PERIODIC SOLUTIONS IN A NONLINEAR FOURTH ORDER DIFFERENTIAL EQUATION II

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ABSTRACT

The hypothesis $\chi(m) \neq 0$ has the following implications

$$\chi(m) = m^4 - a_2 m^2 + a_4 \neq 0$$

or

$$a_2 - a_4 m^2 \neq 0$$

However, the attention of scholars have been on $\chi(m) = m^4 - a_2 m^2 + a_4 \neq 0$, which implies

$$a_2 - \frac{1}{4} a_4^2 > 0 \text{ or } a_4 - \frac{1}{4} a_2^2 < 0.$$ 

But the second condition $a_2 - a_4 m^2 \neq 0$ has been given little or no attention by scholars. In this paper, an existence result has been obtained using the alternative condition

$$\chi(m) = a_2 - a_4 m^2 \neq 0 \text{ or } m^2 \neq a_4^{-1} a_2,$$

along side with other hypotheses.

KEYWORDS: Nonlinear ODE, boundary value problem, a priori bound, fixed point technique.

1. INTRODUCTION

Consider the nonlinear differential equation

$$x^{(4)} + f(x) + g(x) + h(x) + a_4 x = P(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

with boundary conditions

$$D^{(r)} x(0) = D^{(r)} x(2\pi), \quad r = 0, 1, 2, 3, \quad D = \frac{d}{dt}$$

where $a_4$ is a constant, $f = f(x)$, $g = g(x)$, $h = h(x)$, $P = P(t, x, \dot{x}, \ddot{x}, \dddot{x})$ are continuous functions with $P$ $2\pi$ periodic in $t$.

In a special case, consider the constant coefficients equation

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0$$

with the corresponding nonhomogeneous equation

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = P(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

both (3) and (4) subject to the boundary condition (2). The auxiliary equation

$$r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4 = 0$$

of (3) has a root of the form $r = im$ (m an integer) if the equation

$$m^4 - a_2 m^2 + a_4 = 0 \text{ and } m(a_2 - a_4 m^2) = 0$$

are satisfied simultaneously Ezeli (1979). The boundary value problem (3) – (2) has no nontrivial solutions if either

$$\chi(m) = m^4 - a_2 m^2 + a_4 \neq 0$$

or

$$a_2 - a_4 m^2 \neq 0 \text{ or } m^2 \neq a_4^{-1} a_2.$$ 

The equation (6) in its extended forms to nonlinear term has been applicable in the hypotheses for existence of periodic solutions of a fourth order ordinary differential equations. For instance, see Ezeli
In this paper, our interest is on (7), which is new in the literature. Thus, we have the following

**Theorem 1**

Suppose in addition to the basic assumptions on f, g, h, and P

(i) There exist $a_1$, $a_2$ constants such that

$$\frac{f(u)}{u} \geq a_1, \quad u \neq 0$$

(ii) The function $h(x)$ is such that

$$|h'(\tilde{x})| \leq a_2$$

(iii) The function $P$ is bounded and $2\pi$ periodic in $t$.

Then equations (1) – (2) have at least one $2\pi$ periodic solution for arbitrary $g(z)$ and $a_i$.

Remark: This is an extension of Tejumola result for the equation $x^{(4)} + g_1\dddot{x} + g_2\dddot{x} + g_3\dddot{x} + b_4x = P(t, x, \dot{x}, \dddot{x}, \dddot{x})$ for $P \neq 0$ [see Tejumola 2006].

C. **GENERAL COMMENTS ON SOME NOTATIONS**

Throughout the proof which follows, the capitals $C_1$, $C_2$, $C_3$, ..., represent positive constants whose magnitude depend at most on $a_1$, $f$, $g$, $h$, and $P$. The constants $C_1$, $C_2$, $C_3$, ..., retain their identities throughout the proof of theorem 1. The symbols $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$ in respect of the mappings $[0, 2\pi] \rightarrow \mathbb{R}$ shall have their usual meaning

$$\|\theta\|_\infty = \max_{0 \leq t \leq 2\pi} |\theta(t)|, \quad \|\theta\|_1 = \int_0^{2\pi} |\theta(t)|dt, \quad \|\theta\|_2 = \left( \int_0^{2\pi} \theta^2(t)dt \right)^{1/2}$$

D. **PROOF OF THEOREM 1**

The proof of theorem 1 is by the Leray-Schauder fixed point technique (Leray and Schauder, 1934) and we shall consider the parameter $\lambda$ dependent equation, $(0 \leq \lambda \leq 1)$

$$x^{(4)} + f_\lambda(\tilde{x}) + \lambda g(\tilde{x}) + h_\lambda(\dot{x}) + a_i x = \lambda P$$

where

$$f_\lambda(\tilde{x}) = (1 - \lambda)a_1\dddot{x} + \lambda f(\tilde{x})$$

$$h_\lambda(\dot{x}) = (1 - \lambda)a_2\dddot{x} + h(\dot{x})$$

By setting

$$\dddot{x} = y, \quad \dddot{y} = z, \quad \dddot{z} = u, \quad \dddot{u} = -f_\lambda(u) - \lambda g(z) - h_\lambda(y) - a_i x + \lambda P$$

the equation (10) can be written compactly in matrix form

$$\dot{X} = AX + \lambda F(X, t)$$

where

$$X = \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_1 & 0 & 0 & -a_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q \end{pmatrix}$$

with $Q = P - f(u) + a_i u - g(z) - h(y) + a_i x$

Note that equation (80) reduces to a linear equation

$$x^{(4)} + a_1\dddot{x} + a_2\dddot{x} + a_3\dddot{x} + a_4x = 0$$

when $\lambda = 0$ and to equation (1) when $\lambda = 1$. The eigenvalues of the matrix $A$ defined by (13) are the roots of the auxiliary equation Ezeilo (2000)

$$r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0$$

(15)
If equation (15) has no root of the form \( r = im \), then equation (14) together with the boundary condition (2) has no non trivial solutions, since (7) is satisfied Ezeilo (2000). Therefore the matrix \( (t^{2\pi + 1} - I) \), \( I \) being the identity matrix) is invertible. Thus, \( X = X(t) \) is a \( 2\pi \) periodic solution of (12) if and only if (Hale, 1963)

\[
X = \lambda T X,
\]

\( 0 \leq \lambda \leq 1 \)

(16)

where the transformation \( T \) is defined by

\[
(AX)(t) = \int_0^{2\pi} (t^{2\pi} - I)^{-1} t^{2\pi n} F(X(t), S) \, dt
\]

(17)

Let \( S \) be the space of all continuous 4-vector functions \( \tilde{X}(t) = (x(t), y(t), z(t), u(t)) \) which are of period \( 2\pi \) and with norm

\[
\| \tilde{X} \| = \sup_{0 \leq r \leq 2\pi} \left\{ |x(t)| + |y(t)| + |z(t)| + |u(t)| \right\}
\]

(18)

If the operator \( T \) defined by (17) is a compact mapping of \( S \) into itself then it suffices for the proof of theorem 1 to establish a priori bounds \( C_7, C_8, C_4, C_{12} \) independent of \( \lambda \) such that

\[
|x| \leq C_7, \quad |\tilde{x}| \leq C_4, \quad |\tilde{x}| \leq C_4, \quad |\tilde{x}| \leq C_{12}
\]

(19)

see Scheaffer (1955)

4. VERIFICATION OF (19)

Let \( x(t) \) be a possible \( 2\pi \) periodic solution of equation (10). The main tool to be used here in this verification is the function \( V(x, y, z, u) \) defined by

\[
V = \frac{1}{2} \tilde{x}^2 + \int_{0}^{1} g(s) \, ds + a_1 \tilde{x} \tilde{y} - \frac{1}{2} a_2 \tilde{x}^2 + \tilde{x} \tilde{h}(\tilde{x})
\]

(20)

The time derivative \( \dot{V} \) along the solution path of (11) is

\[
\dot{V} = -\tilde{x}\tilde{f}(\tilde{x}) + \tilde{h}'(\tilde{x}) \tilde{y} + \tilde{x} \tilde{h}(\tilde{x})
\]

(21)

Integrating (21) with respect to \( t \) from \( t = 0 \) to \( t = 2\pi \)

\[
\int_0^{2\pi} V \, dt = -\int_0^{2\pi} \tilde{x}\tilde{f}(\tilde{x}) \, dt + \int_0^{2\pi} \tilde{h}'(\tilde{x}) \tilde{y} \, dt + \int_0^{2\pi} \tilde{x} \tilde{h}(\tilde{x}) \, dt
\]

using equation (2), we obtain

\[
\int_0^{2\pi} \tilde{x}\tilde{f}(\tilde{x}) \, dt = \int_0^{2\pi} \tilde{x} \tilde{h}(\tilde{x}) \, dt = \int_0^{2\pi} \tilde{x} \tilde{h}(\tilde{x}) \, dt
\]

(22)

By (8) and (9) (22) implies

\[
\int_0^{2\pi} a \tilde{x} \, dt + \int_0^{2\pi} a \tilde{x} \, dt \leq C \int_0^{2\pi} |\tilde{x}| \, dt
\]

(23)

We have used the boundedness of \( P \) and the fact that \( 0 \leq \lambda \leq 1 \) to achieve (23). In particular

\[
\int_0^{2\pi} a \tilde{x} \, dt \leq C \int_0^{2\pi} |\tilde{x}| \, dt
\]

or

\[
\int_0^{2\pi} \tilde{x} \, dt \leq C \int_0^{2\pi} |\tilde{x}| \, dt
\]

where

\[
C_7 = a, \quad C_8, C_4, C_{12} < 0
\]

Thus,

\[
\int_0^{2\pi} \tilde{x} \, dt \leq C_7 \int_0^{2\pi} |\tilde{x}| \, dt
\]

by Schwartz's inequality. Therefore

\[
\int_0^{2\pi} \tilde{x} \, dt \leq C_7 (2\pi)^{1/2} \left( \int_0^{2\pi} |\tilde{x}| \, dt \right)
\]

(24)

Since \( x(0) = x(2\pi) \), there exists \( \tilde{x}(\tau, \theta) = 0 \) at some \( \tau, \theta \in [0, 2\pi] \) such that
\[ \dot{x}(t) = \ddot{x}(t) + \int_{t}^{\infty} \ddot{x}(s)ds \]

Then

\[ \max_{0 \leq t \leq 2\pi} |\ddot{x}(t)| \leq \int_{0}^{2\pi} |\ddot{x}(t)|dt \]

\[ \leq (2\pi)^{\frac{1}{2}} \left( \int_{0}^{2\pi} (\ddot{x}(t))dt \right)^{\frac{1}{2}} \]

by Schwartz's inequality. By (24)

\[ \max_{0 \leq t \leq 2\pi} |\ddot{x}(t)| \leq (2\pi)^{\frac{1}{2}} C_3 \equiv C_4 \]

Therefore

\[ |\dddot{x}| \leq C_4 \] \hfill (25)

Also since \( x(0) = x(2\pi) \) by (2), there exist \( \ddot{x}(\tau) = 0 \) at some \( \tau \in [0, 2\pi] \) such that

\[ \ddot{x}(t) = \ddot{x}(\tau) + \int_{\tau}^{t} \dddot{x}(s)ds \]

so that

\[ \max_{0 \leq t \leq 2\pi} |\ddot{x}(t)| \leq \int_{\tau}^{2\pi} |\dddot{x}(t)|dt \]

\[ \leq (2\pi)^{\frac{1}{2}} \left( \int_{\tau}^{2\pi} (\dddot{x}(t))dt \right)^{\frac{1}{2}} \]

by Schwartz's inequality. In view of (25), we have

\[ |\dddot{x}| \leq C_4 \] \hfill (26)

Now integrating (10) with respect to \( t \) from \( t = 0 \) to \( t = 2\pi \) and using (2) yields

\[ \int_{0}^{2\pi} a_1 \dot{x}dt = \int_{0}^{2\pi} \lambda P dt - \int_{0}^{2\pi} \lambda f_1(\ddot{x})dt - \int_{0}^{2\pi} \lambda g(\ddot{x})dt - \int_{0}^{2\pi} h_1(\ddot{x})dt \] \hfill (27)

With bounds on \( \dddot{x}, \ddot{x}, \dot{x} \) in (24), (25) and (26) respectively and the boundedness of \( P \) as specified in (iii) of the hypotheses of theorem 1, the right hand side of (27) is bounded.

That is

\[ \int_{0}^{2\pi} \lambda P dt + \int_{0}^{2\pi} f_1(\ddot{x})dt + \int_{0}^{2\pi} g(\ddot{x})dt + \int_{0}^{2\pi} h_1(\ddot{x})dt \leq C \]

Therefore

\[ \int_{0}^{2\pi} a_1 \dot{x}dt \leq C \]

which implies that

\[ \int_{0}^{2\pi} \dot{x}dt \leq C \] \hfill (28)

where \( C = a_1^{-1} C_6, \ a_1 \neq 0 \)

From section 2

\[ x_1 = \int_{0}^{2\pi} \dot{x}dt \]

That is

\[ x_1 \leq C \]

Also,

\[ \max_{0 \leq t \leq 2\pi} |x(t)| \leq \int_{0}^{2\pi} |x(t)|dt \]

implies

\[ \max_{0 \leq t \leq 2\pi} |x(t)| \leq C_7 \] \hfill (29)
Now it remains the fourth inequality in (19) for our theorem 1 to be fully verified. Note that equation (10) can be expressed in the form
\[ x^{(4)} + f_\lambda(\bar{x}) = \eta_0 \] (30)
where
\[ \eta_0 = \lambda P - \lambda g(\bar{x}) - h_\lambda(\bar{x}) - a_\lambda \]
with bounds on \( x, \bar{x}, \dot{\bar{x}} \) in (29), (26) and (25) respectively together with the bounded of \( P \) and the fact that 0 \( \leq \lambda \leq 1 \), then
\[ |\eta_0| \leq C_8 \] (31)
Therefore
\[ x^{(4)} + f_\lambda(\bar{x}) \leq C_{10} \] (32)
Multiplying (32) by \( x^{(4)} \) and integrating with respect to \( t \) from \( t = 0 \) to \( t = 2\pi \) yields
\[ \int_0^{2\pi} x^{(4)}(t) \, dt + \int_0^{2\pi} f_\lambda(\bar{x}) x^{(4)}(\bar{x}) \, dt \leq |\eta_0| \int_0^{2\pi} |x^{(4)}| \, dt \]
Since \( f \) is a continuous function and \( f \) is defined as in section 2, there are constants \( \gamma, C_{11}, \) such that
\[ |x^{(4)}|^2 \leq C_7 |x^{(4)}|^2 + C_{10} |x^{(4)}|^2 \] (33)
Hence
\[ |x^{(4)}|^2 \leq C_{11} \] (34)
from which because of (2) with \( r = 3 \) then
\[ |\bar{x}|^2 \leq C_{12} \] (35)

CONCLUSION

The estimates (25), (26), (29) and (35) verify the inequality (19) and hence the proof theorem 1, which implies existence of Periodic Solutions for equations (1) – (2).

REFERENCES


Ogbug H. M., 2007 Existence of periodic solutions for a certain boundary value problem of a non linear fourth order differential equation, Pacific Journal of Science and Technology USA 8(1) 130-136

