_i} \) so let

\[
K_r = \langle w^{*}_{1}, W^{*}_i ; 1 \leq i \leq r, 3 \leq j \leq r \rangle
\]

where

\[
w^{*}_i = (w_i \otimes 1)^{P_i^{q_i-1}}, 3 \leq i \leq r, w^*_1 = y^{P_i^{q_i-2}}, w^*_j = x^{P_i^{q_i-1}}
\]

and

\[
W^{*}_i = \langle w^{*}_i \otimes t \in T_i \setminus \{1\}, 3 \leq i \leq r \rangle
\]

for some transversal \( T_i, i \) to \( K_i, i \) in \( H_i \).

\[
L_r = \langle (W^{*}_i)^{P_i}, W^{*}_i, W^{*}_j ; 1 \leq i \leq r, 3 \leq j \leq r \rangle
\]

Now we note that

\[
K_r = \langle \Omega_{n_r}(P_i), 1 \leq i \leq r, L_r \rangle
\]

while

\[
L_r = \langle \Omega_{n_r}(P_i), 1 \leq i \leq r, L_r \cap L_{r-1} \rangle
\]

But

\[
\Omega_{n_r}(P_i) \leq \Omega_{n_r}(P_i) \leq C_{n_r}(L_{r-1}), 1 \leq i \leq r
\]

So

\[
L_r < K_r
\]

Clearly \( K_r/L_r \) is cyclic of order \( p_1 \ldots p_r \), generated by

\[
p = w^{*}_1 \ldots w^{*}_r, \text{ where } w^{*}_i = w_i^{*}, 1 \leq i \leq r
\]

In order to show that \( H_r \) is a CS-\( n \) group we start by showing that \( L_r \) is subnormal in \( H_r \). Here we observe that

\[
K_r = P_i^{L_{r-1}}
\]

and because \( L_{r-1} \) is subnormal in \( H_{r-1} \) we have

\[
K_r P_i^{L_{r-1}} = L_{r-1} \cap H_{r-1} = H_r/P_i
\]

Since \( L_r \) is subnormal in \( K_r \), we also have

\[
L_r \cap H_r
\]

So using proposition (2.2), (Makarfi 2005) it only remains to show that for any element \( h \) in \( H_r \) of prime power order

\[
<h> \cap K_r < <h> \cap L_r \Rightarrow h \in L_r
\]

Now, for each \( i \in \{1, 2, \ldots, r\} \)

\[
P_i = \langle w_i \otimes 1 > x W_i^{*}
\]

So we have

\[
K_r \cap P_i = \langle w_i^{*} > x W_i^{*} \text{ and } L_r \cap P_i = \langle (W_i^{*})^{P_i} > x W_i^{*}
\]

Now suppose that \( h \) is a \( p_r \)-element satisfying the hypothesis in (*) and \( Q \) is a Sylow \( p_r \)-subgroup of \( H_r \) containing \( h \). Then

\[
Q \cap L_r \text{ is a Sylow } p_r \text{-subgroup of } L_r \text{ because } L_r \text{ is subnormal in } H_r, \text{ so}
\]

\[
Q \cap L_r = \langle u_r, L_r \rangle
\]

Thus \( Q \cap L_r \) contains a subgroup isomorphic to \( W^{*}_r \), say \( Q_0 \) and this is a direct factor of \( Q \) because both \( Q \) and \( W^{*}_r \) are homocyclic of the same exponent. So

\[
Q = Q_0 \times <u_r>
\]

\[
\therefore Q \cap L_r = Q_0 \times <u_r>, \text{ where } <u_r> = <u_r > \cap L_r
\]

Also \( Q \cap K_r \) is a Sylow \( p_r \)-subgroup of \( K_r \) and

\[
Q \cap L_r \subset Q \cap K_r
\]

\[
Q \cap K_r = Q_0 \times <u_2>, \text{ where } <u_2> = K_r \cap <u_r>
\]

So that \( u_2^{p_i} = u_1 \). If

\[
h \in Q \cap L_r
\]
then \( h = h_1u' \) for some \( h_1 \in Q_0 \) and \( u' \in \langle u \rangle \cdot \langle u_1 \rangle \) so that \( \langle u' \rangle \) contains \( \langle u_2 \rangle \). Therefore \( \langle h \rangle \) contains \( h_1u_2 \) while \( h_1u_2 \not\in L_r \).

and this contradicts the assumption that \( h \) satisfies the hypothesis in (\( * \)). So we conclude that \( h \in L_r \). This establishes (\( * \)) and we must have \( H \) in CSn.

We finally conclude, by induction, that our group is a CSn group.

\[
G = P_n \times H_{n+1}
\]

ACKNOWLEDGEMENT

This author wishes to thank the International Centre for Theoretical Physics, Trieste, Italy, for giving him the facilities to prepare this paper.

REFERENCES

