

CAUCHY PROBLEMS FOR SEMI-LINEAR HYPERBOLIC SYSTEMS WITH THEIR APPLICATIONS

E. E. NDIYO AND E. UKEJE

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ABSTRACT

In this paper, the solution to semi-linear hyperbolic systems of the form $A(x,t)u_t + B(x,t)u_x + c(x,t)u = f(x,t,u)$ in $\Omega(x(0,\infty))$ with $u(x,0) = \phi(x)$ is given. We apply this solution approached to establish the solution representation for any isentropic fluid flow.

KEYWORDS: Hyperbolic Systems, Characteristic curves, eigenvalues, eigenvectors, isentropic fluid.

1.0 INTRODUCTION

Cauchy problems arose naturally from physical phenomena and are of great interest. Different scholars like Lee-Datsin and Yu-wen Tzu (1981), Lagan (1976), Kong and Zhang (1996) had put forward various approaches in order to obtain a better result, by relaxing certain restrictions encountered in earlier results. In all, the characteristics concept appears to be the best and accepted method.

Nevertheless, the work is not complete given the varying physical principles where these Cauchy problems can be applied. The motion of compressible fluids offers instructive and significant illustrations of these characteristic concepts. Various forms expressing their flow exist depending on the space and media, whether in two dimensional space or spherically symmetric flow.

We have as an example the equations

$$\rho u_x + u \rho_x + \rho_t = - \frac{2 \rho u}{x} \quad (1.1)$$

$$\rho u u_x + \rho u_t + c^2 \rho_x = 0 \quad (1.2)$$

Moreso, considering Gas dynamics, either in Lagrangian coordinates or Eulerian coordinates, we have cases where the spatial variables will represent fixed points in space and in another instance, fixed particles.(Courant and Hilbert(1962),Michael and Robert(1993)).

Thus, researches are inconclusive regarding any equation which may likely vary or be used to represent different phenomena like in water waves. In our earlier publication (Ndiyo 2005), we gave estimate solution to semi-linear hyperbolic systems in a generalized (weak) sense. In this paper, we consider in its generality the Cauchy problem of second order semi-linear hyperbolic equation of the form

$$A(x,t)u_{tt} + 2B(x,t)u_{tx} + c(x,t)u_{xx} + D(x,t)u_t + E(x,t)u_x = F(x,t,u) \quad (1.3)$$

$$u(x,0) = \phi(x), u_t(x,0) = \psi(x) \quad (1.4)$$

We reduced equation 1.3 – 1.4 into a first order system and obtained its eigenvalues and left eigenvectors, from which we established its solution representation. We then illustrate this solution using an isentropic fluid flow equation.

2.0 System of First Order and Solution Representation

Given the equation 1.3 – 1.4, let $p(x,t)$ be u_t and $q(x,t) = u_x$. We differentiate and substitute in 1.3 to obtain

$$A(x,t)p_t + B(x,t)p_x + B(x,t)q_t + C(x,t)q_x + D(x,t)p + E(x,t)q = F(x,t,u) \quad (2.1)$$

and

$$q_t - p_x = 0 \quad (2.2)$$

by the continuity of mixed derivatives.

Equations (2.1) and (2.2) becomes a system written

E. E. Ndiyo, Dept. Of Maths/Stats/Computer Science, University Of Uyo, Uyo, Nigeria.

E. Ukeje, Dept. Of Maths/Stats/Comp. Science, Michael Okpara University of Agriculture, Umudike, Nigeria.

$$\begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_t + \begin{pmatrix} B & C \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_x + \begin{pmatrix} D & E \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \quad (2.3)$$

$$p(x,0) = \phi(x), q(x,0) = \psi(x) \quad (2.4)$$

The matrix $k = \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}$ is non-singular and for $A \neq 0$ will have its inverse as $k^{-1} = \begin{pmatrix} 1/A & -B/A \\ 0 & 1 \end{pmatrix}$

The system (2.3) and (2.4) is reduced to $V_t + mV_x + NV = f$

where $m = \begin{pmatrix} 2B/A & C/A \\ -1 & 0 \end{pmatrix}, N = \begin{pmatrix} D/A & E/A \\ 0 & 0 \end{pmatrix}$

and $f = \begin{pmatrix} F(x,t,u) \\ 0 \end{pmatrix}$

The eigenvalues of the principal part 'm' will be

$$\lambda_i = \frac{B \pm \sqrt{B^2 - AC}}{A} \quad i = 1, 2.$$

The resulting eigenvectors matrix which are functions of (x,t) is

$$S = \begin{pmatrix} \frac{-\lambda_1}{\sqrt{1+\lambda_1^2}} & \frac{-\lambda_2}{\sqrt{1+\lambda_2^2}} \\ 1 & 1 \\ \frac{1}{\sqrt{1+\lambda_1^2}} & \frac{1}{\sqrt{1+\lambda_2^2}} \end{pmatrix}$$

$$S^{-1} = \frac{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & \lambda_2 \\ \sqrt{1+\lambda_1^2} & \sqrt{1+\lambda_2^2} \\ -1 & -\lambda_1 \\ \frac{1}{\sqrt{1+\lambda_1^2}} & \frac{1}{\sqrt{1+\lambda_2^2}} \end{pmatrix}$$

The matrix m in a diagonalised form

$$m = S \wedge S^{-1}, \quad \wedge = \text{diag}(\lambda_1, \lambda_2)$$

makes the system to become

$$\begin{pmatrix} p \\ q \end{pmatrix}_t + S \wedge S^{-1} \begin{pmatrix} p \\ q \end{pmatrix}_x + \begin{pmatrix} D/A & E/A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} f(x,t,u) \\ 0 \end{pmatrix} \quad (2.5)$$

Let $\begin{pmatrix} r \\ h \end{pmatrix} = S^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$, we have from (2.5).

$$\begin{pmatrix} r \\ h \end{pmatrix}_t - S_t^{-1} S \begin{pmatrix} r \\ h \end{pmatrix} + \wedge \left\{ \begin{pmatrix} r \\ h \end{pmatrix}_x - S_x^{-1} S \begin{pmatrix} r \\ h \end{pmatrix} \right\}$$

$$+ S^{-1} \begin{pmatrix} D/A & E/A \\ 0 & 0 \end{pmatrix} S \begin{pmatrix} r \\ h \end{pmatrix} = S^{-1} \begin{pmatrix} f(x,t,u) \\ 0 \end{pmatrix} \quad (2.6)$$

Define

$$Q = \begin{pmatrix} A_1^* & A_2^* \\ B_1^* & B_2^* \end{pmatrix} = S^{-1} \begin{pmatrix} D/A & E/A \\ 0 & 0 \end{pmatrix} S - S_t^{-1} S - A(S_x^{-1} S) \quad (2.7)$$

we obtained the system of equations

$$\begin{pmatrix} r \\ h \end{pmatrix}_t + \wedge \begin{pmatrix} r \\ h \end{pmatrix}_x + \begin{pmatrix} A_1^* & A_2^* \\ B_1^* & B_2^* \end{pmatrix} \begin{pmatrix} r \\ h \end{pmatrix} = \begin{pmatrix} \frac{C_1^* f(x,t,u)}{A} \\ \frac{C_2^* f(x,t,u)}{A} \end{pmatrix} \quad (2.8)$$

$$r(x,0) = S^{-1}\phi(x) \quad \text{and} \quad h(x,0) = S^{-1}\psi(x) \quad (2.9)$$

Equation 2.8 and 2.9 becomes system of characteristics equations for the Cauchy problem 1.3 – 1.4. Along each curves c_1 and c_2 corresponding to each eigenvalues $\lambda_i, i = 1,2$ we get

$$\frac{dr}{dt} + \lambda_1 \frac{dr}{dx} + A_1^* r + A_2 h = \frac{C_1^*}{A} f(x,t,u)$$

$$\frac{dh}{dt} + \lambda_2 \frac{dh}{dx} + B_1^* r + B_2^* h = \frac{C_2^*}{A} f(x,t,u)$$

The solution representations are expressed by the corresponding integral taken over the characteristic curves with respect to t for $0 \leq t \leq \xi$ thus;

$$\begin{aligned} r(x,t) = & r(x,0)e^{-\int_0^t A_1^*(x,\tau)d\tau} - \int_0^t \left[A_2^*(x,\xi)e^{-\int_0^\xi A_1^*(x,\tau)d\tau} \right] e^{-\int_0^\xi A_1^*(x,\tau)d\tau} d\xi \\ & + \int_0^+ \left[\frac{C_1^*(x,\varepsilon)}{A(x,\varepsilon)} f(x,\xi,u) e^{-\int_0^\xi A_1^*(x,\tau)d\tau} \right] e^{-\int_0^\xi A_1^*(x,\tau)d\tau} d\varepsilon \end{aligned} \quad (2.10)$$

$$\begin{aligned} h(x,t) = & h(x,0)e^{-\int_0^t B_1^*(x,\tau)d\tau} - \int_0^t \left[B_2^*(x,\eta)(x,\eta)e^{-\int_0^\eta B_1^*(x,\tau)d\tau} \right] e^{-\int_0^\eta B_1^*(x,\tau)d\tau} d\eta \\ & + \int_0^+ \left[\frac{C_2^*(x,\eta)}{A(x,\eta)} f(x,\eta,u) e^{-\int_0^\eta B_1^*(x,\tau)d\tau} \right] e^{-\int_0^\eta B_1^*(x,\tau)d\tau} d\eta \end{aligned} \quad (2.11)$$

we eliminate $h(x,\varepsilon)$ and $r(x,\xi)$ from each of the solution representation as applicable to obtain

$$\begin{aligned} r(x,t) = & r(x,0)e^{-\int_0^t A_1^*(x,\tau)d\tau} + \int_0^t \left[\int_0^\varepsilon B_1^*(x,\eta)r(x,\eta)e^{-\int_0^\eta B_1^*(x,\tau)d\tau} d\eta \right] A_2^*(x,\xi) \\ & - e^{-\int_0^\xi A_1^*(x,\tau)d\tau} d\xi + \int_0^+ \left[\int_0^\xi \frac{C_2^*(x,\eta)}{A(x,\eta)} f(x,\eta,u) e^{-\int_0^\eta B_1^*(x,\tau)d\tau} d\eta \right] A_1^*(x,\xi) \\ & e^{-\int_0^\xi A_1^*(x,\tau)d\tau} d\varepsilon + \int_0^t \frac{C_1^*(x,\varepsilon)}{A(x,\varepsilon)} f(x,\xi,u) e^{-\int_0^\xi B_1^*(x,\tau)d\tau} d\varepsilon \end{aligned} \quad (2.12)$$

$$\begin{aligned} h(x,t) = & h(x,0)e^{-\int_0^t B_1^*(x,\tau)d\tau} + \int_0^t \left[\int_0^\varepsilon A_2^*(x,\xi)h(x,\xi)e^{-\int_0^\xi A_1^*(x,\tau)d\tau} d\xi \right] B_2^*(x,\eta) \\ & e^{-\int_0^\eta B_1^*(x,\tau)d\tau} d\eta + \int_0^+ \left[\int_0^\xi \frac{C_1^*(x,\xi)}{A(x,\xi)} f(x,\xi,u) e^{-\int_0^\xi A_1^*(x,\tau)d\tau} \right] B_1^*(x,\eta) \\ & e^{-\int_0^\eta B_1^*(x,\tau)d\tau} d\eta + \int_0^t \frac{C_2^*(x,\eta)}{A(x,\eta)} f(x,\eta,u) e^{-\int_0^\eta B_2^*(x,\tau)d\tau} d\eta \end{aligned} \quad (2.13)$$

It has been shown earlier by Ndiyo-Ukeje (2004) that these solutions $r(x,t)$ and $h(x,t)$ does exist and are unique within the domain of dependency of the solution.

3.0 Application

Consider the equations

$$\rho_t + (\rho u)_x = 0 \quad (3.1)$$

$$(\rho u)_t + (\rho u^2)_x + \rho_x = 0 \quad (3.2)$$

where ρ is the density and u the velocity of the fluid in the x -direction.

The matrix form is

$$\begin{pmatrix} 1 & 0 \\ u & \rho \end{pmatrix} \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} + \begin{pmatrix} u & \rho \\ u^2 & 2\rho u \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho_x \end{pmatrix} \quad (3.3)$$

represented as

$$A\bar{v}_t + B\bar{v}_x = F$$

where A is non-singular since $\rho \neq 0$ with inverse

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -u/\rho & 1/\rho \end{pmatrix}$$

Multiplying equation 3.3 by A^{-1} , we obtain

$$\begin{pmatrix} \rho_t \\ u_t \end{pmatrix} + \begin{pmatrix} u & \rho \\ 0 & u \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho_x/\rho \end{pmatrix} \quad (3.4)$$

having the eigenvalues as $\lambda_1 = \lambda_2 = u$. which is a case of multiplicity of eigenvalues.

The eigenvector matrix becomes

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

By writing the system in its diagonalised form, we get

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + S \wedge S^{-1} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ -\rho_x/\rho \end{pmatrix} \quad (3.5)$$

$$S^{-1} \begin{pmatrix} \rho \\ u \end{pmatrix}_t + \wedge S^{-1} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = S^{-1} \begin{pmatrix} 0 \\ -\rho_x/\rho \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} + \wedge \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -\rho_x/\rho \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\rho_x}{\rho}(x, t, u) \end{pmatrix} \quad (3.6)$$

Indicating the component form along the characteristic curves, where

$$\frac{d\rho}{dt} = 0$$

making $\rho = \text{constant}$

$$\text{then } \frac{du}{dt} = -\frac{\rho_x}{\rho}(x, t, u) \quad (3.7)$$

From the solution representation, we have that $Q \equiv 0$ from equation 2.7.

Hence

$$u(x, t) = u(x, 0) + \int_0^t -\frac{\rho_x}{\rho}(x, \xi, u) d\xi \quad (3.8)$$

Following the uniqueness theorem criteria by Lagan (1976), this solution does exist and it is unique if the integrand is Lipschitz continuous with respect to u .

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