GLOBAL JOURNAL OF MATHEMATICAL SCIENCES VOL. 5, NO. 1, 2006: 25 - 34
COPYRIGHT (C) BACHUDO SCIENCE CO. LTD. PRINTED IN NIGERIA. ISSN 1596 - 6208

CUMULANT STRUCTURE FOR THE BILINEAR MULTIPLICATIVE SEASONAL ARIMA (0,d,0) x (1,D,1)$_s$ TIME SERIES MODEL

I. S. IWUEZE and D. C. CHIKEZIE

(Received 6 October, 2005; Revision Accepted 31 January, 2006

ABSTRACT

Difference equations are derived for third order moments and cumulants for bilinear multiplicative seasonal ARIMA (0,d,0) x (1,D,1)$_s$ time series model studied by Iwueze and Chikezie (2005). The third order cumulant structure are shown to be the same as the covariance structure. The moments (first, second and third) and cumulants obtained are used for:- (i) determining uniquely the periodicity of the series, (ii) initial estimation of the model parameters, and (iii) determining uniquely the region where the parameters lie. The initial estimates are then used to obtain least squares estimates of the parameters iteratively.

KEYWORDS: *Bilinear models, multiplicative seasonal time series, moments, cumulants, standardized cumulants.*

1. INTRODUCTION

Let Y_t, $t \varepsilon Z$ and e_t, $t \varepsilon Z$ be two stochastic processes defined on some probability space (Ω, $\mathcal{F}$, P), where Z = { ..., -1, 0, 1, ... }. For our purposes, e_t, $t \varepsilon Z$ is taken to be a sequence of independent and identically distributed random variables with $E(e_t) = 0$ and $E(e_t^2) = \sigma^2 < \infty$. Let $X_t = (1-B)^d (1-B^s)^D Y_t$, where $(1-B)^d$ is the regular differencing to remove the stochastic trend (if any) in the series and $(1-B^s)^D$ is the seasonal differencing operator used when the mean of a realization shifts according to a seasonal pattern.

Second order covariance analysis of the bilinear multiplicative seasonal ARIMA (0,d,0) x (1,D,1)$_s$ time series model

$$X_t = \alpha X_{t-s} + \beta e_{t-s} + \gamma X_{t-s} e_{t-s} + e_t \qquad (1.1)$$

have been studied by Iwueze and Chikezie (2005). The second order properties of (1.1) are similar to the linear time series equivalent, with $\gamma = 0$. For the stationary time series X_t, $t \varepsilon Z$ satisfying (1.1) and its linear equivalent, the autocovariances/autocorrelations are zero everywhere except at lags s, 2s, 3s, ... In fact, they have similar second order covariance structure as that of ARIMA (1,d,1), except that the non-zero autocovariances occur at multiples of lag s; which in turn are characterized into seven regions discussed in Section 2.

As has been noted in the literature (see Subba Rao (1981), Akamanam (1983), Iwueze (1989)), second order covariance analysis is not sufficient to distinguish a linear model and a bilinear model. Higher order moments and cumulants are therefore required.

The object of this paper is to derive the third order cumulants of (1.1) and to show that it maintains the known covariance structure at a specified plane. We will also investigate the use of the covariance structure and third order cumulant structure to determine the periodicity s and provide initial estimates of the parameters. The initial estimates are then used to obtain least squares estimates of the parameters iteratively.

2. SECOND ORDER MOMENTS: A REVIEW.

Iwueze and Chikezie (2005) have obtained the following results for (1.1):

1. $\mu = E(X_t) = \sigma^2 \gamma / (1-\alpha), \ |\alpha| < 1.$ (2.1)

2. $E(X_t e_t) = \sigma^2$ (2.2)

3. $E(X_t e_t^2) = \sigma^2 \mu$ (2.3)

4. $E(X_t^2 e_t) = 2\sigma^2 \mu$ (2.4)

5. $E(X_t^2 e_t^2) = \sigma^2 E(X_t^2) + 2\sigma^4$ (2.5)

I. S. Iwueze, Department of Statistics, Faculty of Biological and Physical Sciences, Abia State University, P. M. B. 2000, Uturu. Abia State, Nigeria.
D. C. Chikezie, Department of Statistics, Faculty of Biological and Physical Sciences, Abia State University, P. M. B. 2000, Uturu. Abia State, Nigeria.

6. $\mu_2 = E(X_t^2) = \dfrac{\sigma^2\left[1+\beta^2+2\alpha\beta+2\gamma\mu(1+\alpha+\beta)\right]}{1-\alpha^2-\sigma^2\gamma^2}$ (2.6)

provided that $\alpha^2 + \sigma^2\gamma^2 < 1$.

7. $R(0) = E[(X_t - \mu)^2] = E(X_t^2) - \mu^2$ (2.7)

8. $R(s) = E[(X_t - \mu)(X_{t+s} - \mu)] = E(X_t X_{t+s}) - \mu^2 = \alpha R(0) + \sigma^2(\beta + \gamma\mu)$ (2.8)

9. $$R(k) = \begin{cases} R(0), & k = 0 \\ R(s), & k = s \\ \alpha^{a-1} R(s), & k = as,\ a = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases} \quad (2.9)$$

10. $\rho_k = R(k) / R(0)$ (2.10)

$$= \begin{cases} 1, & k = 0 \\ \rho_s, & k = s \\ \alpha^{a-1} \rho_k, & k = as\ \ a = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases} \quad (2.11)$$

The autocorrelation functions (ac.f's) are characterized into seven regions (see Iwueze and Chikezie (2005) ; which are shown in Table 1.

Table 1: Behaviour of autocorrelations for model (1.1).

Region	Sign and behaviour of ρ_k.
1. $\alpha > 0, \beta > 0, \gamma \neq 0$, such that $\alpha + \beta > 0$.	$\rho_s > 0$ and all non-zero ac.f's positive.
2. $\alpha < 0, \beta > 0, \gamma \neq 0$, such that $\alpha + \beta > 0$.	$\rho_s > 0$ and decays with alternating sign.
3. $\alpha < 0, \beta > 0, \gamma \neq 0$, such that $\alpha + \beta < 0$.	$\rho_s < 0$ and decays with alternating sign.
4. $\alpha < 0, \beta < 0, \gamma \neq 0$, such that $\alpha + \beta < 0$.	$\rho_s < 0$ and decays with alternating sign.
5. $\alpha > 0, \beta < 0, \gamma \neq 0$, such that $\alpha + \beta < 0$.	$\rho_s < 0$ and decays with alternating sign.
6. $\alpha > 0, \beta < 0, \gamma \neq 0$, such that $\alpha + \beta > 0$.	$\rho_s > 0$ and all non=zero ac.f's positive.
7. $\alpha = -\beta, \gamma \neq 0$, such that $\alpha + \beta = 0$.	Not white noise [(a) $\rho_s > 0$ for $\alpha > 0, \beta < 0$ such that $\alpha + \beta = 0$ and $\gamma \neq 0$, (b) $\rho_s < 0$ for $\alpha < 0, \beta > 0$ such that $\alpha + \beta = 0$ and $\gamma \neq 0$].

3. THIRD ORDER MOMENTS AND CUMULANTS

Let us assume that X_t, $t \varepsilon Z$ is a real third order stationary process for which moments up to order 3 exists. By virture of the assumed third order stationarity, the third order moment

$$m(k_1,k_2) = E(X_t X_{t-k_1} X_{t-k_2}) \quad (3.1)$$

depends only on k_1 and k_2 for all admissible values of t, k_1, and k_2.

The third order cumulant

$$\mu(k_1,k_2) = E[(X_t - \mu)(X_{t-k_1} - \mu)(X_{t-k_2} - \mu)] \quad (3.2)$$

is identical to the third order moment about the mean. Simplifying (3.2), we obtain

$$\mu(k_1,k_2) = E(X_t X_{t-k_1} X_{t-k_2}) - \mu [R(k_1) + R(k_2) + R(k_1-k_2)] - \mu^3 . \quad (3.3)$$

If the time series X_t, $t \varepsilon Z$ is a real valued stationary time series, the following symmetric relations

$$\mu(k_1,k_2) = \mu(k_2,k_1) = \mu(-k_1,k_2-k_1) = \mu(k_1-k_2,-k_2) \quad (3.4)$$

hold (see Gabr (1988), Sesay and Subba Rao (1991), Oyet and Iwueze (1993)). It follows from (3.4) that $\mu(k_1,k_2)$ is completely specified over the entire plane by its values in any one of the six sectors shown in Figure 1. In view of the symmetry, we need to

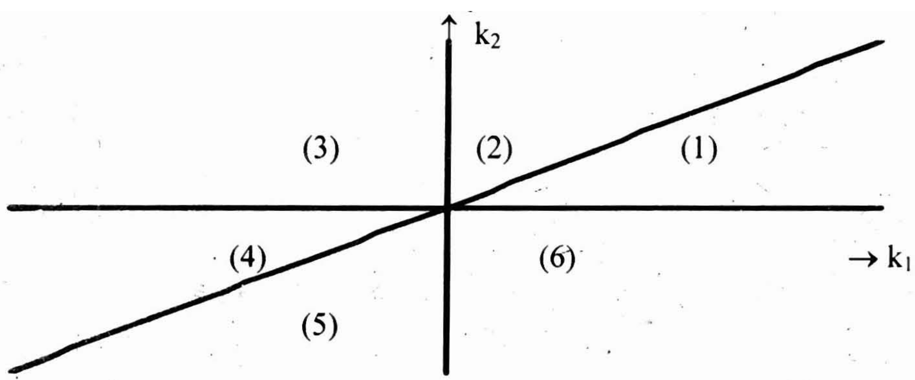

Figure 1: Symmetry of $\mu(k_1,k_2)$.

calculate cumulants for positive lags only. All we want to achieve in this paper is to distinguish between the linear and bilinear forms of (1.1). It is sufficient for our purposes to calculate $\mu(k_1,k_2)$. For $k_1 > 0$, $k_2 > 0$ on the line $k_1 = k_2$ of Figure 1.

In deriving Equation (2.1) through (2.11), Iwueze and Chikezie (2005) assumed that the random variables e_t, $t \varepsilon Z$ are Gaussian with $E(e_t) = 0$, $E(e_t^2) = \sigma^2 < \infty$, so that $E(e_t^3) = 0$, $E(e_t^4) = 3\sigma^4$, $E(e_t^5) = 0$ and $E(e_t^6) = 15\sigma^6$. Also assumed is the fact that by expression (1.1), e_t, is independent of X_h, $h < t$.

Based on these assumptions, one can verify that the following are true:

$$E(X_t e_t^3) = 3\sigma^4 \quad (3.5)$$

$$E(X_t^3 e_t) = 3\sigma^2 \mu_2 \tag{3.6}$$

$$E(X_t^2 e_t^3) = 6\sigma^4 \mu \tag{3.7}$$

$$E(X_t^3 e_t^2) = \sigma^2 E(X_t^3) + 6\sigma^4 \mu = \sigma^2 \mu_3 + 6\sigma^4 \mu \tag{3.8}$$

where $\mu_3 = E(X_t^3)$.

$$E(X_t^3 e_t^3) = 9\sigma^4 \mu_2 + 6\sigma^6 \tag{3.9}$$

$$[1 - \alpha^3 - 3\sigma^2 \alpha \gamma^2] E(X_t^3)$$

$$= 3\sigma^2\gamma [3\sigma^2\gamma^2 + 3\alpha^2 + 2\alpha\beta] \mu_2 + 3\sigma^2 [\alpha + \alpha\beta^2 + 2\alpha^2\beta + 6\sigma^2\alpha\gamma^2 + 6\sigma^2\beta\gamma^2] \mu$$

$$+ 3\sigma^4\gamma [1 + 4\alpha\beta + 3\beta^2 + 2\sigma^2\gamma^2]. \tag{3.10}$$

In obtaining (3.10), we assumed that $\alpha^3 + 3\sigma^2\alpha\gamma^2 < 1$, and this is a sufficient condition for the existence of the third order moment of the time series X_t, $t \in Z$ satisfying (1.1).

Now we obtain the third order cumulants. Limiting our search to the line $k_1=k_2=k$, $k = 0, 1, 2, 3, \ldots$ of Figure 1, we have from (3.3) that

$$\mu(0,0) = E(X_t^3) - 3\mu R(0) - \mu^3 \tag{3.11}$$

$$\mu(k,k) = E(X_t X_{t-k}^2) - \mu [R(0) + 2R(k)] - \mu^3 \tag{3.12}$$

Computation of (3.12) is done by looking at $s = 1, 2, 3, \ldots$. For want of space, we demonstrate the computations when $s = 2$.

When $s = 2$, Equation (1.1) becomes.

$$X_t = \alpha X_{t-2} + \beta e_{t-2} + \gamma X_{t-2} e_{t-2} + e_t \tag{3.13}$$

and

$$X_t X_{t-k}^2 = \alpha X_{t-2} X_{t-k}^2 + \beta e_{t-2} X_{t-k}^2 + \gamma X_{t-2} e_{t-2} X_{t-k}^2 + e_t X_{t-k}^2 \tag{3.14}$$

Based on our assumptions and previous results (Equations (2.1) through (2.9), and (3.5) through (3.10)), we obtain for:

<u>k = 1:</u>

$$E(X_t X_{t-1}^2) = \alpha E(X_{t-2} X_{t-1}^2) + \gamma E(X_{t-2} e_{t-2} X_{t-1}^2) = \alpha E(X_{t-2} X_{t-1}^2) + \sigma^2 \gamma \mu_2 \tag{3.15}$$

$$\Rightarrow \quad \mu(1,1) + \mu [R(0) + 2R(1)] + \mu^3 = \alpha [\mu(1,1) + \mu [R(0) + 2R(1)] + \mu^3]$$

$$+ (1 - \alpha) \mu [R(0) + \mu^2]$$

$$\Rightarrow \quad (1-\alpha)\,\mu(1,1) = 0$$

$$\Rightarrow \quad \mu(1,1) = 0 \tag{3.16}$$

since $\alpha \neq 1$ and $R(1) = 0$ for $s = 2$.

<u>k = 2:</u>

$$E(X_t X_{t-2}^2) = \alpha E(X_t^3) + \beta E(X_t^2 e_t) + \gamma E(X_t^3 e_t) \tag{3.17}$$

$$\Rightarrow \quad \mu(2,2) + \mu[R(0) + 2R(2)] + \mu^3 = \alpha[\mu(0,0) + 3\mu R(0) + \mu^3] + 2\sigma^2\beta\mu$$

$$+ 3\sigma^2\gamma\mu_2$$

$$= \alpha[\mu(0,0) + 3\mu R(0) + \mu^3] + 2\sigma^2\beta\mu$$

$$+ 3(1-\alpha)\mu[R(0) + \mu^2].$$

$$\Rightarrow \quad \mu(2,2) = \alpha\mu(0,0) + 2\sigma^2\gamma R(0) \tag{3.18}$$

<u>k = 3:</u>

$$E(X_t X_{t-3}^2) = \alpha E(X_{t-2} X_{t-3}^2) + \gamma E(X_{t-2} e_{t-2} X_{t-3}^2) = \alpha E(X_{t-2} X_{t-3}^2) + \sigma^2\gamma\mu_2 \tag{3.19}$$

$$\Rightarrow \quad \mu(3,3) + \mu[R(0) + 2R(3)] + \mu^3 = \alpha[\mu(1,1) + \mu[R(0) + 2R(1)] + \mu^3]$$

$$+ (1-\alpha)\mu[R(0) + \mu^2]$$

$$\Rightarrow \quad \mu(3,3) = \alpha\mu(1,1) = 0 \tag{3.20}$$

since $R(3) = \alpha R(1) = 0$ for $s = 2$.

<u>k = 4:</u>

$$E(X_t X_{t-4}^2) = \alpha E(X_{t-2} X_{t-4}^2) + \gamma E(X_{t-2} e_{t-2} X_{t-4}^2) = \alpha E(X_{t-2} X_{t-4}^2) + \sigma^2\gamma\mu_2 \tag{3.21}$$

$$\Rightarrow \quad \mu(4,4) + \mu[R(0) + 2R(4)] + \mu^3 = \alpha[\mu(2,2) + \mu[R(0) + 2R(2)] + \mu^3]$$

$$+ (1-\alpha)\mu[R(0) + \mu^2]$$

$$\Rightarrow \quad \mu(4,4) = \alpha\mu(2,2) \tag{3.22}$$

Continuing, we obtain : $\mu(5,5) = 0$; $\mu(6,6) = \alpha\mu(4,4) = \alpha^2\mu(2,2)$; $\mu(7,7) = 0$; $\mu(8,8) = \alpha\mu(6,6) = \alpha^3\mu(2,2)$; and so on.

Generally, for all values of $s \geq 1$,

$$\mu(k,k) = \begin{cases} E(X_t^3) - 3\mu R(0) - \mu^3, & k = 0 \\ \alpha\mu(0,0) + 2\sigma^2\gamma R(0), & k = s \\ \alpha^{a-1}\mu(s,s), & k = as,\ a = 2, 3, 4, \ldots \\ 0, & \text{otherwise.} \end{cases} \tag{3.23}$$

We introduce a standardized third- order cumulant given by

$$\rho(k,k) = \mu(k,k) \ / \ \mu(0,0) \tag{3.24}$$

$$= \begin{cases} 1, \ k = 0 \\ \alpha + [\, 2\,\sigma^2 \gamma\, R(0) / \mu(0,0)\,], \ k = s \\ \alpha^{a-1}\, \rho(s,s), \ k = as, \ a = 2, 3, 4, \ldots \\ 0, \ \text{otherwise} \end{cases} \tag{3.25}$$

From our analysis so far, we draw the following conclusions

(1). For the linear multiplicative seasonal ARIMA (0,d.0) x $(1,D,1)_s$ models, third order moments and cumulants are zero.

(2). For bilinear multiplicative seasonal ARIMA (0,d,0) x $(1,D,1)_s$ models, third order cumulant structure are similar to the covariance structure (see computations for a simulated example in Table 2). This deviates from previous results obtained by Sesay and Subba Rao (1991) and Oyet and Iwueze (1993).

(3). The special characteristics of our derivations is that

$$\alpha = R(2s) / R(s) = \mu(2s,2s) / \mu(s,s) = \rho_{2s} / \rho_s = \rho(2s,2s) / \rho(s,s) \tag{3.26}$$

Table 2. Sample autocorrelations (r_k) and standardized cumulants ($r(k,k)$) for (1.1) with $\alpha = 0.8$, $\beta = 0.4$, $\gamma = 0.2$, $e_t \sim N(0,1)$, n = 100.

	s = 1		s = 2		s = 3		s = 4		s = 6		s = 12	
k	r_k	r(k,k)	r_k	r(k,k)	r_k	r(k,k)	r_k	r(k,k)	r_k	r(k,k)	r_k	r(k,k)
1	0.84	0.85	0.00	-0.20	-0.18	0.00	-0.34	-0.22	0.07	0.28	-0.11	**-0.27**
2	0.58	0.60	0.86	0.91	-0.21	-0.20	0.32	0.33	-0.10	0.20	0.16	**0.44**
3	0.36	0.43	0.05	-0.18	0.88	0.91	-0.34	-0.25	-0.15	-0.18	-0.01	**-0.07**
4	0.21	0.28	0.65	0.73	-0.17	-0.02	0.85	0.87	-0.09	-0.29	-0.14	**-0.15**
5	0.11	0.18	0.11	-0.10	-0.26	-0.19	-0.33	-0.18	0.01	-0.17	-0.13	**-0.31**
6	0.02	0.11	0.50	0.61	0.72	0.76	0.28	0.19	0.81	0.87	-0.03	**0.05**
7	-0.08	0.03	0.16	-0.01	-0.15	-0.03	-0.34	-0.31	0.12	0.22	-0.12	**-0.03**
8	-0.15	-0.01	0.35	0.47	-0.32	-0.23	0.63	0.64	-0.13	0.02	-0.15	**-0.34**
9	-0.20	0.00	0.22	0.11	0.60	0.61	-0.30	-0.19	-0.25	-0.07	-0.02	**0.19**
10	-0.20	0.02	0.20	0.37	-0.11	-0.02	0.22	-0.02	-0.13	-0.26	0.16	**0.40**
11	-0.18	0.01	0.29	0.25	-0.36	-0.25	-0.30	-0.26	0.04	-0.16	-0.10	**-0.06**
12	-0.14	0.02	0.08	0.24	0.49	0.50	0.44	0.42	0.55	0.50	0.79	**0.88**
13	-0.13	0.01	0.34	0.38	-0.08	-0.02	-0.26	-0.15	0.16	0.11	-0.06	**-0.26**
14	-0.17	-0.04	-0.01	0.08	-0.36	-0.26	0.15	-0.14	-0.15	-0.09	0.13	**0.43**
15	-0.22	-0.09	0.38	0.49	0.43	0.48	-0.19	-0.14	-0.17	0.00	0.05	**-0.12**
16	-0.22	-0.12	-0.06	-0.02	-0.07	-0.05	0.26	0.24	-0.19	-0.12	-0.10	**-0.15**
17	-0.20	-0.15	0.38	0.54	-0.35	-0.26	-0.19	-0.09	0.05	-0.19	-0.16	**-0.47**
18	-0.18	-0.16	-0.08	-0.05	0.40	0.49	0.07	-0.19	0.40	0.34	-0.06	**0.11**
19	-0.17	-0.14	0.36	0.51	-0.06	-0.09	-0.07	-0.06	0.15	0.05	-0.09	**-0.06**
20	-0.12	-0.08	-0.05	-0.07	-0.35	-0.24	0.13	0.14	-0.13	-0.15	-0.14	**-0.30**
21	-0.06	-0.04	0.32	0.49	0.38	0.47	-0.11	-0.06	-0.12	-0.06	-0.02	**0.00**
22	-0.03	-0.10	0.00	-0.07	-0.05	-0.11	-0.02	-0.23	-0.22	-0.03	0.13	**0.31**
23	-0.02	-0.14	0.26	0.46	-0.29	-0.21	0.07	0.04	0.00	-0.20	-0.09	**-0.14**
24	0.03	-0.07	0.03	-0.09	0.36	0.44	0.06	0.04	0.31	[illegible]	0.52	**0.66**
25	0.13	0.11	0.18	0.41	-0.03	-0.05	-0.03	-0.01	0.10	-0.14	-0.04	**-0.18**

4. IDENTIFICATION AND INITIAL ESTIMATES

Specifying the model (1.1) means finding the seasonal lag s and the estimates of the parameters α, β, γ, and σ^2, the residual variance. All proposed methods of identification such as Box and Jenkins (1976) and many others, exploit the use of pattern recognition and we have seen the presence of certain patterns in both the covariance and cumulant structures.

We have derived the covariance and cumulant structures of (1.1) and have shown that the autocovariances/autocorrelations and cumulants/standardized cumulants are zero everywhere except at the multiples of the seasonal lag s. An estimate of the seasonal lag s can be obtained by computing the sample autocorrelation and standardized cumulant of the process and choosing as s the first lag at which the autocorrelation and standardized cumulant are non-zero. Now the following estimates are necessary

$$M_2 = \hat{\mu}_2 = \frac{1}{n}\sum_{t=1}^{n} X_t^2 \tag{4.1}$$

$$M_3 = \hat{\mu}_3 = \frac{1}{n}\sum_{t=1}^{n} X_t^3 \tag{4.2}$$

$$C_k = \hat{R}(k) = \frac{1}{n}\sum_{t=1}^{n-k} (X_t - \bar{X})(X_{t+k} - \bar{X}) \tag{4.3}$$

$$r_k = \hat{\rho}_k = C_k / C_0 \tag{4.4}$$

$$C(k,k) = \hat{\mu}(k,k) = \frac{1}{n}\sum_{t=1}^{n-k} (X_t - \bar{X})(X_{t+k} - \bar{X})^2 \tag{4.5}$$

$$r(k,k) = \hat{\rho}(k,k) = C(k,k)/C(0,0) \tag{4.6}$$

where

$$\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t \tag{4.7}$$

Simulations of Table 2 vividly illustrates the determination of s for n = 100 with $\alpha = 0.8$, $\beta = 0.4$, $\gamma = 0.2$ taking from Region 1 and $e_t \sim N(0,1)$. Our simulations in other regions gave similar results. The program used for simulation and estimation were written in Fortrann 77 by the authors, coupled with MINITAB software which was used to generate the rando data.

The initial estimate of α, the autoregressive parameter is obtained from (3.26) by replacing theoretical values by their sample equivalents. Using second order moments we obtain

$$\hat{\alpha}_1 = C_{2s}/C_s = r_{2s}/r_s \tag{4.8}$$

while, using the third order moments we obtain

$$\hat{\alpha}_2 = C(2s,2s)/C(s,s) = r(2s,2s)/r(s,s) \tag{4.9}$$

Simulations of Tables 3 and 4 illustrate the computations for n = 100 and n = 500 respectively with $\alpha = 0.8$, $\beta = 0.4$, $\gamma = 0.2$, $e_t \sim N(0,1)$.

Table 3: Sample estimates of first, second, third order moments and cumulants for (1.1) with $\alpha = 0.8$, $\beta = 0.4$, $\gamma = 0.2$, $e_t \sim N(0,1)$, n = 100.

s	$\bar{X}$	C_0	C_s	C_{2s}	C(0,0)	C(s,s)	C(2s,2s)	M_2	M_3	$\hat{\alpha}_1$	$\hat{\alpha}_2$
1	0.8455	6.4006	5.3612	3.6873	26.4795	22.5306	15.8566	7.1155	43.3191	0.69	0.70
2	0.7778	5.9596	5.0965	3.8824	17.3143	15.7309	12.6050	6.5645	31.6904	0.76	0.80
3	0.9340	8.0267	7.0998	5.7925	40.3067	36.8797	30.4323	8.8990	63.6120	0.82	0.83
4	0.9168	6.9508	5.9164	4.3821	22.9155	19.9114	14.6923	7.7913	42.8038	0.74	0.74
6	0.9671	5.4059	4.3770	2.9950	11.6665	10.1564	5.8080	6.3411	28.2543	0.68	0.57
12	0.6547	5.8501	4.6278	3.0314	17.0814	15.0341	11.3393	6.2788	28.8531	0.66	0.75

Table 4: Sample estimates of first, second, third order moments and cumulants for (1:1) with $\alpha = 0.8$, $\beta = 0.4$, $\gamma = 0.2$, $e_t \sim N(0,1)$, n = 500.

s	$\bar{X}$	C_0	C_s	C_{2s}	C(0,0)	C(s,s)	C(2s,2s)	M_2	M_3	$\hat{\alpha}_1$	$\hat{\alpha}_2$
1	1.0030	6.4023	5.6143	4.5275	11.6951	10.7565	8.3979	7.4063	31.9691	0.81	0.78
2	0.9787	6.0685	5.2467	4.0445	13.2726	12.6520	10.4601	7.0234	32.0278	0.77	0.83
3	1.0577	7.8689	6.9813	5.6041	27.3553	25.2171	20.6763	8.9877	53.5085	0.80	0.82
4	1.0331	7.3959	6.5226	5.3274	23.1467	21.4463	17.2442	8.4032	47.1715	0.82	0.80
6	0.9431	7.2767	6.2535	4.7511	44.0951	39.1176	29.8999	8.1661	65.5217	0.76	0.76
12	0.9019	6.4099	5.4832	4.2913	19.1481	17.4401	14.2866	7.2234	37.2251	0.78	0.82

We must note that when $\alpha = 0.8$, $\beta = 0.4$, $\gamma = 0.2$, $e_t \sim N(0,1)$, the theoretical moments are [illegible] = [illegible]0000, $\mu_2 = 8.3750$, $\mu_3 = 51.9260$, $R(0) = 7.3750$, $R(s) = 6.5000$, $R(2s) = 5.2000$, $\mu(0,0) = 28.8010$, $\mu(s,s) = 25.9908$. $\mu(2s,2s) = 20.7926$. Tables 3 and 4 show that α_1 and α_2 given by (3.8) and (3.9) respectively, provide initial estimates of α that are close to the true/theoretical value. However, estimates for n = 500 (Table 4) show that $\hat{\alpha}_1 \approx \hat{\alpha}_2 \approx \alpha$; indicating that with large samples, initial estimates of α can be used as the true values. It is also clear from Tables 3 and 4 that sample estimates of second order moments are closer to the true values than the sample estimates of third order moments are to their theoretical equivalents. Based on these observations, initial estimates of the parameters of (1.1) will be obtained using the first and second moments only.

Having determined s and initial estimate of α, we now consider how to obtain the initial estimates of β, γ and σ^2 using the first and second moments. Solving Equations (2.1), (2.6) and (2.7) and replacing theoretical moments with their sample equivalents, we obtain

$$\hat{\sigma}^2 = \frac{\left(1 - \hat{\alpha}^2\right)M_2}{1 + \hat{\beta}^2 + M_2\hat{\gamma}^2 + 2\hat{\alpha}\hat{\beta} + 2\hat{\gamma}\bar{X}\left(1 + \hat{\alpha} + \hat{\beta}\right)} \tag{4.10}$$

$$\hat{\beta} = \frac{C_s - \hat{\alpha}C_0 - \left(1 - \hat{\alpha}\right)\bar{X}^2}{\hat{\sigma}^2} \tag{4.11}$$

$$\hat{\gamma} = \frac{\left(1 - \hat{\alpha}\right)\bar{X}}{\hat{\sigma}^2} \tag{4.12}$$

Having obtained $\hat{\alpha}$, we adopt an iterative procedure called "Linearly convergent process" by Box and Jenkins (1976, p202) to obtain initial estimates of β, γ, and σ^2. We compute the estimates $\hat{\sigma}^2$, $\hat{\beta}$, $\hat{\gamma}$ in this precise order using the iteration (4.10), (4.11) and (4.12). The parameters $\hat{\beta}$ *and* $\hat{\gamma}$ are set equal to zero to start the iteration and the values of $\hat{\sigma}^2$, $\hat{\beta}$ *and* $\hat{\gamma}$ to be used in any subsequent calculation are the most up to date values available. For example, using Table 3 for s = 12, we obtain

$\hat{\alpha} = 0.66$, $\hat{\sigma}^2 = 3.54/\left(1 + \hat{\beta}^2 + 1.32\hat{\beta} + 6.28\hat{\gamma}^2 + 2.17\hat{\gamma} + 1.31\hat{\beta}\hat{\gamma}\right)$, $\hat{\beta} = 0.62\,\hat{\sigma}^2$, $\hat{\gamma} = 0.22/\hat{\sigma}^2$. Table 5 shows how the iteration converged for s = 12 of Table 3. Using similar procedure, initial estimates are obtained for various values of s considered for data of Table 3 and the results are shown in Table 7. Our initial values are close to the true values, demonstrating the workability of outlined procedure for determination of initial estimates.

Table 5: Convergence of initial estimates of σ^2, β and γ for s = 12 of data of Table 3.

Iteration	$\hat{\sigma}^2$	$\hat{\beta}$	γ
0		0.00	0.00
1	3.54	0.18	0.06
2	2.47	0.25	0.09
3	2.12	0.29	0.11
4	1.94	0.32	0.11
5	1.88	0.33	0.12
6	1.82	0.34	0.12
7	1.80	0.35	0.12
8	1.78	0.35	0.13
9	1.74	0.36	0.13
10	1.72	0.36	0.13
11	1.72	0.36	0.13

5. LEAST SQUARES ESTIMATES

Having determined s and the initial estimates of the paramerters including the residual variance, we consider how to obtain the final estimates when we have a realization { $X_1, X_2, \ldots, X_n$ } of the time series X_t, $t \in Z$. To obtain the

Table 6: Convergence of final estimates of α, β, γ and σ^2 for s = 12 of Table 3 using the initial estimates $\hat{\alpha} = 0.66$, $\hat{\beta} = 0.36$, $\hat{\gamma} = 0.13$ *and* $\hat{\sigma}^2 = 1.72$ of Table 5.

Iteration k	Parameters			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\sigma}^2$
0	0.6600	0.3600	0.1300	1.7200
1	0.6262	0.7438	0.2091	1.1154
2	1.3347	0.3880	0.2234	2.8839
3	0.7836	0.2657	0.2441	1.6967
4	0.7723	0.4236	0.1990	1.0623
5	0.7918	0.4254	0.1907	0.9922
6	0.7893	0.4299	0.1872	0.9862
7	0.7892	0.4302	0.1869	0.9859
8	0.7892	0.4302	0.1869	0.9859

Table 7: Initial estimates of α, β, γ and σ^2 for the different values of s for data of Table 3.

s	Initial estimates			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\sigma}^2$
1	0.70	0.60	0.20	1.08
2	0.76	0.36	0.16	1.17
3	0.82	0.26	0.12	1.39
4	0.74	0.50	0.21	1.00
6	0.68	0.34	0.27	1.17
12	0.66	0.36	0.13	1.72

Table 8: Final estimates of α, β, γ and σ^2 for the different values of s for data of Table 3.

s	Final estimates			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\sigma}^2$
1	0.7809±0.0706	0.2746±0.0675	0.2466±0.0226	0.9383
2	0.7170±0.0789	0.4401±0.0329	0.1905±0.0067	0.9630
3	0.7605±0.0947	0.4049±0.0475	0.1910±0.0168	0.9790
4	0.8010±0.0723	0.3307±0.0714	0.2254±0.0263	0.9707
6	0.7744±0.0886	0.3986±0.1270	0.2140±0.0577	0.9413
12	0.7892±0.0796	0.4302±0.0820	0.1869±0.0257	0.9859

final estimates of the parameters, we proceed as in Subba Rao (1981), Gabr and Subba Rao (1981), Iwueze (2004) and apply the methods of least squares to minimize

$$S(\theta) = \sum_{t=1}^{n} X_t \qquad (5.1)$$

with respect to the parameters $\theta = (\theta_1 = \alpha, \theta_2 = \beta, \theta_3 = \gamma)^T$ When minimizing S(θ) with respect to θ, the normal equations are nonlinear in θ. The solution of these equations require the use of nonlinear algorithm such as Newton-Raphson. The Newton-Raphson iterative procedure usually converge, but to obtain a good set of final estimates it is necessary that we have a good set of initial values of the parameters.

The problem of obtaining the initial estimates of the parameters was discussed in Section 3. Using the Newton-Raphson iterative procedure and the initial estimates tabulated in Table 7, we fit the model (1.1) to the sets of data whose second and third moments are described in Tables 2 and 3. Table 8 gives the final estimates and the values in parenthesis below the parameter estimates are the associated standard errors. Adequacy of fit was based on the randomness of the residuals by comparing the ac.f of the estimated residuals with $\pm 2/\sqrt{n}$ [Chatfield (1980)].

6. CONCLUDING REMARKS

We have derived the third order moments and cumulants of the model (1.1). Our results show that for (1.1), the third order cumulant structure are similar to the covariance structure of zero values everywhere except at the multiples of the seasonal lag s. Based on this similarity, we obtained initial estimates of the parameters using the first and second moments. This method gave initial estimates that are close to the true values. The initial estimates were then used in the Newton-Raphson iterative procedure to obtain the least squares estimates. These final estimates were almost the same value as the true values proving that the entire procedure of finding the initial estimates and achieving the final estimates are adequate.

REFERENCES.

Akamanam, S. I., 1983. Some contributions to the study of bilinear time series models. Ph.D Thesis, University of Sheffield.

Box, G. E. P. and Jenkins, G. M., 1976. Time series analysis, forecasting and control, Holden-Day, San-Francisco.

Chatfield, C., 1980. The analysis of time series: An introduction, Chapman and Hall, London.

Gabr, M. M., 1981. On the third order moments structure and bispectral analysis of some bilinear time series, Journal of Time Series Analysis, 9:11-20.

Gabr, M. M. and Subba Rao, T. (1981). The estimation and prediction of subset bilinear time series models with applications, Journal of time series analysis, 3, pp.155-171

Iwueze, I. S., 1989. On the invertibility of bilinear time series models, STATISTICA, Anno XLIX, n.3, pp. 441-446.

Iwueze, I. S., 2004. Some examples of modeling with super-diagonal bilinear moving average time series, Journal of the Nigerian Association of Mathematical Physics, 8: 125-130.

Iwueze, I. S. and Chikezie, D. C., 2005. A bilinear representation of the multiplicative seasonal ARIMA (0,d,0) x $(1,D,1)_s$ time series model, Accepted for publication in Global Journal of Mathematical Sciences, 4: 1 & 2.

Oyet, A. J. and Iwueze, I. S., 1993. Difference equations for the third order moments and cumulants of seasonal bilinear time series, Journal of the Nigerian Mathematical Society, 12, pp. 3-10.

Sesay, S. A. O. and Subba Rao, T., 1991. Difference equations for higher order moments and cumulants for the bilinear time series model BL(p,0,p,1), Journal of Time Series, 12: 156-176.

Subba Rao, T., 1981. On the theory of bilinear time series, Journal Royal Stat. Society, Series B, 43, pp. 244-255.