SOME OPTIMALITY CONDITIONS FOR THE EXISTENCE OF OPTIMIZERS OF CERTAIN CLASS OF LINEAR PROGRAMMING PROBLEMS

M. U. UMOREN

(Received 16 December 1999, Revision accepted 6 October 2000)

Abstract

It is well known that the optimum of a Linear Programming problem occurs at an extremum point of the feasible region. This paper considers some other optimality conditions for the existence of optimizers of Linear Programming (LP) problems based on the principles of optimal experimental design. It is shown in this work, for example, that:

(i) The optimizer of an LP Problem occurs at a point, which the d-function is minimum within the feasible region.

(ii) The d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space.

(iii) The d-function at the minimizer $x^*$ is the maximum of the minimum d-functions for $k$ different iterations. This is a first-order necessary condition for the existence of a minimizer of an LP problem.

Key words: Linear Programming, d-function, Linear Exchange Algorithm, Optimal Experimental Design.

Introduction

Linear programming (LP) problems are characterised by a linear objective function in $n$ non-negative variables constrained by a set of $m$ (usually, $m < n$) linearly independent equations. These constraints which may be linear equality or linear inequality define a convex feasible region.

We remark that since the objective function of an LP problem is non-stochastic, we shall throughout this work denote $x^* M^i (\xi^n), x \in S_x$ to be the d-function rather than variance of the objective function; $S_x$ is the column space of the information matrix at the $k$th iteration; i.e $d(f(x)) = d(x, \xi^n) = x^* M^i (\xi^n) x$, $x \in S_x \subset X$ where $\xi^n$ is the design measure of the exact design and $X$ is the feasible region.

We further remark that in developing the optimality conditions, some restriction has been placed on the objective function $f(x)$; i.e $f(x) \geq 0$, $x \in X$. Thus, LP problems for which $f(x)$ can change sign from positive to negative or from negative to positive within the feasible region are not covered here.

Each of the optimality conditions discussed in this paper is based on the Linear Exchange Algorithm (LEA), itself a line search algorithm, for solving Linear Programming Problems developed by making use of the principles of experimental design. The basic steps involved in the LEA are given as follows:

1. At the boundary of $X$, take $n_0$ support points for the initial design matrix $X_{00} = \{x_{10}, x_{20}, \ldots, x_{m0}, \ldots, x_{n0}\}$

2. Such that $\det (X_{00}^t X_{00}) = 0$ and $n + 1 \leq n_0 \leq \frac{n}{2} (n + 1)$

3. Make a move in the gradient direction to the point

$$
\begin{align*}
x_k & = x_{0k} - \alpha k g, \quad k = 1, 2, \ldots, n, \\
g & = \left( \frac{\partial f(x)}{\partial x} \right)_{x \in S_x}, \quad j = 1, 2, \ldots, n, \\
\alpha_{0k} & = \min \left( \frac{a^t (x_{0k} - b)}{a^t g} \right), \quad j = 1, \ldots, m
\end{align*}
$$

M. U. UMOREN, Department of Mathematics, Statistics, and Computer Science, University of Uyo, Uyo, Nigeria
Sz: if \( x_k = x^* \), the minimizer, stop otherwise replace \( x_{mk} \) in \( X_0 \) with \( x_k \); i.e. define
\[
X_{m+1} = (x_1, x_2, \ldots, x_{m+1}, \ldots, x_{n+1})'
\]
and \( x_{m+1} = x_{m+1}/n \), where \( x_{mk} \) is such that \( f(x_{mk}) > f(x_k), i = 1, 2, \ldots, n \); \( i \neq m \).

Sz: set \( k + 1 = k \) and return to step Sz.

The sequence terminates whenever
\[
\frac{|f(x_{k+1}) - f(x_k)|}{|f(x_k)|} < \epsilon, \epsilon > 0
\]
or
\[
|x_0 - M_k x_k| < \delta, \delta > 0;
\]

\( M_k \) is the information matrix at the \( k \)th iteration. That is, the sequence terminates whenever the difference between the d-function at the starting point \( x_0 \) and the d-function at \( x_k \), the end point of the \( k \)th iteration is small; i.e. no significant move is made from \( x_0 \) to \( x_k \). A detailed discussion on the operation of this algorithm is given in Umoren (2000).

In developing the optimality conditions, we require the following lemma that is useful in obtaining the inverses and determinants of information matrices for line search algorithms.

**Lemma 1.1**: Let \( B = A + uv^T \) be an \( m \times m \) matrix such that \( \det(A) \neq 0 \), \( u \) and \( v \) are \( m \) component non-zero vectors, then
\[
B^{-1} = A^{-1} - A^{-1}u'(1 + v'A^{-1}u)^{-1}v'A^{-1} \\
(1.1.1)
\]
and
\[
\det(B) = \det(A)(1 + v'A^{-1}u) \\
(1.1.2)
\]

The above lemma in linear algebra has been referred to or proved by many authors including Rao (1965), Raghavarao (1971), hence the proof is here omitted.

**Lemma 1.2**: Given the line search equation
\[
x_{j+1} = x_j - \rho_0 d_j; \quad d_j = 1
\]
where \( d_j \) and \( \rho_0 \) are respectively the direction of search and step length at the \( j \)th iteration.

Let \( M_{ij}, M_{j+1}, \in M^{nxn} \) be information matrices at the \( j \)th and \( (j+1) \)th iteration respectively, \( M^{xn} \) is the set of all information matrices. Then

(i) \[
det(M_{j+1}) = det(M_j)(1 + n\rho_0^2 d_j'M_j^{-1}d_j) \quad \text{or}
\]
\[
det(M_{j+1}) = 1 + w_j; \quad w_j = n\rho_0^2 d_j'M_j^{-1}d_j \quad \text{det}(M_j)
\]

(ii) \[
M_{i+1}^{-1} = M_i^{-1} - \frac{x_i}{\rho \circ d_i}; \quad \rho = n\rho_0^2 d_i'M_i^{-1}d_i
\]

**Proof**: (i) At \( x_k \), the column space of the design matrix \( X_k \) is spanned by the vector \( x_k = (x_1^*, x_2^*, \ldots, x_n^*) \); i.e. \( x_k \in L(X_k) \). Similarly at \( x_{k+1} \) the column space of the design matrix \( X_{k+1} \), \( (n \times n) \) is spanned by the vector \( x_{k+1} = (x_{k+1,1}, x_{k+1,2}, \ldots, x_{k+1,n}) \). Therefore
\[
x_{k+1} = (x_1 - \rho d_1, x_2 - \rho d_2, \ldots, x_n - \rho d_n)
\]
and
\[
x_{k+1}X_{k+1}' = X_kX_k' - \rho X_k1d_1' - \rho X_k1d_1'I - \rho^2 n\rho d_i'd_i'
\]
\[
= X_kX_k' + \rho^2 n\rho d_i'd_i' - \rho^2 n\rho d_i'd_i' \\
i.e. \quad M_{k+1} = M_k + \rho^2 n\rho d_i'd_i'
\]

or
\[
det(M_{k+1}) = det(M_k)(1 + \rho^2 n\rho d_i'd_i')
\]

and
\[
det(M_{k+1}) = 1 + w_j; \quad w_j = \rho^2 n\rho d_i'd_i'
\]

On application of (1.1.2) in (1.2.1) we have
\[
M_i^{-1} = M_i^{-1} - \frac{n\rho_0^2 d_i'd_i'M_i^{-1}d_i'}{1 + \rho_0^2 d_i'd_i'}
\]
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Definition 1.3: Let \( f(x) \) be the objective function of an LP problem and let \( x^* \) be the optimizer (maximizer or minimizer), \( x \in X \). Then

\[
f(x^*) \begin{cases} 
\geq f(x), & \text{if } f(x) \text{ is to be maximized} \\
\leq f(x), & \text{if } f(x) \text{ is to be minimized}
\end{cases}
\]

Pazman (1986) has established a functional relationship between the square of a response function and the variance of the function. That is, the variance of the BLUE for a linear functional \( h \) defined on \( \Omega \) is

\[
\text{Var}(h) = \text{Var}(y_n M_n y) = y^* M^1 y
\]

\[
= \max \left\{ (u^* \hat{a})^2, \quad u, \hat{a} \in R^n, M \hat{a} \neq 0 \right\}
\]

(1.1.4)

\[
= \max \left\{ 2u^* \hat{a} - \hat{a}^* M \hat{a}, \quad u, \hat{a} \in R^n \right\}
\]

(1.1.5)

if \( h(w) = u^* w, \quad w \in \Omega \), where \( y_n \in L(M) \) and \( u = M y_n \Rightarrow y_n = M^{-1} u \).

Correspondingly, since the objective function of an LP problem, namely \( f(x) = c^* x \) is a linear functional, the relationship between its square and its d-function can be derived from (1.1.4) and (1.1.5) as

\[
df(x) = x^* M x = \max \left\{ (c^* x)^2, \quad c, x \in R^n, M c \neq 0 \right\}
\]

\[
= \max \left\{ \frac{f^2(x)}{c^* M c}, \quad c, x \in R^n, M c \neq 0 \right\}
\]

(1.1.6)

\[
= \max \left\{ 2c^* x - c^* M c, \quad c, x \in R^n \right\}
\]

(1.1.7)

2.1 Optimality Conditions

It is well known from the point of view of mathematics that, at the optimizer \( x^* \), the gradient of the objective function vanishes; i.e.

\[
\frac{\partial f(x)}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n
\]

From definition (1.3), if \( x^* \) is a minimizer and \( f(x) \geq 0 \), then

\[
f(x^*) \leq f(x) \iff f^2(x^*) \geq f^2(x)
\]

(2.1.1)

Let us now lay the foundation for the development of the optimality conditions. We first show that in the LEA, the sequence moves from a point of relatively high d-function to a point of lower d-function.

Theorem 2.2: Given the line search sequence

\[
x_k = \bar{x}_0 - \alpha c g; \quad \bar{x}_0 = X_0 ^{1/n_0}
\]

Then

\[
x_k M_k x_k \leq \bar{x}_0 M_k \bar{x}_0, \quad x_k \in S_k
\]

Where \( x_k \) and \( M_k \) are the end points and information matrix at the kth iteration and \( S_k \) is the column space of the information matrix at the kth iteration.

Proof: \( x_k = \bar{x}_0 - \alpha c g \)

\[
x_k M_k x_k = (\bar{x}_0 - \alpha c g) M_k (\bar{x}_0 - \alpha c g)
\]

\[
= \bar{x}_0 M_k \bar{x}_0 - 2 \alpha c \bar{x}_0 M_k g + \alpha^2 c^2 g M_k g
\]

\[
= \bar{x}_0 M_k \bar{x}_0 - 2 \bar{x}_0 ^* \bar{x}_0 + c ^* M_k c
\]
on setting \( \alpha \hat{\mathbf{M}}^{-1} \mathbf{g} = \mathbf{c} \), the coefficient of the objective function.

R.H.S.

\[
\Rightarrow \begin{align*}
\mathbf{x}_{\alpha k} \cdot \hat{\mathbf{M}}_{\alpha k}^{-1} \mathbf{x}_{\alpha k} & \geq 2\mathbf{c} \cdot \mathbf{x}_{\alpha k} - \mathbf{c} \cdot \mathbf{Mc} \\
\Rightarrow \mathbf{x}_{\alpha k} \cdot \hat{\mathbf{M}}_{\alpha k}^{-1} \mathbf{x}_{\alpha k} &= \max \{ 2\mathbf{c} \cdot \mathbf{x}_{\alpha k} - \mathbf{c} \cdot \mathbf{Mc} \} \geq 0
\end{align*}
\]

from (1.1.5). Therefore,

\[
\mathbf{x}_k \cdot \hat{\mathbf{M}}^{-1} \mathbf{x}_k \leq \mathbf{x}_{\alpha k} \cdot \hat{\mathbf{M}}_{\alpha k}^{-1} \mathbf{x}_{\alpha k}
\]

We now show that if the starting point of the sequence is a weighted average, then the value of the objective function at the kth iteration is less than its value for any other \( \mathbf{x} \in S \subset \tilde{\mathbf{X}} \). But before that let us state the following lemma that is fundamental to the proof of the theorem that follows.

**Lemma 2.3:** Given

\[
\mathbf{x}_i \cdot \hat{\mathbf{M}}_{i}^{-1} \mathbf{x}_i \leq \mathbf{x}_i \cdot \hat{\mathbf{M}}_{i}^{-1} \mathbf{x}_j, \quad \mathbf{x}_i, \mathbf{x}_j \in S_i
\]

Then

\[
f_i(\mathbf{x}_i) \leq f_i(\mathbf{x}_j) \iff f_i(\mathbf{x}_i) \leq f_i(\mathbf{x}_j)
\]

where \( \mathbf{M}_i \) is the information matrix at the kth iteration.

**Proof:**

\[
\begin{align*}
\mathbf{x}_i \cdot \hat{\mathbf{M}}_{i}^{-1} \mathbf{x}_i & \leq \mathbf{x}_j \cdot \hat{\mathbf{M}}_{j}^{-1} \mathbf{x}_j \\
\Rightarrow \text{tr}(\mathbf{M}_i^{-1} \mathbf{x}_i \mathbf{x}_i^\top) & \leq \text{tr}(\mathbf{M}_j^{-1} \mathbf{x}_j \mathbf{x}_j^\top) \\
\Rightarrow \mathbf{x}_i \mathbf{x}_i^\top & \leq \mathbf{x}_j \mathbf{x}_j^\top \\
\Rightarrow f_i(\mathbf{x}_i) & \leq f_j(\mathbf{x}_j) \iff f_i(\mathbf{x}_i) \leq f_i(\mathbf{x}_j)
\end{align*}
\]

**Theorem 2.3:** Given the line search sequence

\[
\mathbf{x}_k = \mathbf{x}_{\alpha k} - \alpha d_k, \quad \mathbf{x}_{\alpha k} = \sum \alpha_i \mathbf{x}_i, \quad \alpha \geq 0; \quad \sum \alpha_i = 1
\]

and \( \mathbf{x}_i \)'s are independent. Then

\[
f(\mathbf{x}_k) \leq f(\mathbf{x}_i), \quad \mathbf{x} \in S_k
\]

**Proof:** The proof is by induction

Set \( n_0 = 2 \)

\[
\mathbf{x}_{\alpha k} = \alpha \mathbf{x}_1 + (1 - \alpha) \mathbf{x}_2, \quad \alpha \geq 0
\]

From Theorem 2.2

\[
d_k = \mathbf{x}_k \cdot \hat{\mathbf{M}}_k^{-1} \mathbf{x}_k \leq d_{10k} = \mathbf{x}_{\alpha k} \cdot \hat{\mathbf{M}}_{\alpha k}^{-1} \mathbf{x}_{\alpha k}
\]

\[
d_{10k} = \alpha^2 \mathbf{x}_1 \cdot \hat{\mathbf{M}}_1^{-1} \mathbf{x}_1 + (1 - \alpha)^2 \mathbf{x}_2 \cdot \hat{\mathbf{M}}_2^{-1} \mathbf{x}_2
\]

On setting \( \mathbf{x}_i \cdot \hat{\mathbf{M}}_i^{-1} \mathbf{x}_i = \mathbf{d}_i, \quad i = 1, 2 \), we have

\[
d_{10k} = \alpha^2 \mathbf{d}_1 + (1 - \alpha)^2 \mathbf{d}_2
\]

Choose \( \alpha \) to minimize \( d_{10k} \); i.e.

\[
\frac{\partial d_{10k}}{\partial \alpha} = 2\alpha \mathbf{d}_1 + 2(1 - \alpha) \mathbf{d}_2 = 0
\]

\[
\alpha = \frac{d_2}{d_1 + d_2}
\]

Therefore

\[
d_{10k} = \left( \frac{d_2}{d_1 + d_2} \right)^2 \mathbf{d}_1 + \left( \frac{d_1}{d_1 + d_2} \right)^2 \mathbf{d}_2
\]

\[
= \left( \frac{d_1 d_2}{(d_1 + d_2)^2} \right) (d_1 + d_2) = \frac{d_1 d_2}{d_1 + d_2} \leq d_1, \quad d_2
\]

Hence \( d_k \leq d_1, \quad d_2 \)

From lemma 2.3 the above inequality implies

\[
f_k(\mathbf{x}_k) \leq f_k(\mathbf{x}_i), \quad f_k(\mathbf{x}_i)
\]

Set \( n_0 = 3 \)

Define \( \mathbf{x}_{10k} = \alpha \mathbf{x}_1 + (1 - \alpha) \mathbf{x}_2 \), and \( \mathbf{x}_{20k} = \beta \mathbf{x}_1 + (1 - \beta) \mathbf{x}_2, \beta \geq 0 \).

Then

\[
d_{20k} = \beta^2 \mathbf{d}_1 + (1 - \beta)^2 \mathbf{d}_2
\]

The value of \( \beta \) which minimizes \( d_{20k} \) is \( \beta = \frac{d_2}{d_1 + d_2} \). Therefore
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\[ d_{23} = \left( \frac{d_3}{d_1 + d_3} \right)^2 d_{12} + \left( \frac{d_{12}}{d_1 + d_3} \right)^2 d_3 \]

\[ = \frac{d_{12} d_3}{(d_1 + d_3)^2} \approx d_{12}, d_3 \]

but \( d_{23} \leq d_{12} = \alpha^2 d_1 + (1 - \alpha)^2 d_2 \leq d_1, d_2 \)

Hence,

\[ d_{23} \leq d_1, d_2, d_3 \Rightarrow d_0 \leq d_1, d_2, d_3 \]

From Lemma 2.3, the above inequality implies

\[ f(x_0) \leq f(x_1), f(x_2), f(x_3) \]

Hence, by induction

\[ f(x_k) \leq f(x_{k-1}), f(x_{k-2}), \ldots, f(x_0) \]

We now show that if the value of the objective function is minimum at \( x_k \) within the experimental space \( S_x \), then the \( d \)-function at \( x_k \) is minimum for all \( x \in S_x \).

Theorem 2.4: Given a line search equation

\[ \bar{x}_k = x_k^* - \alpha \nabla f \quad ; \quad x_k = \sum x_k \alpha, \quad \alpha \geq 0 \]

Then

\[ \bar{x}_k' M_k^{-1} x_k \leq x_k' M_k^{-1} x_k, \quad x_k \in S_x \subset \bar{X} \]

Proof: Define \( f(x_k) = c' x_k \quad \Leftrightarrow \quad f^2(x_k) = x_k' c c' x_k \) and

\[ f(x) = c' x \quad \Leftrightarrow \quad f^2(x) = x' c c' x \]

From Theorem 2.2,

\[ f(x_k) \leq f(x) \quad \Leftrightarrow \quad f^2(x_k) \leq f^2(x), \quad x \in S_x \]

\[ \Rightarrow x_k' c c' x_k \leq x' c c' x \]

\[ \Rightarrow c' x_k x_k' c \leq c' x x' c \]

\[ \Rightarrow x_k x_k' \leq x x' \]

\[ \Rightarrow \text{tr}(x_k x_k' M_k^{-1}) \leq \text{tr}(x x' M_k^{-1}) \]

\[ \Rightarrow x_k' M_k^{-1} x_k \leq x' M_k^{-1} x, \quad x \in S_x \]

Having in view the functional relationship between the square of the objective function and its \( d \)-function (see 1.1.6) we give the following theorem which states that the minimizer of an LP problem occurs at the point where the \( d \)-function is minimum within the feasible region.

Theorem 2.5: Given \( M^* = \lim_{k \to \infty} M_k \), where \( M^* \) is the information matrix at the point

\[ x_k \]

of convergence of the sequence. If \( x^* \) is a minimizer, then

\[ x^* M^*^{-1} x^* \leq x_k' M_k^{-1} x_k, \quad x \in \bar{X} \]

and if \( x^* \) is a maximizer

\[ x^* M^*^{-1} x^* \geq x_k' M_k^{-1} x_k, \quad x \in \bar{X} \]

Proof: Let \( f(x^*) \) be represented by a linear functional \( x^* c \). Then

\[ f^2(x^*) = x^* c c' x^*; \quad f^2(x) = x' c c' x \]

\[ f(x_k) \leq f(x) \quad \Leftrightarrow \quad x_k' c c' x_k \leq x' c c' x \quad \text{(i)} \]

Define \( f(x) = x c \), where \( X \) is the design matrix, so that \( c = (X' X)^{-1} X' f(x) \).
Then (i) becomes
\[ x^* - (X^*X)^{-1}X^*f(x^*) \leq x^* - (X^*X)^{-1}X^*f(x) \leq (X^*X)^{-1}x \]
\[ \Rightarrow x^* - (X^*X)^{-1}x^* \leq x^* - (X^*X)^{-1}x \]
\[ \Rightarrow x^* - M^{-1}x^* \leq x^* - M^{-1}x \]
\[ \Rightarrow x^* - M^{-1}x^* \leq x^* - M^{-1}x \quad , x \in \bar{X} \]

Similarly, \( f^*(x^*) \geq f^*(x) \)
\[ \Rightarrow x^* - M^{-1}x^* \geq x^* - M^{-1}x \quad , x \in \bar{X} \]

We now state a first order necessary condition for the existence of optimizers of LP problems. That is, we show that if the d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space, then the d-function at the minimizer \( x^* \) is the maximum of the minimum d-functions at \( k \) different iterations. But before that, let us state a lemma that is fundamental to the proof of the theorem that follows:

**Lemma 2.6:** Given the line search sequences
\[ x_k = x_0 - \alpha_k g_k, \quad x_0 = X_0 \cdot 1/n0 \]
and
\[ x_{k+1} = x_k + \beta_k g_{k+1}, \quad x(0+1) = X_0 \cdot 1/n0 \]

Then \( M_k \geq M_{k+1} \). Then
\[ x^* \geq x_k \]

**Proof:** By the exchange rule \( x_{k+1} \leq x_k \).

Let
\[ x_{k+1} = x_k - \alpha_k \]

Then
\[ x^* \geq x_{k+1} = x_k - \alpha_k \]

On setting \( c = 2\alpha_k M_k \) we have
\[ x^* \geq x_{k+1} = x_k - \alpha_k \]

R.H.S. \( \Rightarrow x^* \geq x_{k+1} = 2\alpha_k x_k - c \cdot M_k \]

\[ \Rightarrow \max \{ 2\alpha_k x_k - c \cdot M_k \}], \quad c \in \bar{R}^n \]

Therefore
\[ x^* \geq x_{k+1} = x_k - \alpha_k \]

**Theorem 2.6:** First - Order Necessary Condition.

Given \( x^* \) to be a minimizer of an LP problem, Then,
\[ x^* - M^{-1}x^* = \max_{k} \{ x^* - M_k^{-1}x, \quad x \in \bar{X} \} \]

is a first - order necessary condition to be satisfied by \( x^* \); \( M \) is the information matrix at the point where \( x_k \) converges to \( x^* \), \( x_k \) and \( M_k \) are the end point and information matrix at the kth iteration.

**Proof:** Let \( \lim_{k \to \infty} M_k = M^{-1} \)

i.e \( M_k \geq M_2 \geq \ldots \ldots \geq M^{-1} \) and let
\[ M_k = M_{k-1} + \delta_k d_k, \quad \delta_k \geq 0 \]

From lemma 1.2.
\[ M_k^{-1} = M_{k-1}^{-1} - \frac{\delta_k d_k}{\lambda_k} \]
\[ v_k = \frac{n \lambda^2 \alpha_k}{(1 + \lambda_k)^2} M_k \cdot d_k, \quad v_k = \frac{n \lambda^2 \alpha_k}{(1 + \lambda_k)^2} M_k \cdot d_k \]
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\[ x_k \cdot M_k \cdot x = x_k \cdot [M \cdot x_k \cdot x_k] \]
\[ \Rightarrow x_k \cdot M_k \cdot x_k = x_k \cdot M_k \cdot x_k - x_k \cdot \nu x_k \cdot x_k \]
\[ \Rightarrow x_k \cdot M_k \cdot x_k \geq x_k \cdot M_k \cdot x_k \]

since \( x_k \cdot \nu x_k \cdot x_k \geq 0 \)
\[ \Rightarrow x^* \cdot M_k \cdot x^* \geq x_k \cdot M_k \cdot x_k \]

since, from lemma 2.6, \( x^* \cdot M_k \cdot x_k \cdot x_k \cdot x_k \cdot x_k \cdot x_k \), then,
\[ x^* \cdot M_k \cdot x_k = \max(x_k \cdot M_k \cdot x_k) \]

But from Theorem 2.4
\[ x_k \cdot M_k \cdot x_k = \min(x_k \cdot M_k \cdot x_k), \quad x \in S_k \]

Therefore
\[ x^* \cdot M_k \cdot x^* = \max \min(x_k \cdot M_k \cdot x_k, \quad x \in \bar{X}) \]

Using the fact that
\[ \min f(x) = \max(-f(x)) \]
we have
\[ x^* \cdot M_k \cdot x^* = \min \max(x_k \cdot M_k \cdot x_k, \quad x \in \bar{X}) \]

as a first - order necessary condition to be satisfied for the existence of optimizers of LP problems. That is, the optimizer of an LP problem has both the maxmin and minmax properties, which could be reached either through minimization or maximization.

### 3.0 Conclusion: Illustration

The optimality conditions considered in this paper are here demonstrated with an illustration. It is here reminded that the development of the optimality conditions are based on the Linear Exchange Algorithm (LEA), a method of solving LP problems. Applying the basic steps of the LEA to the problem

\begin{align*}
\text{minimize} & \quad f(x) = 3x_1 + 2x_2 \\
\text{subject to} & \quad 2x_1 + x_1 \geq 6 \\
& \quad x_1 + x_2 \geq 4 \\
& \quad x_1 + 2x_2 \geq 6 \\
& \quad x_1, x_2 \geq 0
\end{align*}

(see Umoren, 2000), we give Table 3.1 for \( k = 1,2,3,4 \) iterations.

| Table 3.1: d functions at the support points and end points for four different iterations of the Linear Exchange Algorithm |
|------------------|------------------|------------------|------------------|
| \( x_1 \) | \( x_2 \) | \( d(x, \bar{x}) \) | \( x_1 \) | \( x_2 \) | \( d(x, \bar{x}) \) |
|------------------|------------------|------------------|------------------|
| 1.00 | 4.00 | 0.2625 | 1.00 | 4.00 | 0.3152 |
| 3.00 | 1.50 | 0.3124 | 3.00 | 1.50 | 0.2752 |
| 4.00 | 1.00 | 0.4252 | 2.24 | 1.88 | 0.2861 |
| \( x_1 \) | 2.24 | 1.88 | 0.2382 | \( x_2 \) | 1.85 | 2.31 | 0.2695 |
| \( x_0 \) | 2.67 | 2.17 | 0.3309 | \( x_0 \) | 2.08 | 2.46 | 0.3245 |
| \( x_1 \) | \( x_2 \) | \( d(x, \bar{x}) \) | \( x_1 \) | \( x_2 \) | \( d(x, \bar{x}) \) |
| 1.00 | 4.00 | 0.3620 | 2.24 | 1.88 | 0.3428 |
| 2.24 | 1.88 | 0.3256 | 1.85 | 2.31 | 0.3228 |
| 1.85 | 2.31 | 0.3694 | 1.65 | 2.76 | 0.3344 |
| \( x_0 \) | 1.65 | 2.70 | 0.3204 | \( x_4 \) | 1.87 | 2.27 | 0.3216 |
| \( x_0 \) | 1.70 | 2.75 | 0.3361 | \( x_0 \) | 1.91 | 2.30 | 0.3331 |
Using the results in table 3.1, we notice the following:

(i) The $d$-function at the end point of the $k$th iteration is minimum compared to the $d$-functions at the support points of the design matrix.

(ii) The sequence \( \left\{ x_k M^{-1} x_k \right\}_{k=1}^n \) is non-decreasing; i.e. the $d$-function at the minimizer $x_k = x^*$ is the maximum of the minimum $d$-functions for the $k$ different iterations.

References


