CONVOLUTION PROPERTIES ASSOCIATED WITH CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT

Murugusundaramoorthy and Magesh (2004) introduced the subclasses $TS(\alpha, \beta)$ and $TS_p(\alpha, \beta)$ of uniform convex functions and starlike functions with negative coefficients where they obtained some results. Our aim here is to investigate the convolution properties associated with the subclasses $TS(\alpha, \beta)$ and $TS_p(\alpha, \beta)$ respectively by applying certain techniques based especially upon the Cauchy-Schwarz and Holder inequalities. Some consequences are also discussed.

KEY WORDS: Analytic, Convolution, Convex, Starlike, Univalent.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARY RESULTS.

Denoted by $S$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the unit disk $E = \{z: |z| < 1\}$. Also, ST and CV are the subclasses of $S_p$ that are respectively starlike and convex.

A function is uniformly convex (uniformly starlike) in $E$ if $f(z)$ is in CV(ST) and has the property that for every circular arc $\gamma$ contained in $E$, with centre $e$ also in $E$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(e)$. The class of uniformly convex functions is denoted by UCV and the class of uniformly starlike functions by UST.

It is well known that

$$f \in UCV \iff \frac{|zf'(z)|}{|f'(z)|} \leq \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\}$$

see Ma and Minda (1992). Rongning (1993) introduced a new subclass of starlike functions related to UCV defined as

$$f \in S_p \iff \frac{|zf'(z)|}{|f(z)|} \leq \Re \left\{ \frac{zf'(z)}{f(z)} \right\}.$$ 

We note here that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$.

Later, Rongning (1993) generalized the class $S_p$ by introducing a parameter $\alpha$, $-1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \iff \frac{|zf'(z)|}{f(z)} \leq \Re \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\}.$$ 

Also Murugusundaramoorthy and Magesh (2004) introduced subclasses $TS(\alpha, \beta)$ and $TS_p(\alpha, \beta)$.

Here we let $TS(\alpha, \beta) = S(\alpha, \beta) \cap T$ where $T$, the subclass of $S$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad \forall n \geq 2.$$ 

This class of functions was introduced and studied by Silverman(1995). Silverman and Silvia (1997). And also let $TS_p(\alpha, \beta)$ denote the class of function, $f(z)$ in $TS(\alpha, \beta)$ and be of the form
\[ f(z) = z - \frac{b(1-\alpha)(c)}{(2 + \beta - \alpha)(\alpha)} z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (a_n \geq 0, \; 0 \leq b \leq 1). \]  \tag{1.3}

The main aim of this paper is to investigate the convolution properties associated with the subclasses \( TS^*(\alpha, \beta) \) and \( TS^*_n(\alpha, \beta) \).

A necessary and sufficient condition for a function \( f(z) \) of the form (1.2) to be in the class \( TS^*(\alpha, \beta) \), \(-1 \leq \alpha < 1, \beta \geq 1 \) is that

\[ \sum_{n=2}^{\infty} \frac{\gamma(n+1+\gamma)}{\gamma(n+1)} a_n \leq 1 - \alpha. \]

Also, \( TS^*_n(\alpha, \beta) \) denote the class of functions \( f(z) \) in \( TS^*(\alpha, \beta) \) and of the form

\[ f(z) = z - \frac{b(1-\alpha)(c)}{(2 + \beta - \alpha)(\alpha)} z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (a_n \geq 0, \; 0 \leq b \leq 1) \]

if and only if

\[ \sum_{n=2}^{\infty} \frac{\gamma(n+1+\gamma)}{\gamma(n+1)} a_n \leq (1-b)(1-\alpha) \quad (-1 \leq \alpha < 1, \; \beta \geq 0) \]

To do this we need the following preliminary results which we shall state without proof.

Theorem A: Murugusundaramoorthy and Magesh (2004). A necessary and sufficient condition for \( f(z) \) of the form (1.2) to be in the class \( TS(\alpha, \beta) \), \(-1 \leq \alpha < 1, \; \beta \geq 0 \) is that

\[ \sum_{n=2}^{\infty} \frac{\gamma(n+1+\gamma)}{\gamma(n+1)} a_n \leq 1 - \alpha \]  \tag{1.4}

Theorem B: Murugusundaramoorthy and Magesh (2004). Let function \( f(z) \) be defined by (1.3) then \( f(z) \in TS_n(\alpha, \beta) \) if and only if

\[ \sum_{n=3}^{\infty} \frac{\gamma(n+1+\gamma)}{\gamma(n+1)} a_n \leq (1-b)(1-\alpha) \]  \tag{1.5}

Finally, for functions \( f_j(z) \in S \) \((j = 1, \ldots, m)\) given by

\[ f_j(z) = z - \sum_{n=1}^{m} a_{n,j} z^n \quad (j = 1, \ldots, m) \]  \tag{1.6}

the Hadamard product (or convolution) is defined by

\[ (f_1 \ast \ldots \ast f_m)(z) = z - \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} a_{n,j} \right) z^n \]  \tag{1.7}

2. **CONVOLUTION PROPERTIES**

Theorem 2.1: If \( f_j(z) \in TS^*(\alpha, \beta) \) \((j = 1, \ldots, m)\), then

\[ (f_1 \ast \ldots \ast f_m)(z) \in TS^*(\rho, \beta) \]

where

\[ \rho = 1 - \frac{(n-1)c_{n-1} \prod_{j=1}^{m} (1-\alpha)}{(a_{n-1} \prod_{j=1}^{m} \frac{\gamma(n+1+\gamma)}{\gamma(n+1)} \alpha_j)} \quad (n-1)c_{n-1} \prod_{j=1}^{m} \frac{\gamma(n+1+\gamma)}{\gamma(n+1)} \alpha_j \]

\[ (1-\alpha) \]

2.1
The result is sharp for the functions \( f_j(z) \) given by

\[
f_j(z) = z - \left( \frac{1 - \alpha_j}{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})} \right) z^n
\]

2.2

Proof. Following the work of Owa [1.2 (1992)], Owa and Srivastava (2003), we use the principle of mathematical induction in our proof of Theorem 2.1.

Let \( f_1(z) \in TS^*(\alpha_1, \beta) \) and \( f_2(z) \in TS^*(\alpha_2, \beta) \). Then the inequality

\[
\sum_{n=2}^{\infty} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \leq 1 - \alpha_j \quad (j = 1, 2)
\]

that is, for \( m = 1 \), we see that \( \rho = \alpha_1 \). For \( m = 2 \) Theorem A gives

\[
\sum_{n=2}^{\infty} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \leq 1 \quad (j = 1, 2)
\]

2.3

Thus by applying the Cauchy-Schwarz inequality we have

\[
\left| \sum_{n=2}^{\infty} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \right|^2 \leq \left( \sum_{n=2}^{\infty} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \right) \left( \sum_{n=2}^{\infty} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \right)
\]

Therefore, if

\[
\sum_{n=2}^{\infty} \frac{n - \delta}{1 - \delta} (a_{m,1})(a_{m,2}) \leq \sum_{n=2}^{\infty} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,j} \frac{a_{n,1}a_{n,2}}{(c)_{n-1}^2}
\]

that is, if

\[
\sqrt{(a_{m,1})(a_{m,2})} \leq \frac{1 - \delta}{n - \delta} \sqrt{\frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} \frac{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}{1 - \alpha_j} a_{n,1}a_{n,2}}
\]

Then, \( (f_1 \ast f_2)(z) \in TS^*(\delta, \beta) \)

We also note that the inequality (2.3) yields

\[
\sqrt{a_{n,j}} \leq \sqrt{\frac{1 - \alpha_j}{n(1 + \beta) - (\alpha_j + \beta)(a_{n-1})}} \quad (n = 2, 3, ..., j = 1, 2)
\]

Consequently, if
\[
\sqrt{\frac{(1-\alpha)(1-\alpha_i)}{n(1+\beta)-(\alpha_i+\beta)(1-\alpha_i)(1-\alpha_i)}} \leq \frac{1-\delta}{n-\delta} \sqrt{\frac{n(1+\beta)-(\alpha_i+\beta)(1-\alpha_i)(1-\alpha_i)}{(1-\alpha)(1-\alpha_i)}} \tag{c}_{n-1}
\]
That is, if
\[
\frac{n-\delta}{1-\delta} \leq \frac{n(1+\beta)-(\alpha_i+\beta)(1-\alpha_i)(1-\alpha_i)}{(1-\alpha)(1-\alpha_i)} \tag{c}_{n-1}
\]
then, we have \((f_i \ast f_j)(z) \in TS^*(\delta, \beta)\). It follows from (2.4) that
\[
\delta \leq 1 - \frac{n-1}{n(1+\beta)-(\alpha_i+\beta)(1-\alpha_i)(1-\alpha_i)} \tag{c}_{n-1}
\]
which shows that \((f_i \ast f_j)(z) \in TS^*(\delta, \beta)\), where
\[
\delta = 1 - \frac{n-1}{n(1+\beta)-(\alpha_i+\beta)(1-\alpha_i)(1-\alpha_i)} \tag{c}_{n-1}
\]
Therefore, the result is true for \(m = 2\).
Next we suppose that the result is true for any positive integer \(m\). Then we have
\[
(f_i \ast f_j \ast \ldots \ast f_m)(z) \in TS^*(\tau, \beta)
\]
where
\[
\tau = 1 - \frac{n-1}{(n+1)(1-\alpha_i)} \prod_{j=1}^{n-1} (1-\alpha_j)
\]
and \(\rho\) is given by (2.1). After a simple calculation we obtain
\[
\rho = 1 - \frac{n-1}{(n+1)(1-\alpha_i)} \prod_{j=1}^{n+1} (1-\alpha_j)
\]
This shows that the result is true for \(m+1\). Therefore, by mathematical induction the result is true for any positive integer \(m\).
Further, taking the function \(f_i(z)\) defined by (2.2) we have
\[
(f_i \ast \ldots \ast f_m)(z) = z - \left(\frac{1-\alpha_i}{n(1+\beta)-(\alpha_i+\beta)} \frac{(c)_{n-1}}{(a)_{n-1}}\right) z^n = z - A_n z^n
\]
where
\[
A_n = \prod_{j=1}^{n} \frac{1-\alpha_i}{n(1+\beta)-(\alpha_i+\beta)} \frac{(c)_{n-1}}{(a)_{n-1}}
\]
It follows that
\[ \sum_{n=1}^{\infty} \left( \frac{n(1+\beta) - (\rho + \beta)(c)_{n-1}}{1 - \rho} \right) A_n = 1 \]

This evidently complete the proof of Theorem 2.1.
Letting \( \alpha_j = \alpha (j = 1, \ldots, m) \) in Theorem 2.1, we have

**Corollary A:** If \( f_j(z) \in TS' (\alpha, \beta) \) \( (j = 1, \ldots, m) \), then

\[ (f_1 \ast f_2 \ast \ldots \ast f_m)(z) \in TS' (\rho, \beta) \]

where

\[ \rho = 1 - \frac{(n-1)(c)_{n-1}(1-\alpha)^n}{(\alpha)_{n-1}(n(1+\beta) - (\alpha + \beta))^{n-1} - (1-\alpha)^n(c)_{n-1}} \]

The result is sharp for the functions \( f_j(z) \) \( (j = 1, 2, \ldots, m) \) given by

\[ f_j(z) = z - \frac{1-\alpha}{n(1+\beta) - (\alpha + \beta)(\alpha)_{n-1}} z^n \]

(j = 1, 2, \ldots, m)

Setting \( \alpha = -1 \) and \( \beta = 0 \) in Corollary A to obtain

**Corollary B:** If \( f_j(z) \in TS' (1, 0) \) \( (j = 1, 2, \ldots, m) \), then

\[ (f_1 \ast f_2 \ast \ldots \ast f_m)(z) \in TS' (\rho, 0) \]

where

\[ \rho = 1 - \frac{2^n(n-1)(c)_{n-1}}{(n+1)^n(\alpha)_{n-1} - 2^n(c)_{n-1}} \]

The result is sharp for the functions \( f_j(z) \) \( (j = 1, 2, \ldots, m) \) given by

\[ f_j(z) = z - \left( \frac{2}{n+1} \frac{(c)_{n-1}}{(\alpha)_{n-1}} \right) z^n \]

(j = 1, 2, \ldots, m)

Setting \( \beta = 0 \) in Theorem 2.1, we have

**Corollary C:** If \( f_j(z) \in TS' (\alpha, 0) \) \( (j = 1, 2, \ldots, m) \), then

\[ (f_1 \ast f_2 \ast \ldots \ast f_m)(z) \in TS' (\rho, 0) \]

where

\[ \rho = 1 - \frac{(n-1)(c)_{n-1} \prod_{r=1}^{m} (1 - \alpha_r)}{(\alpha)_{n-1} \prod_{r=1}^{m} (n - \alpha_r) - (c)_{n-1} \prod_{r=1}^{m} (1 - \alpha_r)} \]

The result is sharp for the functions
\[ f_j(z) = z - \left( \frac{1 - \alpha_j}{n - \alpha_j} \right) (c)_{n-1} \left( \frac{c}{a} \right)^{z^*} \]  
\[ (j = 1, 2, \ldots, m) \]

By fixing the second Coefficient, and putting \( \beta = 0 \) we have the following

**Corollary D:** If \( f_j(z) \in TS^*(\alpha_j, \beta) \)  
\[ (f_1 \ast f_2 \ast \ldots \ast f_n)(z) \in TS^*(\rho, \beta) \]

where

\[ \rho = 1 - \frac{c \prod_{j=1}^{m} (1 - \alpha_j)}{a \prod_{j=1}^{m} (2 - \alpha_j) - c \prod_{j=1}^{m} (1 - \alpha_j)} \]

The result is sharp for the functions \( f_j(z) \)  
\[ (j = 1, 2, \ldots, m) \]

After fixing second coefficient as in Theorem B, we have the next Theorem

**Theorem 2.2:** If \( f_j(z) \in TS^*_n(\alpha_j, \beta) \)  
\[ (j = 1, 2, \ldots, m) \text{, then} \]

\[ (f_1 \ast \ldots \ast f_n)(z) \in TS^*_n(\rho, \beta) \]

where

\[ \rho = 1 - \frac{(n - 1)(1 - b)(c)_{n-1} \prod_{j=1}^{m} (1 - \alpha_j)}{(a)_{n-1} \prod_{j=1}^{m} \left[ n(1 + \beta) - (\alpha_j + \beta) \right] - (1 - b)(c)_{n-1} \prod_{j=1}^{m} (1 - \alpha_j)} \]  \[ 2.5 \]

The result is sharp for the functions \( f_j(z) \)  
\[ (j = 1, 2, \ldots, m) \text{, given by} \]

\[ f_j(z) = z - \left( \frac{(1 - b)(1 - \alpha_j)(c)_{n-1} (c)_{n-1}}{n(1 + \beta) - (\alpha_j + \beta)(a)_{n-1}} \right) z^* \]  
\[ (j = 1, 2, \ldots, m) \]  \[ 2.6 \]

**Proof:** Following the same method as in Theorem 2.1 with some simple calculation the result follows.

Letting \( \alpha_j = \alpha (j = 1, 2, \ldots, m) \) in Theorem 2.2; we have

**Corollary E:** If \( f_j(z) \in TS^*_n(\alpha, \beta) \)  
\[ (j = 1, 2, \ldots, m) \text{, then} \]

\[ (f_1 \ast \ldots \ast f_n)(z) \in TS^*_n(\rho, \beta) \]

where
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\[ \rho = 1 - \frac{(n-1)(1-h)(c_{n-1})(1-\alpha)^n}{(a_{n-1})[n(1+\beta)-(\alpha+\beta)]^{n}-(1-h)(1-\alpha)^n(c_{n-1})} \]

The result is sharp for the functions \( f_j(z) \) \( (j = 1,2,\ldots,m) \) given by

\[ f_j(z) = z - \frac{(1-h)(1-\alpha)(c_{n-1})}{n(1+\beta)-(\alpha+\beta)(a_{n-1})} z^n \quad (j = 1,2,\ldots,m) \]

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