NUMERICAL ALGORITHM FOR DIGITAL IMAGE ENHANCEMENT AND NOISE MINIMIZATION.

L. N. EZEAKO and K. R. ADEBOYE
(Received 1 August 2007. Revision Accepted 17 March 2008)

ABSTRACT

We adopt the approach of Vogel and Oman, 1998 and introduce a Lagrange multiplier Ibiejugba, 1985, to obtain an appropriate discrete energy which we minimize, in order to minimize equivalently, the unwanted vibration (noise) associated with a digitally transmitted image. An iterative algorithm is developed for this minimization and the convergence of the algorithm is proved analytically.

KEY WORDS: Digital Image Enhancement. Noise Minimisation

1. INTRODUCTION

An image is a bounded gray level function, \( g \Omega \rightarrow [0,1] \), where \( \Omega \) is a “screen” which is usually an open domain in \( \mathbb{R}^2 \) e.g a rectangle \( (0,1) \times (0,1) \). \( g(x) = A(x) + n(x) \), in practice, where \( A \) is a linear operator say, from \( L^2(\Omega) \) to \( L^2(\Omega) \). \( u(x) \) is a good image and \( n(x) \) is a vibration (noise) Rudin, Osher and Fatemi, 1992. We would need to find the best function \( u \) among all possible \( u \), satisfying:

\[
\begin{align*}
\int_{\Omega} Au(x) - g(x) \, dx &= 0 \\
\int_{\Omega} |Au(x) - g(x)|^2 &= \sigma^2
\end{align*}
\]

(1)

Where \( 0 \) is the mean and \( \sigma^2 \) is the variance. We adopt the approach of Rudin, Osher and Fatemi, 1992, who proposed the “total variation” of the function of \( u \) as a measure of the optimality of the image. This criterion is approximately the integral \( \int_{\Omega} \nabla u(x) \, dx \)

The main advantage is that this integral can be defined for functions which have discontinuities along hypersurfaces (in two-dimensional images, along one-dimensional curves). This is essential to get a correct representation of the edges in an image to facilitate pattern recognition etc.

The main task is to minimize the integral \( \int_{\Omega} \nabla u(x) \, dx: u \text{satisfies (1)} \) \quad \text{(P1)}

2. A DISCRETE ENERGY APPROACH TO THE MAIN TASK

We consider problem (P1) in dimension 2 and endeavour to compute a solution. We adopt the approach of Vogel and Oman, 1996, 1998. We assume the existence of a Lagrange multiplier \( \lambda > 0 \) (see Ibiejugba, 1985) such that (P1) is equivalent to the problem:

\[ \min_{u \in B^1(\Omega)} \left\{ Du(\Omega) + \lambda \int_{\Omega} |Au(x) - g(x)|^2 \, dx \right\} \]

(2.1)

\[ (P_\lambda) \]

Assumptions

(i) The operator \( A \) satisfies \( A1 = 1 \) (i.e. the image of a constant function is the same function)

(ii) The initial data satisfies \( \int_{\Omega} g(x) - f_{1\Omega} \, dx \geq \sigma^2 \)

L. N. Ezeako, Department of Mathematics/ Computer Sciences, Federal University of Technology Minna, Nigeria
K. R. Adeboye, Department of Mathematics / Computer Science, Federal University of Technology Minna, Nigeria
(iii) There exists a \( u \) satisfying equation (1) such that \( \| Du(z) \| < \varepsilon \).

2.2 Discretization

By making all the assumptions in section 2.1 the minimizer of (P) automatically satisfies \( \int_0^1 u(x) \psi(x) \ dx \) see Chambolle and Lions, 1997, for details. We discretize (P) assuming that \( u \) and \( g \) are discretized on the same square lattice, \( i, j = 1, \ldots, L \). The functions \( u \) and \( g \) are thus approximated by the discrete matrices:

\[
U = (U_i)_{1 \leq i \leq L} \text{ and } G = (G_j)_{1 \leq j \leq L}
\]

The term \( \int_0^1 |Au(x) - g(x)|^2 \ dx \) is replaced by a term \( \lambda \sum_{i,j} |(AU)_{ij} - g_{ij}| \) in this discrete setting. Hence \( A \) denotes a linear operator of \( R^L \to R^L \) and \( (AU)_{ij} \).

The discrete energy we thus need to minimize is

\[
E(U) = \sum_{i,j} |(U_{i+1,j} - U_{i,j})|^2 + |(U_{i,j+1} - U_{i,j})|^2 + \lambda \sum_{i,j} |(AU)_{ij} - g_{ij}|
\]

(2.3)

2.3 Remark

Our first reaction is to minimize (P) by the gradient method e.g. CGM and ECGM (see Ibejigba, 1985, Ibejigba and Abiola 1985 a & b). But the strong nonlinearity of (P) and moreso the derivative \( D(U) \), pose serious problems. The simplest of these problems is the nonexistence of the derivative of the absolute value \( |x| \) at \( x = 0 \).

(Even though we can overcome this problem by replacing \( |x| \) with \( \sqrt{\beta + x^2} \), where \( \beta \) is a small parameter, the overall minimization process is cumbersome).

2.4 The Minimization Method

We adopt a method that is common in the image processing literature. (see Chambolle, 1997, Rudin et al., 1992 for example). Observe that for every \( x \in \mathbb{R}, x \neq 0, |x| \min(\frac{x^2}{2} + \frac{|x|^2}{2}, 2|\nabla V|) \), the minimum being reached for

\[
V = \frac{x}{|x|}
\]

we thus introduce the function \( f(x, y) = \frac{xy^2}{2} + \frac{1}{2y^2} \) and a new field

\[
V = (V_i)_{1 \leq i \leq L} \text{ with } (V_{i+1,j}, V_{i,j}) \in \mathbb{R} \times \mathbb{R}^2 \text{ (of positive real numbers)}
\]

and a new energy

\[
F(U, V) = \sum_{i} (f'(V_{i+1,j} - U_{i,j}) + f'(V_{i,j} - U_{i,j}) + \frac{1}{2} \sum_{i,j} |(AU)_{ij} - g_{ij}|^2)
\]

\[
= \sum_{i} \left( \frac{1}{2} |V_{i+1,j} - U_{i,j}|^2 + \frac{1}{2} |V_{i,j} - U_{i,j}|^2 \right) + \frac{1}{2} \sum_{i,j} |(AU)_{ij} - g_{ij}|^2
\]

and we notice that, \( \min \ V F(U, V) = E(U) \), the minimum being reached for

\[
V^{\infty} = \frac{1}{|U_{i+1,j} - U_{i,j}|} \text{ (or at } + \infty \text{ if } U_{i+1,j} = U_{i,j}) \text{ and}
\]

\[
V^{\infty} = \frac{1}{|U_{i,j}|} \text{ (or at } - \infty \text{ if } U_{i,j} = 0)
\]

We choose some starting values \( U^0, V^0 \) and compute for every \( n > 1 \)

\[
U^{n+1} = \arg \min_{U} F(U, V^n)
\]

And

\[
V^{n+1} = \arg \min_{V} F(U^{n+1}, V)
\]

The idea is that as \( n \) becomes large, \( U^n \) will converge to the minimizer of the problem (P). This is actually true if we slightly modify this algorithm (and the function \( E(U) \) which we minimize).

So we choose \( k > 0 \) and introduce the convex closed set
\[ K_\varepsilon = \{ V : \varepsilon \leq V, \ldots, \varepsilon \leq V \}, \quad \text{and} \quad \forall i, j \in \mathbb{R}^M \}

(M = (L-1) \times L + L \times (L-1))

We define a new energy \( E_\varepsilon (U) = \min_{V \in K_\varepsilon} F(U, V) \)

It is easy to compute \( E_\varepsilon \), explicitly because:

\[
E_\varepsilon = \sum_{i,j} (j_\varepsilon(U_{i,j} - U_{i,j}) + j_\varepsilon(U_{i,j} - U_{i,j}) + \lambda \sum_{i,j} (AU_{i,j} - \mu_{i,j})^2
\]

where \( j_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} x^2 + \frac{\varepsilon}{2} & \text{if } |x| \leq \varepsilon \\
\frac{\varepsilon}{2} x^2 + \frac{1}{2\varepsilon} & \text{if } |x| > \varepsilon \end{cases} \)

Define

\[
\phi_\varepsilon(x) = \left( \varepsilon - \frac{1}{|x|} \right) \times \frac{1}{\varepsilon} = \begin{cases} \frac{1}{|x|} & \text{if } \varepsilon \leq |x| \leq \frac{1}{\varepsilon} \\
\frac{|x|}{\varepsilon} & \text{if } |x| \geq \frac{1}{\varepsilon} \end{cases}
\]

Then \( \phi_\varepsilon(x) \) is the unique value in \( \left[ \varepsilon, \frac{1}{\varepsilon} \right] \) such that \( j_\varepsilon(x, \phi_\varepsilon(x) \)

We deduce that the unique \( V \in K_\varepsilon \) for which

\[ E_\varepsilon(U) = \min_{V \in K_\varepsilon} F(U, V) \]

\[ (U^n + V^n) \text{ is given by } V^n_{i,j} = \phi_\varepsilon(U^n_{i,j} - \frac{1}{\varepsilon}) \text{ and } \]

\[ V^n_{i,j} = \phi_\varepsilon(U^n_{i,j} - \frac{1}{\varepsilon}) \text{ for every } i,j \]

In this case, we set \( \phi_\varepsilon(U) = V \). This defines a continuous function \( \phi_\varepsilon : \mathbb{R}^N \to K_\varepsilon \subset \mathbb{R}^M \).

The Algorithm, now consists in computing for every \( n \geq 1 \), the starting values \( U^0, V^0 \) being chosen:

\[ U^n = \arg \min_{V \in K_\varepsilon} F(U, V^{n-1}) \]

and

\[ V^n = \arg \min_{V \in K_\varepsilon} F(U^n, V) = \phi_\varepsilon(U^n) \]

3. Analytical Proof of the Convergence of the Numerical Algorithm for Noise Minimization

i.e., \( U^n = \arg \min_{i} F(U, V^{n-1}) \) And

\[ V^n = \arg \min_{i} F(U^n, V) = \phi_\varepsilon(U^n) \]

Proof

Let \( l_n \) be the vector in \( \mathbb{R}^N \) defined by \( l_n[i,j] = 1 \) for every \( i,s, j,s \) (where \( N = L \times L \) is the dimension of the space e.g. the metric space, where \( U \) resides).

We assume that the image of a constant function is the same function. That is, given the linear operator \( A \).

\( A \) is \( l_n \).

Conjecture

There exist \( U, V = \Phi_\varepsilon(U) \)

such that as \( n \to \infty \), \( U^n \to U \) and \( V^n \to V \) and \( U \) is (the) minimizer of \( E_\varepsilon \).

Proof of Conjecture

Lemma 1: We claim that there exists \( 0 < \alpha < \beta \) such that the second derivatives \( D^2_{\alpha}, F \) and \( D^2_{\beta}, F \) satisfy
for every 

\[ U \in \mathcal{K}_\varepsilon \text{ that is } U \in \mathbb{R}^n, \forall \varepsilon \in \mathbb{R}^n \text{ and } \eta \in \mathbb{R}^n, \text{ we have :} \]

\[ \alpha \eta^2 \leq \langle D_{i,h}^2 F(U,V) \eta \rangle \leq \beta \eta^2 \]

and

\[ \alpha \eta^2 \leq \langle D_{i,h}^2 F(U,V) \eta \rangle \leq \beta \eta^2 \]

Proof (see Vogel and Oman, 1993.) We also recall the following “Poincare inequality” (in finite dimension): There exist a constant \( C > 0 \) such that for every \( \varepsilon \in \mathbb{R}^n = \mathbb{R}^{1,0} \) such that \( \sum_{i=1}^{n} \varepsilon_{i} = 0 \)

\[ (3.1) \quad \sum_{i,j=1}^{\infty} \left| \varepsilon_{i,j} \right|^2 \leq C \left( \sum_{i,j \in \mathbb{Z}} \left| \varepsilon_{i+1,j} - \varepsilon_{i,j} \right|^2 + \sum_{i,j \in \mathbb{Z}} \left| \varepsilon_{i,j+1} - \varepsilon_{i,j} \right|^2 \right) \]

We note that for every \( U, V \in \mathcal{K}_\varepsilon \) and \( \varepsilon \in \mathbb{R}^n \),

\[ \langle D_{i,h}^2 F(U,V) \varepsilon \rangle \geq \varepsilon \sum_{i,j} \left[ \left| \varepsilon_{i+1,j} - \varepsilon_{i,j} \right|^2 + \left| \varepsilon_{i,j+1} - \varepsilon_{i,j} \right|^2 \right] + \left| A \varepsilon \right|^2 \]

In particular, letting \( m(\varepsilon) = \frac{1}{N} \sum_{i,j} \varepsilon_{i,j} \) be the average of \( \varepsilon \) we have (since \( A \xi = \xi \))

\[ \langle D_{i,h}^2 F(U,V) \varepsilon \rangle \geq \left| A \varepsilon \right|^2 = \left| A(\varepsilon - m(\varepsilon) I_N) + m(\varepsilon) I_N \right| \geq \left| m(\varepsilon) I_N \right| - \left| A \right| \left| \varepsilon - m(\varepsilon) I_N \right| \]

But by equation (3.1)

\[ \left| \varepsilon - m(\varepsilon) I_N \right| \leq c \sum_{i,j} \left[ \left| \varepsilon_{i+1,j} - \varepsilon_{i,j} \right|^2 + \left| \varepsilon_{i,j+1} - \varepsilon_{i,j} \right|^2 \right] \leq \left( \frac{1}{\varepsilon} \right) \langle D_{i,h}^2 F(U,V) \varepsilon \rangle \]

Therefore

\[ \left| m(\varepsilon) I_N \right| \leq c \sqrt{ \langle D_{i,h}^2 F(U,V) \varepsilon \rangle } \] (here \( c \) denotes any positive constant that does not depend on \( U, V, \xi \)). Moreover, by using equation (3.1) again,

\[ c \langle D_{i,h}^2 F(U,V) \varepsilon \rangle \geq \left| \varepsilon - m(\varepsilon) I_N \right| \]

Since \( \eta_N \) and \( \varepsilon - m(\varepsilon) I_N \) are orthogonal we deduce that \( \left| \varepsilon \right|^2 \leq c \langle D_{i,h}^2 F(U,V) \varepsilon \rangle \).

Lemma 2:  For every \( n \geq 1 \)

\[ E_n(U^{n-1}) - E_n(U^n) \geq \frac{\alpha}{2} \left( \left| U^{n-1} - U^n \right|^2 + \left| V^{n-1} - V^n \right|^2 \right) \]

Proof:  For every \( n \geq 1 \)

\[ D_{v} F(U^n, V^{n-1}) = 0 \text{ while} \]

\[ \langle D_{v} F(U^n, V^n), V - V^n \rangle \geq 0 \text{ for every } V \in \mathcal{K}_\varepsilon \]

By Lemma 1, we deduce that,

\[ F(U^n, V^{n-1}) = F(U^n, V^n) + \langle D_{v} F(U^n, V^n), V^{n-1} - V^n \rangle \]

\[ + \int_0^1 \left[ \left( 1 - t \right) D_{v} F(U^n, V^n + t(V^{n-1} - V^n) \right) \left( V^{n-1} - V^n \right) dt \]

\[ \geq F(U^n, V^n) + \frac{\alpha}{2} \left| V^{n-1} - V^n \right|^2 \]

In a similar way, we prove that

\[ F(U^{n-1}, V^{n-1}) \geq F(U^n, V^n) + \frac{\alpha}{2} \left| U^{n-1} - U^n \right|^2 \]

Since \( E_n(U^n) = F(U^n, V^n) \), this lemma is proved.
Remark:

By construction, the sequence 
\[ E_k(U^n) = F(U^n, V^n) \]
must decrease and it is bounded from below. It goes to some constant \( c \) and 
\[ E_k(U^{n+1}) - E_k(U^n) \to 0 \]
Thus \( U^{n+1} \to U^* \) and \( V^{n+1} \to V^* \) go to zero as \( n \to \infty \).

Also from Lemma 1, we notice that 
\( E_k \) is coercive, which implies that for every \( c > 0 \), the set \( \{ E_k \leq c \} \) is bounded in \( \mathbb{R}^N \). It is also closed and hence compact. Thus we may extract a subsequence \( U^{n_k} \) and find a \( U \in \mathbb{R}^N \) such that as \( k \to \infty \), \( U^{n_k} \to U \).

By continuity \( V^{n_k} = \Phi_k(U^{n_k}) \to \Phi_k(U) \), and we let \( V = \Phi_k(U) \).

We also have \( D_jF(U^{n_k}, V^{n_k}) = 0 \) and since \( V^{n_k} - V^* \to 0 \) (by lemma 2)
\( V^{n_k} \to V \), so that by continuity, \( D_jF(U, V) = 0 \)

Proof of Conjecture: Let \( h \in \mathbb{R}^N \) and \( t > 0 \)

Letting \( V_t = \Phi_k(U + th) \to V \) as \( t \to 0 \)

We have \( E_k(U_t + th) - E_k(U_t) = F(U_t + th, \Phi_k(U_t + th)) - F(U_t, V) \)

\[ = (F(U_t + th, V_t) - F(U_t, V_t)) + (F(U_t, V_t) - F(U_t, V)) \]

Since \( V_t \in K_k \),
\[ F(U_t, V_t) = F(U_t, V), \text{ so that } E_k(U_t + th) - E_k(U_t) \geq F(U_t + th, V_t) - F(U_t, V) \]

Hence, \( F(U_t + th, V_t) = F(U_t, V) = \langle D_j F(U_t, V_t), h \rangle = \frac{1}{2} \int_0^1 (1 - s)(D_j F(U_t, V_t), h, h) \rangle dt \)

\[ = \lim_{t \to 0} \frac{E_k(U_t + th) - E_k(U_t)}{t} \geq \langle D_j F(U_t, V_t), h \rangle = 0 \]

Since \( h \) is arbitrary, \( D_j E_k(U_t) = 0 \)

4. CONCLUSION

Since \( E_k \) is strictly convex \( \Rightarrow \) for every \( U, U' \) and
\[ 0 < \theta < 1, E_k((1 - \theta)U + \theta U') \leq \theta E_k(U) + (1 - \theta)E_k(U') \]

Unless \( U = U' \) it has a unique minimizer characterized by the equation \( D_j E = 0 \). We deduce that \( \bar{U} \) is the UNIQUE MINIMIZER OF \( E \). This achieves the proof of our CONJECTURE.

By the uniqueness of this minimizer, any subsequence of \( \langle U \rangle \) must converge to the same value \( \bar{U} \), so that
the whole sequence \( U^n \) converges to \( \bar{U} \).

Similarly, \( V^n \) converges to \( \overline{V} \).

REFERENCES

Chambolle, A. and Lions, P., 1997, Image Recovery Via Total Variation Minimization And Related Problems, Numer Math, 76(2)167-188,


