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# EXACT SOLUTIONS OF THE SPHERICALLY SYMMETRIC MULTIDIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR.

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## ABSTRACT

The complete orthonormalised energy eigenfunctions and the energy eigenvalues of the spherically symmetric isotropic harmonic oscillator in N dimensions, are obtained through the methods of separation of variables. Also, the degeneracy of the energy levels are examined.

**KEYWORDS**: - Schrödinger Equation, Isotropic harmonic oscillator, Hyperspherical harmonics, Multidimensional space.

## INTRODUCTION

Over years there has been much discussion about the problems involving multidimensional spaces (N dimensions), an exact general solution of the 3 – (and in a generalized form N-) dimensional Schrödinger equation are of considerable interest in Quantum mechanics. Chatterjee, (1990); Mavromatis, (1992). Fukutaka and Kashiwa, (1987) considered the formulation of the path integrals and their quantization on N-dimensional sphere, spherically symmetric distribution of a perfect fluid solutions in higher dimensions (N dimensions) have been considered by Krori et al., (1989). Also Blinder and Pollock, (1989) discussed the N-dimensional Coulomb Green's functions using fractional derivatives. Of recent, some aspects of the N-dimensional hydrogen atom are investigated by Al-Jaber, (1998).

In the case of harmonic oscillator in higher dimensions, Davtyan et al., (1987) discussed the transformation of the 5 –dimensional hydrogen atom to the 8-dimensional isotropic harmonic oscillator, the mapping of the Coulomb problem into the oscillator has been described by Bateman et al., (1992). Recently, a simple transformation is obtained that relates the arbitrary dimensional Coulomb problem and the N-dimensional isotropic oscillator by Mavromatis, (1997).

It is the purpose of this paper to obtain the exact solutions of the spherically symmetric isotropic harmonic oscillator in N-dimensional spaces analytically which are in agreement with the solutions obtained by Bakhrakh et al., (1972) using Green's function method and Grosche and Steiner, (1998) using path integral method. In section 2, the formulation and the method of solution are discussed. The results and discussion i.e. the energy eigenvalues, the energy eigenfunctions of the problem and the degeneracy of the oscillator states are contained in section 3. In section 4, we have conclusion.

## THE FORMULATION AND THE METHOD OF SOLUTION.

To discuss the Schrödinger equation in N dimensions, we need to introduce the polar coordinates which are defined as a simple generalization of the procedure in three dimensions, namely; with the 'polar' type angles $\theta_1, \theta_2, \ldots, \theta_{N-2}$ and $\phi$ ' azimuth' angle, so that just as we have $z = r \operatorname{Cos}\theta$, $x = r \operatorname{Sin}\theta \operatorname{Cos}\phi$, $y = r \operatorname{Sin}\theta \operatorname{Sin}\phi$ in 3-dimensions, we shall now have (Grosche and Steiner 1998; Mavromatics, 1998)

$x_1 = r \operatorname{Cos} \theta_1,$

$x_2 = r \operatorname{Sin}\theta_1 \operatorname{Cos}\theta_2,$

$x_3 = r \operatorname{Sin}\theta_1 \operatorname{Sin}\theta_2 \operatorname{Cos}\theta_3,$

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$x_{N-1} = r \sin\theta_1 \cos\theta_2 \ldots\ldots \sin\theta_{N-2} \cos\phi$
$x_N\, r \sin\theta_1 \sin\theta_2 \ldots\ldots \sin\theta_{N-2} \cos\phi.$

Where $0 \le \theta_i \le \pi$ ( i = 1,2,......, N – 2), and $0 \le \phi = \theta_{N-1} \le 2\pi$, $r = \sum_{i=1}^{N} \sqrt{(x_i)^2} \ge 0$.

Then, the Schrodinger equation in N-dimensions becomes (see Mavromatis, 1998; Al-Jaber, 1998).

$$-\frac{\hbar^2}{2\mu} \Delta_N \Psi + V\Psi = E\Psi \tag{1}$$

In the N-dimensional space, the Laplacian operator in polar coordinates ($r, \theta_1, \theta_2, \ldots \theta_{N-2}, \phi$) of $\Re^N$ is

$$\Delta_N = r^{1-N} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial}{\partial r}\right) - \frac{\Lambda^2(\Omega)}{r^2}. \tag{2}$$

Where $\Lambda^2(\Omega)$ is the grand orbital operator on the unit sphere $S_{N-1}$ (hypersphere) and $\Omega$ denotes a set of hyperspherical variables $\theta_1, \theta_2, \ldots, \theta_{N-2}, \phi$ ( i.e. the set of hyperangles) with equation (1), the N-dimensional spherically symmetric isotropic harmonic oscillator potential equation is given as;

$$-\frac{\hbar^2}{2\mu}\left[r^{1-N} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial}{\partial r}\right) - \frac{\Lambda^2(\Omega)}{r^2}\right]\Psi + \frac{1}{2}\mu\omega^2 r^2 \Psi = E\Psi. \tag{3}$$

Where $\mu$ is the reduced mass, $\omega$ is the angular frequency and other terms have their usual meaning.

The method of separation of variable is used to reduce equation (3) into two separate ordinary differential equations i.e. putting

$$\Psi(r,\Omega) = R(r)\, Y_\ell^m(\Omega). \tag{4}$$

And the equations are;

$$\Lambda^2(\Omega)\, Y_\ell^m(\Omega) - \beta Y_\ell^m(\Omega) = 0. \tag{5}$$

$$-\frac{\hbar^2}{2\mu}\left[r^{1-N} \frac{d}{dr}\left(r^{N-1} \frac{d}{dr} R(r)\right) - \frac{\beta}{r^2} R(r)\right] + \frac{1}{2}\mu\omega^2 r^2 R(r) = ER(r). \tag{6}$$

Where $\beta$ is a separation constant whose values (which are the eigenvalues of $-\Lambda^2$ ) are: (Shimakura, 1992).

$$\beta = \ell(\ell + N - 2);\ \ell = 0, 1, 2, \ldots\ldots \tag{7}$$

Then, the wave function for the eigenfunctions of the $\Lambda^2$ operator are called the "hyperspherical harmonies", and their eigenvalues equation reads: $\Lambda^2(\Omega)Y_l^m(\Omega) = l(l+N-2)Y_l^m(\Omega)$ where $l = 0,1,2,......$ denotes the order of the harmonics. The second index m stands for the set of additional quantum numbers required to index the (generally very large) degeneracy of harmonics with the same value of $l$. The harmonics $Y_l^m(\Omega)$ are multidimensional extensions of the familiar spherical harmonics. The hyperspherical harmonics can be quite elaborate to evaluate and work with, but fortunately we do not need many of their detailed properties. These functions have been thoroughly studied in the literature and for this reason we refer the interested reader to the following few literature for details: (Grosche and Steiner, 1995 and 1998; Ruan and Bao, 1997; Avery, 1989 and Barnea, 1999).

# 3 RESULTS AND DISCUSSION

## 3.1 THE HYPERRADIAL EQUATION AND THE EIGENVALUES.

In the equation (6), R (r) is the hyperradial part of the wave function and E is the energy eigenvalue. With the substitution

$$R(r) = r^{\frac{(1-N)}{2}} G(r).$$

we obtained the convenient form of equation (6) as

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} - \frac{(k-1)(k-3)}{4r^2}\right]G(r) + \frac{1}{2}\mu\omega^2 r^2 G(r) = EG(r).. \qquad (9)$$

With $k = 2l + N$ (Chatterjee, 1990), the substitution of a dimensionless variable $t = \frac{\mu\omega r^2}{\hbar}$ changes equation (9) to equation in t-space as

$$t\frac{d^2G(t)}{dt^2} + \frac{1}{2}\frac{dG(t)}{dt} - \frac{(k-1)(k-3)}{16t}G(t) - \frac{t}{4}G(t) + \frac{E}{2\omega\hbar}G(t) = 0. \qquad (10)$$

This equation is readily transformed into the standard form through the substitution.

$$G(t) = \exp\left(-\frac{t}{2}\right)t^{\frac{(k-1)}{4}}U(t). \qquad (11)$$

as the (Kummer's differential equation) confluent hypergeometric equation.

$$t\frac{d^2U(t)}{dt^2} + \left(\frac{k}{2} - t\right)\frac{dU(t)}{dt} - \left[\frac{k}{4} - \frac{E}{2\omega\hbar}\right]U(t) = 0. \qquad (12)$$

The solution of equation (12) is regular at $U = 0$ or $t = 0$ which is a degenerate hypergeometic function: (Abramowitz and Stegun 1970; Flügge, 1994; Merzbacher, 1970)

$$U(t) = {}_1F_1\left[\frac{k}{4} - \frac{E}{2\omega\hbar}, \frac{k}{2}; t\right]. \tag{13}$$

The wave function can only be normalized (for large, r U (t) would diverge as exp ( -t), thus. preventing normalization of the wave function) i.e. the wave function is not squarely integrable unless $\frac{k}{4} - \frac{E}{2\omega\hbar}$ is a negative integer. Hence, we have

$$\frac{k}{4} - \frac{E}{2\omega\hbar} = -n_r,\ r = 0,1,2,\ldots\ldots\ldots \tag{14}$$

the energy eigenvalues for the spherically symmetric isotropic harmonic oscillator potential in the arbitrary-dimensional space is

$$E_{n_r} = \hbar\omega\left(2n_r + \ell + \frac{N}{2}\right) = \hbar\omega\left(n + \frac{N}{2}\right), \tag{15}$$

Where $n_r$ is the radial quantum number and $n$ is the principal quantum number.

## 3.2 THE HYPERRADIAL EIGENFUNCTIONS.

From equation (6), the unnormalized energy eigenfunctions of the hyperradial Schrödinger equation with spherically symmetric isotropic harmonic oscillator potential is

$$R_{n_r\ell}(r) = C_{n_r\ell}\left(\frac{\mu\omega}{\hbar}\right)^{\frac{(2\ell+N-1)}{4}} r^{\ell}\exp\left(-\frac{\mu\omega r^2}{2\hbar}\right)\ {}_1F_1\left[n_r, \ell + \frac{N}{2}; \frac{\mu\omega r^2}{\hbar}\right], \tag{16}$$

Where $n_r$ is the normalization factor whose value is determined from the requirement (i.e. the inner product of the hyperradial functions)

$$(R_{n_r\ell}(r), R_{n_r\ell}(r))\, r^{N-1} = \int_0^{\infty} \left|R_{n_r\ell}(r)\right|^2 r^{N-1} dr = 1. \tag{17}$$

Therefore,

$$C_{n_r\ell} = \frac{\Gamma\left(n_r + \ell + \frac{N}{2}\right)}{n_r!\left(\ell + \frac{N-2}{2}\right)!}\sqrt{2\left(\frac{\mu\omega}{\hbar}\right)^{\frac{1}{2}} \frac{n_r!}{\Gamma\left(n_r + \ell + \frac{N}{2}\right)}} \tag{18}$$

With equation (18), and by using the relation

$${}_1F_1(-\gamma, m+1; Z) = \frac{\gamma! m!}{\gamma + m}!\, L_\gamma^m(Z), \quad \text{Equation (16) becomes}$$

$$R_{n_r\ell}(r) = \sqrt{\frac{2\mu\omega}{\hbar r^{N-2}} \frac{n_r!}{\Gamma(n_r + \ell + \frac{N}{2})}}\left(\frac{\mu\omega r^2}{\hbar}\right)^{\ell + \frac{N-2}{2}} \exp\left[-\frac{\mu\omega r^2}{2\hbar}\right] L_{n_r}^{\ell+\frac{N-2}{2}}\left(\frac{\mu\omega r^2}{\hbar}\right). \tag{19}$$

Equation (19) is the normalized energy eigenfunctions of hyperradial Schrödinger equation with spherically symmetric isotropic harmonic oscillator potential. This result is in agreement with the result obtained by Grosche and Steiner (1998) using path integral method. Also, Bakhrakh et al.,(1972) using Green's function method.

Now, we are in a position to write the complete energy eigenfunctions of the N-dimensional spherically symmetric isotropic harmonic oscillator, namely:

$$\Psi_{n_r\ell m}\left(r,\theta_1,\theta_2,\ldots\ldots,\theta_{N-2},\phi\right)=R_{n_r\ell}(r)\,Y_\ell^m\left(\theta_1,\theta_2,\ldots\ldots,\theta_{N-2},\phi\right). \tag{20}$$

Where $R_{n_r\ell}$ is given by equation (19), and $Y_\ell^m\left(\theta_1,\theta_2,\ldots\ldots,\theta_{N-2},\phi\right)$ is hyperspherical harmonics of degree $\ell$ on the $S^{N-1}$-sphere (Grosche and Steiner 1995). One should note that equation (2) reduces to the well-known energy eigenfunctions for the 3-dimensional spherically symmetric isotropic harmonic oscillator.

## 3.3 DEGENERACY OF THE OSCILLATOR STATES

The purpose here is to give a simple prescription for determining the degeneracy of any energy level of the spherically symmetric isotropic harmonic oscillator potential in N-dimensions. One should recall that for any spherically symmetric isotropic harmonic potential in N-dimensions (such as the one under consideration) the Schrödinger equation could be separated into an ordinary differential equation for the radial part and a partial differential equation for the angular part. The solutions to the angular part are the hyperspherical harmonics $Y_\ell^m$ If the potential has no symmetries beyond rotational invariance, the degeneracies of energy levels are therefore the multiplicities for the hyperspherical harmonics for fixed $\ell$.
In the familiar Cartesian coordinates, if each degree of freedom i holds $n_i$ quanta, then the total energy is

$$E_n=\hbar\,\omega\sum_i^N\left(n_i+\frac{1}{2}\right);\; n=\sum_i n_i \tag{21}$$

The degeneracy of levels with energy equation (21) equals the number of ways in which n quanta can be distributed among N degrees of freedom, which is given by the coefficient ,

$$^{n+(N-1)}C_n=\frac{\left[n+(N-1)\right]!}{n!\left[N-1\right]!} \tag{22}$$

In hyperspherical language, the energy eigenvalues are given by equation (15). For a given energy, we thus make the identification $2n_r+\ell=n.$. Avery, (1989) gives an expression for the number of hyperspherical harmonics with a given value of $\ell$.

$$\frac{\left(N+2\ell-2\right)\left(N+\ell-3\right)!}{\ell!\left(N-2\right)!}={}^{\ell+N-2}C_\ell+{}^{(\ell-1)+(N+2)}C_{(\ell-1)}. \tag{23}$$

To obtain the complete degeneracy, we need to sum this expression for all allowed values of $\ell$, consistent with $2n_r+\ell=n.$. Note that $\ell$ is restricted to values with the same parity as n, but the form of equation (23) guarantees that values of $\ell$, even odd, will appear in the sum. The total degeneracy is thus

$$\sum_{\ell=0}^{n} {}^{\ell+N-2}C_{\ell} = {}^{n+N-1}C_{n}. \qquad (24)$$

We obtained equation (24) from the help of the properties of binomial coefficients Jeffrey, (1995); Bohn et al., (1998).

Table 1: Degeneracies of the multidimensional spherically symmetric harmonic oscillator (N – 2 to 10) for principal quantum number ( n = 1 to 5).

| N | n=1 | n=2 | n=3 | n=4 | n=5 |
|---|---|---|---|---|---|
| 2 | 2 | 3 | 4 | 5 | 6 |
| 3 | 3 | 6 | 10 | 15 | 21 |
| 4 | 4 | 10 | 20 | 35 | 56 |
| 5 | 5 | 15 | 35 | 70 | 126 |
| 6 | 6 | 21 | 56 | 126 | 252 |
| 7 | 7 | 28 | 84 | 210 | 462 |
| 8 | 8 | 36 | 120 | 330 | 792 |
| 9 | 9 | 45 | 165 | 495 | 1287 |
| 10 | 10 | 55 | 220 | 715 | 2002 |

## 4 CONCLUSION

Some aspects of the spherically symmetric isotropic harmonic oscillator in N dimensions are investigated. The energy eigenvalues are obtained which are dimensional dependents.
The set of the complete orthonormalized energy eigenfunctions are obtained as a linear combination of the hyperradial solutions and the hyperspherical harmonics of degree $\ell$ on the $S^{N-1}$- sphere. Also, the energy eigenfunctions are N dependent. It is noticed that the aspects considered here reduces to the well-known energy eigenvalues, or eigenfunctions for 3-dimensional spherically symmetric isotropic harmonic oscillator when N=3. We have also considered the degeneracy of the energy levels, which increases with the dimension N.

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