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MODELLING FLOW WITH HYDRAULIC JUMP IN A PARABOLIC CHANNEL WITH ABRUPT CHANGE IN SLOPE

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ABSTRACT

A mathematical model for dredging (excavating) a parabolic open channel with hydraulic jump and abrupt change in slope is developed using the conditions of geometrical and dynamical similarities coupled with continuity conditions. The model is then applied to a numerical example and new parameters like new depth, new area of cross section, new hydraulic mean depth, new discharge, etc., of the new (excavated) channel are determined and compared with those of the original channel. A further application of the model in Bernoulli's equation enables other parameters like energy dissipated in the jump, jump efficiency, relative energy loss and power loss to be also determined in the two channels and compared.

KEYWORDS: Parabolic open channel, dredging, hydraulic jump, mathematical model.

INTRODUCTION

An open channel is a conduit for flow with a free surface, e.g. canals, rivers and pipes which are not running full, Chanson (2004) and Chow (1959). Recently, Eyo and Udoh (2007) compared the hydraulic performances of three open channel sections using a combination of Darcy's formula and the conditions for best hydraulic performances for the channels. Their conclusion was that the rectangular section of the channel is hydraulically and also economically better than the trapezoidal and triangular sections, while the triangular section is a very inefficient section hydraulically. Also, in studying a jet-assisted hydraulic jump in an open channel, France (1981) investigated the stability of the hydraulic jump and the effectiveness of the jets over a wide range of operating conditions. He observed that the stabilization of the jump is dependent on a number of parameters but concluded however that the angle of inclination of the jets has the most pronounced effect. Other researchers in open channel flows include, notably, Baddour and Abbink (1983), Chanson (2008), Hornung et al. (1995), Leuthensser and Kartha (1972), Moramarco et al. (2004), Nasser et al. (1980), Scott-Moncrieff (1974), Wilson (1977).

The present work deals with excavation of a parabolic open channel with hydraulic jump and abrupt change in slope. Here the flow is non-uniform and steady. Mathematical model governing the excavation of the channel is developed. From the numerical results, for a channel flow problem, some parameters exhibit certain characteristics.

DEVELOPMENT OF MATHEMATICAL MODEL FOR DREDGING THE PARABOLIC CHANNEL

Throughout, the two systems in dredging an open channel shall be denoted by the symbols O and N, where

System O = original open channel (i.e. open channel before dredging)

System N = new open channel (i.e. open channel after dredging)

Also, we shall use the subscripts 1 and 2 to denote the conditions upstream and downstream of the jump respectively.

Mathematical model for the original parabolic channel

Upstream parameters

(i) Cross sectional area $(A_1)_0$

From Chow, 1959, and in view of our modeling notation

$$(A_1)_0 = \frac{2}{3} B (h_1)_0 \qquad (2.1)$$

where B is the top width of the channel and is constant and h is the depth (see Figure below).

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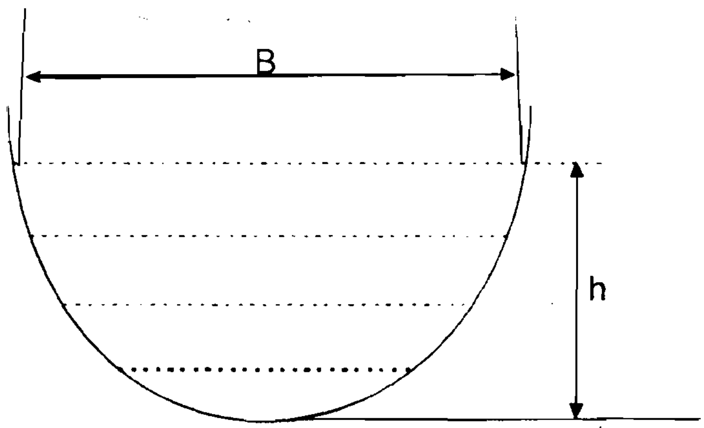

Figure 1: Parabolic section of an open channel.

(ii) Wetted perimeter $(P_1)_0$

Also, from Chow, 1959 and by our notation

$$(P_1)_0 = B + \frac{8}{3}\frac{(h_1)_0^2}{B} \qquad (2.2)$$

(iii) Hydraulic mean depth $(R_1)_0$

$$(R_1)_0 = \frac{(A_1)_0}{(P_1)_0} = \frac{\frac{2}{3}B(h_1)_0}{B + \frac{8}{3}\frac{(h_1)_0^2}{B}} = \frac{2B^2(h_1)_0}{3B^2 + 8(h_1)_0^2} \qquad (2.3)$$

(iv) Mean velocity $(u_1)_0$

From Manning's formula Chow, 1959 and using our notation,

$$(u_1)_0 = \frac{1}{n}\left[(R_1)_0\right]^{2/3}\left[(i_1)_0\right]^{1/2} \qquad (2.4)$$

Here 'i' denotes the slope of the channel.

(v) Discharge Q_0

$$Q_0 = (A_1)_0(u_1)_0 = \frac{2}{3}B(h_1)_0 \cdot \frac{1}{n}\left[(R_1)_0\right]^{2/3}\left[(i_1)_0\right]^{1/2} \qquad (2.5)$$

(vi) Froude number $(F_1)_0$ see Chow, 1959

We note that the jump is characterized by the upstream Froude number F_1, where

$$F_1 = \frac{u_1}{\sqrt{gh_1}} \qquad (2.6)$$

Thus, using our modeling notation

$$(F_1)_0 = \frac{(u_1)_0}{\sqrt{g(h_1)_0}} \qquad (2.7)$$

Downstream parameters

(vii) Downstream depth $(h_2)_0$

The equation for conjugate depths of the jump is given by see Chow, 1959

$$\left(\frac{h_1}{h_2}\right)^{5/2} = 1 + \frac{5}{2} \times F_1^2\left(1 - \left(\frac{h_1}{h_2}\right)^{3/2}\right) \qquad (2.8)$$

From (2.8) we obtain, in view of our notation

$$\left(\frac{(h_1)_0}{(h_2)_0}\right)^{5/2} = 1 + \frac{5}{2} \times (F_1)_0^2 \left(1 - \left(\frac{(h_1)_0}{(h_2)_0}\right)^{3/2}\right) \qquad (2.9)$$

for the determination of $(h_2)_0$ by means of successive approximation

(viii) Cross sectional area $(A_2)_0$

$$(A_2)_0 = \frac{2}{3} B (h_2)_0 \qquad (2.10)$$

(ix) Wetted perimeter $(P_2)_0$

$$(P_2)_0 = B + \frac{8}{3} \cdot \frac{(h_2)_0^2}{B} \qquad (2.11)$$

(x) Hydraulic mean depth $(R_2)_0$

$$(R_2)_0 = \frac{\frac{2}{3} B (h_2)_0}{B + \frac{8}{3} \frac{(h_2)_0^2}{B}} = \frac{2B^2 (h_2)_0}{3B^2 + 8 (h_2)_0^2} \qquad (2.12)$$

(xi) Mean velocity $(u_2)_0$

From Chow, 1959, and by our notation

$$(u_2)_0 = \frac{1}{n} \left[(R_2)_0\right]^{2/3} \left[(i_2)_0\right]^{1/2} \qquad (2.13)$$

(xii) Discharge Q_0

By continuity

$$Q_0 = (A_1)_0 (u_1)_0 = (A_2)_0 (u_2)_0 = \frac{2}{3} B (h_2)_0 \cdot \frac{1}{n} \left[(R_2)_0\right]^{2/3} \left[(i_2)_0\right]^{1/2} \qquad (2.14)$$

(xiii) Froude number $(F_2)_0$

$$(F_2)_0 = \frac{(u_2)_0}{\sqrt{g (h_2)_0}} \qquad (2.15)$$

Mathematical model for the new parabolic channel

Similarity conditions

For dynamical similarity conditions the Froude numbers should be identical at all corresponding points in the original and new (excavated) channels. See Chow, 1959

Thus

for the upstream sections of the two channels

$$\frac{(u_1)_0^2}{g (h_1)_0} = \frac{(u_1)_N^2}{g (h_1)_N} \qquad (2.16)$$

and for the downstream sections

$$\frac{(u_2)_0^2}{g (h_2)_0} = \frac{(u_2)_N^2}{g (h_2)_N} \qquad (2.17)$$

The conditions of geometrical similarity for the two channels require

$$\frac{(h_1)_0}{(h_2)_0} = \frac{(h_1)_N}{(h_2)_N} \qquad (2.18)$$

Upstream parameters

(xiv) Cross sectional area $(A_1)_N$

$$(A_1)_N = \frac{2}{3} B (h_1)_N \qquad (2.19)$$

(xv) Wetted perimeter $(P_1)_N$

$$(P_1)_N = B + \frac{8}{3} \cdot \frac{(h_1)_N^2}{B} \qquad (2.20)$$

(xvi) Hydraulic mean depth $(R_1)_N$

$$(R_1)_N = \frac{(A_1)_N}{(P_1)_N} = \frac{2B^2(h_1)_N}{3B^2 + 8(h_1)_N^2} \tag{2.21}$$

(xvii) Mean velocity $(u_1)_N$

From similarity condition (2.16)

$$(u_1)_N = \left(\frac{(u_1)_0^2 \cdot g(h_1)_N}{g(h_1)_0} \right)^{1/2} \tag{2.22}$$

(xviii) Discharge Q_N

$$Q_N = (A_1)_N (u_1)_N = \frac{2}{3} B(h_1)_N \cdot \left(\frac{(u_1)_0^2 \cdot g(h_1)_N}{g(h_1)_0} \right)^{1/2} \tag{2.23}$$

(xix) Froude number $(F_1)_N$

From (2.6)

$$(F_1)_N = \frac{(u_1)_N}{\sqrt{g(h_1)_N}} \tag{2.24}$$

(xx) Slope $(i_1)_N$

Again, from Chow, 1959

$$(i_1)_N = \left(\frac{n.Q_N}{(A_1)_N \times \left[(R_1)_N\right]^{2/3}} \right)^2 \tag{2.25}$$

Downstream parameters

(xxi) Depth $(h_2)_N$

From geometrical condition (2.18)

$$(h_2)_N = \frac{(h_2)_0 . (h_1)_N}{(h_1)_0} \tag{2.26}$$

(xxii) Cross-sectional area $(A_2)_N$

$$(A_2)_N = \frac{2}{3} B(h_2)_N \tag{2.27}$$

(xxiii) Wetted perimeter $(P_2)_N$

$$(P_2)_N = B + \frac{8}{3} . \frac{(h_2)_N^2}{B} \tag{2.28}$$

(xxiv) Hydraulic mean depth $(R_2)_N$

$$(R_2)_N = \frac{(A_2)_N}{(P_2)_N} = \frac{2B^2(h_2)_N}{3B^2 + 8(h_2)_N^2} \tag{2.29}$$

(xxv) Mean velocity $(u_2)_N$

From (2.17)

$$(u_2)_N = \left(\frac{(u_2)_0^2 \cdot g(h_2)_N}{g(h_2)_0} \right)^{1/2} \tag{2.30}$$

(xxvi) Discharge Q_N

Again, by continuity

$$Q_N = (A_1)_N (u_1)_N = (A_2)_N (u_2)_N = \frac{2}{3} B(h_2)_N \cdot \left(\frac{(u_2)_0^2 . g(h_2)_N}{g(h_2)_0} \right)^{1/2} \tag{2.31}$$

(xxvii) Froude number $(F_2)_N$

Again, from (2.6)

$$(F_2)_N = \frac{(u_2)_N}{\sqrt{g(h_2)_N}} \tag{2.32}$$

(xxviii) Slope $(i_2)_N$

Finally, from Chow, 1959

$$(i_2)_N = \left(\frac{n.Q_N}{(A_2)_N . [(R_2)_N]^{2/3}} \right)^2 \tag{2.33}$$

MODEL FOR ENERGY LOSS (HEAD LOSS), JUMP EFFICIENCY, RELATIVE ENERGY LOSS AND POWER LOSS

The energy loss h_f occurring between the two sections of the channel as determined from Bernoulli's equation for any streamline between points 1 and 2 of the hydraulic jump Chow, 1959 is

$$h_f = \left(\frac{u_1^2}{2g} + h_1 \right) - \left(\frac{u_2^2}{2g} + h_2 \right) \tag{3.1}$$

or

$$h_f = E_1 - E_2 \tag{3.2}$$

where

$$E_1 = \frac{u_1^2}{2g} + h_1 \tag{3.3}$$

and

$$E_2 = \frac{u_2^2}{2g} + h_2 \tag{3.4}$$

Here E_1 and E_2 denote the specific energies before and after the jump respectively. It is elementary to see that (3.1) gives after simplification

$$h_f = \frac{(h_2 - h_1)^3}{4h_1 h_2} \tag{3.5}$$

From (3.5) we obtain by virtue of our modeling notation

Energy loss in the original channel $(h_f)_O$

$$(h_f)_O = \frac{[(h_2)_O - (h_1)_O]^3}{4(h_1)_O (h_2)_O} \tag{3.6}$$

Energy loss in the new channel $(h_f)_N$

$$(h_f)_N = \frac{[(h_2)_N - (h_1)_N]^3}{4(h_1)_N (h_2)_N} \tag{3.7}$$

Also, by virtue of our model, (3.3) and (3.4) yield

Jump efficiency for the original channel $\frac{(E_2)_O}{(E_1)_O}$

$$\frac{(E_2)_O}{(E_1)_O} = \frac{\frac{(u_2)_O^2}{2g} + (h_2)_O}{\frac{(u_1)_O^2}{2g} + (h_1)_O} \tag{3.8}$$

Jump efficiency for the new channel $\frac{(E_2)_N}{(E_1)_N}$

$$\frac{(E_2)_N}{(E_1)_N} = \frac{\frac{(u_2)_N^2}{2g} + (h_2)_N}{\frac{(u_1)_N^2}{2g} + (h_1)_N} \quad (3.9)$$

Relative energy loss for the original channel:

$$\frac{(E_1)_0 - (E_2)_0}{(E_1)_0} = 1 - \frac{(E_2)_0}{(E_1)_0} \quad (3.10)$$

Relative energy loss for the new channel

$$\frac{(E_1)_N - (E_2)_N}{(E_1)_N} = 1 - \frac{(E_2)_N}{(E_1)_N} \quad (3.11)$$

Finally, from (3.6) and (3.7) we obtain respectively

Power loss for the original channel P_0:

$$P_0 = \rho g\, Q_0\, (h_f)_0 \quad (3.12)$$

Power loss for the new channel P_N:

$$P_N = \rho g\, Q_N\, (h_f)_N \quad (3.13)$$

Here ρ = fluid density, g = gravitational acceleration, Q_0 and Q_N are as above. Thus, the expressions (2.16) – (2.33) constitute the model for the new parabolic open channel with jump. Also, while the expressions (3.6), (3.8), (3.10) and (3.12) constitute respectively the model for energy loss, jump efficiency, relative energy loss and power loss for the original channel, the expressions (3.7), (3.9), (3.11) and (3.13) constitute, on the other hand, the model for determining respectively the energy loss, jump efficiency, relative energy loss and power loss in respect of the new channel.

APPLICATION OF THE MODEL TO NUMERICAL EXAMPLE

Consider, for example, a parabolic channel with hydraulic jump having a top width of 6m, Manning's coefficient Chow, 1959 n is 0.012. The channel has an abrupt change in the channel slope from 0.115112 to 0.000538 and the depth of water before the jump occurs being 0.45m. Using the model we wish to determine, after dredging the channel, parameters like (a) the new downstream depth, (b) the new cross sectional areas, (c) the new mean velocities, (d) the new discharge,
(e) the new Froude numbers, (f) the new energy dissipated in the jump, (g) the new jump efficiency, (h) the relative energy loss and (i) the new power loss due to the jump, if the excavation must be to the depth of 1.5m upstream.

Solution

Original channel

Upstream data

From the problem

$B = 6m$, $(h_1)_0 = 0.45m$, $n = 0.012$,
$(i_1)_0 = 0.115112$

Using these data appropriately in the model expressions (2.1), (2.2), (2.3), (2.4), (2.5) and (2.7) we obtain respectively:

$(A_1)_0 = 1.80m^2$, $(P_1)_0 = 6.09m$, $(R_1)_0 = 0.2955m$, $(u_1)_0 = 12.55m/s$, $Q_0 = 22.59m^3/s$, $(F_1)_0 = 5.9757$

Downstream data

Here

$B = 6m$, $n = 0.012$, $(i_2)_0 = 0.000538$

Moreover, appropriate substitution of the above data in the expressions (2.9) – (2.15) yields respectively:

$(h_2)_0 = 2.64m$, $(A_2)_0 = 10.582m^2$, $(P_2)_0 = 9.11m$, $(R_2)_0 = 1.1615m$, $(u_2)_0 = 2.135m/s$, $Q_0 = 22.59m^3/s$, $(F_2)_0 = 0.4191$

New channel

Upstream data

Here

$B = 6m$, $(h_1)_N = 1.5m$, $n = 0.012$

Substituting the above data appropriately in the model (2.19) – (2.25) gives respectively

$(A_1)_N = 6m^2$, $(P_1)_N = 7m$, $(R_1)_N = 0.8771m$, $(u_1)_N = 22.93m/s$,
$Q_N = 137.58m^3/s$, $(F_1)_N = 5.9757$, $(i_1)_N = 0.92914$

Downstream data

Here

B = 6m, n = 0.012

Similarly, the parameters $(h_2)_N$, $(A_2)_N$, $(P_2)_N$, $(R_2)_N$, $(u_2)_N$, Q_N, $(F_2)_N$ and $(i_2)_N$ for the new channel are determined respectively by appropriate substitution of the above data in (2.26) – (2.33). The results is

$(h_2)_N = 8.82m$, $(A_2)_N = 35.28m^2$, $(P_2)_N = 40.57m$, $(R_2)_N = 0.8695m$,
$(u_2)_N = 3.90m/s$, $Q_N = 137.58m^3/s$, $(F_2)_N = 0.4191$, $(i_2)_N = 0.002636$.

Furthermore, substituting the above data appropriately in the model expressions (3.6), (3.8), (3.10) and (3.12) yields respectively the energy loss, jump efficiency, relative energy loss and power loss for the original channel. Thus

$$(h_f)_0 = 2.22m, \quad \frac{(E_2)_0}{(E_1)_0} = 33.92\%,$$

$$1 - \frac{(E_2)_0}{(E_1)_0} = 0.6608, \quad P_0 = 492.63KW$$

Finally, the energy loss, jump efficiency, relative energy loss and power loss for the new channel are determined respectively via appropriate substitution of the above data in the expressions (3.7), (3.9), (3.11) and (3.13). Thus, we obtain

$$(h_f)_N = 7.4094m, \quad \frac{(E_2)_N}{(E_1)_N} = 33.92\%$$

$$1 - \frac{(E_2)_N}{(E_1)_N} = 0.6608, \quad P_N = 9997.19KW$$

RESULTS

The new parameters (a) – (i) that are determined for the new channel are shown in Table 2 and can be compared with their counterparts for the original channel in Table 1 (see Tables 1 and 2 below)

Table 1: Result for the Original Channel

	Original Channel with Jump	
	Upstream Parameters	Downstream Parameters
Bed slope	0.115112	0.000538
Manning's n	0.012	0.012
Top width	6m	6m
Depth h	0.45	2.64m
Area of cross section	1.80m²	10.582m²
Wetted perimeter	6.09m	9.11m
Hydraulic mean depth	0.2955m	1.1615m
Mean velocity u	12.55m/s	2.135m/s
Discharge Q	22.59m³/s	22.59m³/s
Froude number	5.9757	0.4191
Energy loss	2.223m	
Jump efficiency	33.92%	
Relative energy loss	0.6608	
Power loss	492.63KW	

Table 2: Result for the Original Channel

	New (Excavated) Channel with Jump	
	Upstream Parameters	Downstream Parameters
Bed slope	0.092914	0.002636
Manning's n	0.012	0.012
Top width	6.0m	6.0m
Depth h	1.50m	8.82m
Area of cross section	6.0m^2	35.28m^2
Wetted perimeter	7.0m	40.57m
Hydraulic mean depth	0.8571m	0.8696m
Mean velocity u	22.93m/s	3.90m/s
Discharge Q	137.58m^3/s	137.58m^3/s
Froude number	5.9757	0.4191
Energy loss	7.4094m	
Jump efficiency	33.92%	
Relative energy loss	0.6608	
Power loss	9997.19KW	

DISCUSSION AND CONCLUSION

Tables 1 and 2 show respectively the results of the analysis of the flow problem for the original and new (excavated) channels. Comparison of the two Tables indicates that the downstream parameters, namely, the downstream depth, area of cross section, wetted perimeter and hydraulic mean depth are generally greater in the original and new channels than the upstream ones. This is in agreement with the model (2.9) – (2.12) and (2.26) – (2.29) of the original and new channels respectively. In particular, these same parameters upstream and downstream are greater in the new channel than their counterparts in the original one. This is also in agreement with the model (2.1) – (2.3) (upstream), (2.9) – (2.12) (downstream) of the original channel and (2.19) – (2.21) (upstream), (2.26) – (2.29) (downstream) of the new channel. However, this trend is reversed in the case of the Froude number which is lower in the downstream section in both channels than in the upstream section; but the striking thing here is that the upstream Froude numbers are equal in both channels just like the downstream Froude numbers. This agrees with the model (2.16) and (2.17). We also observe that the upstream and downstream mean velocities in the new channel are respectively greater than the upstream and downstream velocities in the original channel (see Tables 1 and 2). Another feature is that, whereas the energy loss and power loss are greater in the new channel than in the original one, the jump efficiency and relative energy loss, on the other hand, remain unchanged in both channels. Furthermore, while the upstream bed slope in the original channel is greater than the one in the new channel, the downstream bed slope in the original channel, however, becomes smaller than its counterpart in the new channel.

Finally, from Tables 1 and 2 it becomes very clear that the new channel maintains a higher water level than the original one. This high water level in the new channel, apart from enhancing navigation, can be harnessed for water distribution purposes and for mixing of chemicals used for water purification or waste water treatment

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