ON THE STABILITY OF EVOLUTION EQUATIONS

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ABSTRACT

A Quasi-Linear Hyperbolic Evolution problem in a Banach space with m-accretive operator is considered and conditions which guarantee asymptotic stability of its solution in a dense subset of the space are given.

KEY WORDS: Quasi-Linear, Hyperbolic, Evolution Problem

1.0 INTRODUCTION

An evolution equation as given below will be considered

\[
\frac{du}{dt} = Au(t) + f(u,t)
\]  

(1.1)

where \( f(u,t) : X \times R_+ \to X \) a Banach Space. Merenkov (1993), investigated the stability for this evolution equation using Lyapunov's functionals where \( f(u,t) : X \times R_+ \to X \) is continuous with respect to \( u \) for almost all \( t \in R_+ \), and strongly measurable with respect to \( t \) for all \( u \in X, X \) a Banach space with dual space \( X' \), inner product \( \langle \cdot , \cdot \rangle \) and norm \( \| \cdot \| \), with the operator \( A \) the generator of a strongly continuous semigroup \( T(t) \) of bounded linear operators on \( X \). Let \( \nu(s) = u(t+s), s \in [0, \gamma], \lambda > 0 \) be the section of the function \( u \).

Definition 1 (Merenkov 1993, Kartsators and Parrott 1982): The zero solution of equation (1.1) is said to be Lyapunov stable if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( (u,t) \in X \times R_+ \), \( \| u(t) \| < \delta \Rightarrow \| u(t) \| < \varepsilon \).

Definition 2 (Merenkov 1993, Kartsators and Parrott 1982): The zero solution of equation (1.1) is said to be asymptotically stable if it is Lyapunov stable and in addition, there exists \( h > 0 \) such that for all \( u \in X, \| u(0) \| < h \Rightarrow \lim_{t \to \infty} \| u(t) \| = 0 \).

Merenkov (1993) established that if this evolution equation referred to above is constructed from a parabolic or a hyperbolic partial differential equation, then the semigroup \( S(t) \) usually satisfies the condition: for all \( u \in X \) there exists \( M,u \in R_+ \) such that \( \| S(t)u \| \leq Me^{\lambda t} \| u \| \). Also if \( w > 0 \) and \( f(u,t) \) is a bounded function, then it is possible to establish asymptotic stability of the zero solution using

\[
u(t) = S(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)f(u(s),s)ds
\]  

(1.2)

Egwurube and Garba (2003), considered a quasi-linear hyperbolic differential equation which was transformed into the evolution equation:

\[
\frac{du}{dt} + Au(t) = 0, u(0) = u_0
\]  

(1.3)

defined on a Banach space \( L^1[0,1] \) with \( D(A) = C[0,1] \) and proved that the operator \( A \) is m-accretive and that it does admit a solution. This paper considers this quasi-linear hyperbolic evolution problem and gives conditions which guarantee asymptotic stability of its solutions in \( C[0,1] \); a dense subset of the Banach space.

2.0 MAIN RESULTS

Let \( u^* \) be the steady state solution of (1.3) above and assume \( x = u - u^* \) so that

\[
\frac{dx}{dt} + A x(t) = h(x), x(0) = x_0
\]  

(2.2)

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where $A$ is a bounded linear operator and $h(x)$ represents the non-linear term. By Ladeira and Tanaka (1997) and the method of variation of parameter (Boyce and DiPrima, 1977), the solution of (2.2) is

$$x(t) = e^{At}x_0 + \int_{t}^{\tau} e^{A(t-\tau)} h(x(\tau)) \, d\tau \quad (2.3)$$

Comparing (2.3) with (1.2) and observing that $\exp(A \cdot t)$ is a strongly continuous semigroup, we proceed to establish asymptotic stability of the zero solution of (1.3).

**Lemma 1** (Liad, 2003): Let $A$ be a continuous linear operator on a Banach space $E$ then there exists a bounded operator $(\xi I - A)^{-1}$, $I$ being the identity operator if $|\xi| > \|A\|$ and $(\xi I - A)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\xi^n} A^n$ with a unique resolvent $H = \left( \frac{1}{\xi} + \delta \left( \frac{1}{\xi^2} \right) \right) \delta$ as $|\xi| \to \infty$.

Applying Laplace transform to (2.3) with respect to $t$ gives

$$x(\xi) = (\xi I - A)^{-1} \{ x_0 + h(x(\xi)) \}.$$

where $x(\xi)$ is the Laplace transform of $x(t)$ and $h(x(\xi))$ is the Laplace transform of $h(x(t))$.

**Theorem 2:** Suppose

(i) $|h(x(\xi))| \leq \delta |x(\xi)|, \delta > 0$

(ii) that the unique resolvent of $(\xi I - A)^{-1}$ exists when $\text{Re} \xi < 0$

(iii) that there exists $\delta > 0$ such that $\|x(0)\| < \delta$

then the solution of the evolution equation (1.3) is asymptotically stable on $C[0,1]$.

**Proof:**

$$|x(\xi)| \leq \|x_0\| + \|h(x(\xi))\|$$

Suppose $|h(x(\xi))| \leq \delta |x(\xi)|, \delta > 0$ then $|x(\xi)| \leq \|x(\xi)\|$.

Hence $|x(\xi)| \leq \|x_0\| + \|h(x(\xi))\|$.

Thus $|x(\xi)| \leq \|x_0\| + \|h(x(\xi))\|$. The unique resolvent of $(\xi I - A)^{-1}$ exists when $\text{Re} \xi < 0$ and can be represented by $H$.

Taking the inverse Laplace transform of the above and denoting the inverse Laplace transform of $H$ by $\tilde{H}$ where

$$\tilde{H} = X(t + \gamma) \text{ where } X(t + \gamma) = \begin{cases} 0, t < \gamma \\ 1, t \geq \gamma \end{cases}$$

Suppose also that $T(\xi) = (\xi I - A)^{-1}$ then $T(\xi) = \xi^{-1} I + o(\xi^{-2})$ as $|\xi| \to \infty, \alpha \leq \text{Re} \xi < 0$.

Therefore $T(t) = O(e^{\alpha t})$ as $t \to \infty$ and

$$|x(t)| \leq e^{\alpha t} |x_0| \left( 1 + \int_{t}^{\infty} e^{-\alpha \tau} X(\tau + \gamma) \, d\tau \right)$$

$$= e^{\alpha t} |x_0| \left( 1 + \frac{\int_{0}^{\infty} e^{-\alpha \tau} \, d\tau}{\alpha} \right) = e^{\alpha t} |x_0| \left( 1 + \frac{1 - e^{-\alpha t}}{\alpha} \right), t \geq 0$$

so that in $C[0,1]$ $\lim_{t \to \infty} \|x(t)\| = 0$. Hence the solution is asymptotically stable in $C[0,1]$.

**3.0 CONCLUSION**

Thus the solutions of the quasi-linear evolution problem in a dense subset of $L^1[0,1]$, are bounded in the neighbourhood of the origin and converge to the origin as $t$ approaches infinity.
REFERENCES


