RESULTS ON AN INTEGRAL INEQUALITY OF THE OPIAL-TYPE

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ABSTRACT
We obtain integral inequalities which are Opial-type inequalities, mainly by using Jensen’s inequality for the case of convex function.

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INTRODUCTION
Opial ([8]) established the following interesting integral inequality:
Let $x(t) \in C[0,b]$ be such $x(0) = x(b) = 0$ and $x(t) > 0$ in $(0,b)$, then
\[
\int_a^b |x(t)x'(t)| \, dt \leq \frac{b}{4} \int_a^b (x'(t))^2 \, dt \tag{1}
\]
where $\frac{b}{4}$ is in the best possible constant.

In 1967 Maroni [5] obtained a generalized Opial’s inequality by using Hölder’s inequality with indices $\mu$ and $\nu$. The result obtained is the following:

**Theorem 1:**
Let $p(t)$ be positive and continuous on $[\alpha, \tau]$ with $\int_\alpha^\tau p^{-\mu}(t) \, dt < \infty$, where $\mu > 1$, $x(t)$ be absolutely function on $[\alpha, \tau]$ and $x(0)=0$. Then, the following inequality holds.
\[
\int_\alpha^\tau |x(t)x'(t)| \, dt \leq \frac{1}{2} \left( \int_\alpha^\tau p^{-\mu}(t) \, dt \right)^{\frac{2}{\mu}} \left( \int_\alpha^\tau p(t) |x'(t)|^\nu \, dt \right)^{\frac{2}{\nu}} \tag{2}
\]
where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Equality holds in (2) if $c \int_\alpha^\tau p^{-\mu}(s) \, ds$.

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Theorem 2: [2] Assume that

(i) \( x(t) \) is absolutely continuous in \([a, \tau]\) and \( x(\alpha) = 0 \)

(ii) \( f(t) \) is continuous, complex-valued, defined in the range of \( x(t) \) and for all real for \( t \) of the form \( t(s) = \int_a^s |x(u)| \, du : f([t]) \) for all \( t \) and \( f(t) \) is real \( t > 0 \) and is increasing there,

(iii) \( p(t) \) is positive, continuous and \( \int_a^t p^{1-\mu}(t) \, dt < \infty \), where \( \frac{1}{\mu} + \frac{1}{\nu} = 1 \). Then the following inequality holds.

\[
\int_a^t f(x(t)) x'(t) \, dt \leq F \left( \left( \int_a^t p^{1-\mu}(t) \, dt \right)^{\frac{1}{\mu}} \left( \int_a^t x'(t)^\nu \, dt \right)^{\frac{1}{\nu}} \right)
\]

where \( F(t) = \int_0^t f(t) \, ds, t > 0 \). Equality holds in (3) iff \( x(t) = \int_a^t p^{1-\mu}(s) \, ds \).

The aim of this paper is to generalize Maroni and Calvert results using Jensen’s inequality.

2. Some Adaptations of Jensen’s inequalities:

Let \( \phi \) be continuous and convex function and let \( h(s, t) \) be a non negative function and \( \lambda \) be non decreasing function. Let \(-\infty \leq \xi(t) \leq \eta(t) < \infty \) and suppose \( \phi \) has a continuous inverse \( \phi^{-1} \) (which is necessarily concave). Then,

\[
\phi^{-1} \left( \frac{\int_{\xi(t)}^{\eta(t)} h(s, t) \, d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)} \right) \leq \phi^{-1} \left( \int_{\xi(t)}^{\eta(t)} \frac{h(s, t)}{d\lambda(s)} \right)
\]

with the inequality reversed if \( \phi \) is concave. The inequality (4) above is known as Jensen’s inequality for convex function. Setting \( \phi(u) = u^{1-\mu}, \xi(t) = 0, \) and \( \eta(t) = t \) in (4), then we obtain

\[
(f(t))^\frac{\mu}{\nu} = \left( \phi^{-1} \left( \int_{\xi(t)}^{\eta(t)} h(s, t) \, d\lambda(s) \right) \right)^\frac{1}{\nu} \leq \phi^{-1} \left( \int_{\xi(t)}^{\eta(t)} \frac{h(s, t)}{d\lambda(s)} \right)
\]

3 MAIN RESULT:

Before stating our main result in this section, we shall need the following useful Lemma:

Lemma 1:

Let \( x(t), \lambda(t) \) and \( f(u) \) be absolutely continuous and non decreasing functions on \([a, b]\) for \( 0 \leq a \leq b < \infty \) with \( f(t) > 0 \). Let \( \alpha, \beta, k \) and \( \varepsilon \) be real numbers such that \( \delta, \geq 0, \varepsilon \geq 0 \) and also Let \( P(x), \) and \( R(t) \) be non negative and measurable function on \([a, b]\) such that

\[
|x'(t)| \times f \left( \int_a^b x'(t) R(t) \, d\lambda(t) \right) \leq \lambda(t)^{1-\delta-\varepsilon} y(t) \delta R(t)^{1-\varepsilon} \lambda'(t)^{-1} y'(t).
\]

Then, the following inequality holds:

\[
\int_a^b |x'(t)| \times f \left( \int_a^t |x'(t)| \, dt \right) \leq \int_a^b y(t)^{\varepsilon} \, dy'(t).
\]
Proof:

Setting $h(s, t) = x'(t)R(t)$ in (5), we have

\[
(f(t))^\epsilon = \left( f\left(\int_0^t |x'(t)R(t)| d\lambda(t)\right) \right)^\frac{1}{\epsilon} \leq \left( \int_0^t \frac{|x'(t)R(t)|}{\lambda(t)} d\lambda(t) \right)^\frac{1}{\epsilon}. \tag{8}
\]

By setting $f(\lambda(t)) = \lambda(t)^{1-\delta}$ in (8) yeilds

\[
\frac{f\left(\int_0^t |x'(t)R(t)| d\lambda(t)\right)}{\lambda(t)^{1-\delta}} \leq \frac{\left( \int_0^t |x'(t)R(t)| d\lambda(t) \right)^\frac{1}{\epsilon}}{\lambda(t)^{1-\delta}}. \tag{9}
\]

Hence

\[
f\left(\int_0^t |x'(t)R(t)| d\lambda(t)\right) \leq \lambda(t)^{1-\delta-\epsilon} \left( \int_0^t |x'(t)R(t)| d\lambda(t) \right)^\frac{1}{\epsilon} \lambda(t)^{1-\delta} y(t)^{\epsilon}. \tag{10}
\]

Now let

\[
y(t) = \int_0^t |x'(t)R(t)|^{\frac{1}{1-\delta}} \lambda'(t) \tag{11}
\]

then

\[
y'(t) = f\left(\int_0^t |x'(t)R(t)|\right)^{\frac{1}{1-\delta}} \lambda'(t). \tag{12}
\]

That is,

\[
y'(t)^{1-\delta} = f\left(\int_0^t |x'(t)R(t)|\right) \lambda'(t)^{1-\delta}. \tag{13}
\]

Using the fact that $f(u) = u^{1-\delta}$ to have

\[
y'(t)^{1-\delta} = |x'(t)|^{1-\delta} R(t)^{1-\delta} \lambda'(t)^{1-\delta}. \tag{14}
\]

\[
|x'(t)| = R(t)^{-\delta} \lambda'(t)^{-\delta} y'(t), \tag{15}
\]

Combining both (10) and (15) to yeilds, inequality (6) and the proof is complete.

\[
|x'(t)| f\left(\int_0^t |x'(t)R(t)| d\lambda(t)\right) \leq \lambda(t)^{1-\delta-\epsilon} y(t)^{\delta} R(t)^{1-\delta} \lambda'(t)^{-\delta} y'(t)
\]

Remarks 1:

By setting $f(u) = u^{1-\delta}, R(t) = P(t)^{-\frac{1}{\delta-1}}, \lambda'(t)^{\frac{1}{\delta-1}}, 1 - \delta = \epsilon$ in Lemma 1 yeilds

\[
|x'(t)| f\left(\int_0^t |x'(t)R(t)| d\lambda(t)\right) \leq \lambda(t)^{1-\delta-\epsilon} y(t)^{\delta} \times P(t)^{1-\frac{1}{\delta-1}} \lambda'(t)^{\frac{1}{\delta-1}} y'(t). \tag{16}
\]

Integrating both sides of inequality (16) over $[a, b]$ with the respect to $t$, to get
\[ \int_a^b |x'(t)| \times f \left( \int_a^b |x'(t)| \, dt \right) \leq \int_a^b y(t)^\varepsilon y'(t) \, dt. \quad (17) \]

That is,
\[ \int_a^b |x'(t)| \times f \left( \int_a^b |x'(t)| \, dt \right) \leq \int_a^b y(t)^\varepsilon dy'(t). \quad (18) \]

\[ \int_a^b |x'(t)| \times f \left( \int_a^b |x'(t)| \, dt \right) \leq \frac{y(t)^{\varepsilon+1}}{\varepsilon + 1} \quad (19) \]

Setting \( \int_0^t |x'(t)| \, dt = x(t) \) By using H"older's inequality with \( \alpha \) and \( \beta \) we obtain
\[ \frac{1}{\varepsilon + 1} y(b)^{\varepsilon+1} = \frac{1}{\varepsilon + 1} \left( \int_a^b |x'(t)| \, dt \right)^{\varepsilon+1} = \frac{1}{\varepsilon + 1} \left( \int_a^b R^{\varepsilon+1} (t) R^{\varepsilon+1} |x'(t)| \, dt \right)^{\varepsilon+1} \]

\[ \leq \frac{1}{\varepsilon + 1} \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon} \left( \int_a^b R(t) |x'(t)|^\alpha \, dt \right)^{\varepsilon+1} \quad (20) \]

Combining inequality (19) and (20) to obtain the Opial's Type inequality of the following
\[ \int_a^b x'(t) f(x(t)) \, dt \leq \frac{1}{\varepsilon + 1} \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon+1} \left( \int_a^b R(t) |x'(t)|^\beta \, dt \right)^{\varepsilon+1} \quad (21) \]

which gives
\[ \int_a^b |x'(t)| x(t)^{-\delta} \, dt \leq \frac{1}{\varepsilon + 1} \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon+1} \left( \int_a^b R(t) |x'(t)|^\beta \, dt \right)^{\varepsilon+1} \quad (22) \]

**Remark 2:**

For \( \varepsilon = 0 \) in inequality (22) yields
\[ \int_a^b |x'(t)| x(t)^{\varepsilon+1} \, dt \leq \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon} \left( \int_a^b R(t) |x'(t)|^\beta \, dt \right)^{\varepsilon+1} \quad (23) \]

Putting \( \varepsilon = 1 \), and \( \delta = 0 \) in inequality (22) reduces to
\[ \int_a^b |x'(t)| x(t) \, dt \leq \frac{1}{2} \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon} \left( \int_a^b R(t) |x'(t)|^\beta \, dt \right)^{\varepsilon+1} \quad (24) \]

which generalizes inequality (2).
If \( \varepsilon = 1 \) and \( \alpha = 0 \) in inequality (22) yields
\[ \int_a^b |x'(t)| x(t) \, dt \leq \frac{1}{2} \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon} \left( \int_a^b R(t) |x'(t)|^\beta \, dt \right)^{\varepsilon+1} \quad (25) \]

If \( \varepsilon = 0 \) in inequality (22) becomes
\[ \int_a^b |x'(t)| x(t) \, dt \leq \left( \int_a^b R^{\varepsilon+1} (t) \, dt \right)^{\varepsilon} \left( \int_a^b R(t) |x'(t)|^\beta \, dt \right)^{\varepsilon+1} \quad (26) \]

In inequality (18) if we set \( 1 - \delta = \varepsilon \) becomes
\[ \int_a^b |x'(t)| \times f \left( \int_a^b |x'(t)| \, dt \right) \leq \int_a^b y(t)^{-\delta} y'(t) \, dt. \quad (27) \]
That is,
\[ \int_a^b |x'(t)| \times f\left(\int_a^b |x'(t)| \, dt\right) \leq \int_a^b f(y(t)) \, dy(t). \]  \tag{28} 

Getting \( \int_a^b |x'(t)| \, dt = x(t) \)

Using Hölder’s inequality with indices \( \alpha \) and \( \beta \), we have
\[ \int_a^b x'(t) \, dt = \int_a^b R^{1-\alpha}(t) R^{\beta} |x'(t)| \, (t) \, dt \leq \left( \int_a^b R^{1-\alpha}(t) \, dt \right)^{\frac{1}{\alpha}} \left( \int_a^b R(t) \, |x'(t)|^\beta \, dt \right)^{\frac{1}{\beta}} \]  \tag{29} 

Combining (28) and (29) to obtain the following inequality
\[ \int_a^b |x'(t)| \times f\left(\int_a^b |x'(t)| \, dt\right) \leq f\left(\left( \int_a^b R^{1-\alpha}(t) \, dt \right)^{\frac{1}{\alpha}} \left( \int_a^b R(t) \, |x'(t)|^\beta \, dt \right)^{\frac{1}{\beta}}\right) \]  \tag{30} 

that is, inequality that generalizes inequality (3)
\[ \int_a^b |x'(t)| f(x(t)) \, dt \leq f\left(\left( \int_a^b R^{1-\alpha}(t) \, dt \right)^{\frac{1}{\alpha}} \left( \int_a^b R(t) \, |x'(t)|^\beta \, dt \right)^{\frac{1}{\beta}}\right) \]  \tag{31} 

Similarly, we need the following Lemma to obtain a new Opial’s type inequality using Jensen’s inequality for the case of convex function.

**Lemma 2:**
Let \( x(t), \lambda(t), f(u), R(t), I, k \) and \( o \geq 0 \) and \( \rho \geq 0 \) be as in Lemma 1 such that
\[ |x'(t)| \times f\left(\int_0^t |x'(t)| R(t) \, d\lambda(t)\right) \leq y'(t)R(t)^{-1}\lambda'(t)^{-1}y(t)^{\delta^{-1}} \]  \tag{32} 

Then, the following inequality holds:
\[ |x'(t)| f\left(\int_0^t |x'(t)| \, dt\right) \leq y(t)^{\delta^{-1}} y'(t). \]  \tag{33} 

**Proof:**
The proof is similar to the proof of Lemma 1.
Since \( f(u) = u^{\delta^{-1}} \), inequality (8) becomes
\[
\left( f\left( \int_0^t |x'(t)| R(t) \, d\lambda(t) \right) \right)^{\frac{1}{\delta^{-1}}} \leq \left( \int_0^t f\left( |x'(t)| R(t) \right) \, d\lambda(t) \right)^{\frac{1}{\delta^{-1}}}. \]  \tag{34} 

\[
f\left( \int_0^t |x'(t)| R(t) \, d\lambda(t) \right) \leq \left( \int_0^t f\left( |x'(t)| R(t) \right) \, d\lambda(t) \right)^{\frac{1}{\delta^{-1}}}. \]  \tag{35} 

\[ f\left(\int_a^b x'(t)R(t)\,d\lambda(t)\right) \leq \lambda(t)\delta^{1+1-\delta} \left(\int_0^t f\left(\left| x'(t)R(t) \right| \right) \frac{1}{\delta} \, d\lambda(t)\right)^{\delta-1} . \]  

(36)

\[ f\left(\int_a^b x'(t)R(t)\,d\lambda(t)\right) \leq \left(\int_0^t f\left(\left| x'(t)R(t) \right| \right) \frac{1}{\delta} \, d\lambda(t)\right)^{\delta-1} . \]  

(37)

\[ y'(t) = |x'(t)| \, R(t) \lambda'(t) \]  

(38)

\[ y'(t) R(t)^{-1} \lambda'(t)^{-1} = |x'(t)| \]  

(39)

Combining (37) and (39) to have

\[ |x'(t)| \times f\left(\int_a^b x'(t)R(t)\,d\lambda(t)\right) \leq y'(t) R(t)^{-1} \lambda'(t)^{-1} y(t)^{\delta-1} \]  

(40)

This completes the proof of the Lemma.

Consider all conditions of Remark 1

\[ |x'(t)| \times f\left(\int_a^b x'(t)P(t)^{-\frac{1}{\delta}} \, P(t)^{\frac{1}{\delta}} dt\right) \leq y'(t) y(t)^{\delta-1} P(t)^{\frac{1}{\delta-1}} P(t)^{\frac{1}{\delta}} y(t) . \]  

(41)

\[ |x'(t)| f\left(\int_a^b |x'(t)| \, dt\right) \leq y(t)^{\delta-1} y'(t) \]

Then, putting \( \int_a^b |x'(t)| \, dt = x(t) \) and integrate both side of inequality above, over \([a,b]\) with the respect to \( t \), yields

\[ \int_a^b |x'(t)| x(t)^{\delta-1} \, dt \leq \int_a^b y(t)^{\delta-1} y(t) \, dt = \frac{1}{\delta} y(b)^{\delta} \]  

(42)

\[ = \frac{1}{\delta} \left(\int_a^b |x'(t)| \, P(t)^{-\frac{1}{\delta}} \, P(t)^{\frac{1}{\delta}} dt\right)^{\frac{1}{\delta}} \]  

(43)

\[ = \frac{1}{\delta} \left(\int_a^b P(t)^{-\frac{1}{\delta}} \, dt\right)^{\frac{1}{\delta}} \left(\int_a^b |x'(t)| \, P(t)^{\frac{1}{\delta}} \, dt\right)^{\frac{1}{\delta}} . \]  

(44)

We are able to generalized inequality (2), (3) and some remarks on Opial- type inequalities.

**REFERENCES**


