A NEW WEIBULL EXPONENTIATED INVERTED WEIBULL DISTRIBUTION FOR MODELLING POSITIVELY-SKEWED DATA

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(Received 12 November 2020; Revision Accepted 27 November 2020)

ABSTRACT

An Exponentiated Inverted Weibull Distribution (EIWD) has a hazard rate (failure rate) function that is unimodal, thus making it less efficient for modeling data with an increasing failure rate (IFR). Hence, the need to generalize the EIWD in order to obtain a distribution that will be proficient in modeling these types of dataset (data with an increasing failure rate). This paper therefore, extends the EIWD in order to obtain Weibull Exponentiated Inverted Weibull (WEIW) distribution using the Weibull-Generator technique. Some of the properties investigated include the mean, variance, median, moments, quantile and moment generating functions. The explicit expressions were derived for the order statistics and hazard/failure rate function. The estimation of parameters was derived using the maximum likelihood method. The developed model was applied to a real-life dataset and compared with some existing competing lifetime distributions. The result revealed that the (WEIW) distribution provided a better fit to the real life dataset than the existing Weibull/Exponential family distributions.

KEYWORDS

Hazard function, Log-likelihood, Failure rate, Moment generating function, Order statistics

INTRODUCTION

Generalization of a probability distribution is a common technique used for introducing flexibility to classical distributions through the introduction of parameters or reduction of some identified redundant parameters. According to Alzaatreh, Lee and Famoye, the Weibull distribution was named after Walodi Weibull and has been used to model some families of distributions by several authors with the intention of generating more flexible distributions [1]. The Weibull-X family was proposed by [1] while Bourguignon, Silva and Cordeiro proposed the Weibull-G family [2]. Several flexible distributions have been developed from these important families of distributions. Alzaatreh, Famoye and Lee developed Weibull-Pareto [3] while Aljarrah, Famoye and Lee developed a new Weibull-Pareto distribution [4]. Merovci and Elbatal developed Weibull Rayleigh distribution [5]. Oguntunde, Balogun, Okagbeue and Bishop developed the Weibull Exponential distribution [6] while Yahaya and Sa’ad introduced the Weibull-Burr XII distribution [7].

By means of the inverse transformation of variables, some baseline distributions have been inverted and generalized. Some of the existing results revealed that inverted distributions when extended can also provide a more flexible distribution than the baseline distribution. Voda introduced the use of Inverse Rayleigh (IR) distribution [8] while the Inverse Exponential (IE) and their study revealed that the IW is useful for modeling systems with failure rates common in biological and reliability studies. Drapella and Mudholkar and Kolia as well studied the Inverse Weibull Distribution and projected complementary Weibull and reciprocal Weibull as another names for the model [9], [10] while Khan, Pasha, and Pasha studied the flexibility of IW distribution [11]. De Gusmao, Ortega, and Cordeiro proposed the Generalized Inverse Weibull (GIW) distribution [12] while Elbatal and Muhammed later developed the Exponentiated Generalized Inverse Weibull (EGIW) Distribution [13]. It was also observed that generalization of distribution using the inverse of baseline distributions continues to achieve more attention among researchers in recent times.

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times as evident in the following studies. Fatima, and Ahmad proposed the Weighted inverse Rayleigh distribution [14] while Oguntunde, Adejumo and Owoloko proposed the Weibull Inverted Exponential distribution [15]. The odd Frechet Inverse Weibull distribution was developed by [16], odd Frechet Inverse Rayleigh by [17], Inverse Weibull Inverse Exponential (IW-IE) distribution by [18], while [19] extended Inverse Rayleigh using the Half-Logistics transformation. The IW is a useful model that attracts the attention of several researchers, it has been studied by several authors and also extended as Exponentiated Inverted Weibull (EIW) distribution by [20]; the authors introduced the standard two-parameter exponentiated Inverted Weibull (EIW) distribution. [21] studied (EIW) distribution by investigating methods of parameter estimations using classical likelihood and Bayesian estimators for samples from complete and Type II censoring scheme. [22] extended (EIW) distribution by carrying out a comparative study of some estimation methods some of which are the MLE, least square, least line based on grouped data.

THEORY / CALCULATIONS

THE WEIBULL DISTRIBUTION

The cumulative distribution function (cdf) of the Generalized Weibull family distribution is defined as:

\[
F(x) = \int_0^{\frac{\ln x}{\beta}} g(t)^{\frac{\ln x}{\beta}} dt
\]

(1)

\[
F(x) = 1 - e^{-\left[\frac{\ln (x)}{\beta}\right]^\theta}
\]

(2)

while the corresponding density function is given as:

\[
f(x) = a\beta g(x) \frac{[\ln (x)]^{\beta-1}}{[1-\ln (x)]^{\beta+1}} e^{-\left[\frac{\ln (x)}{\beta}\right]^\theta}
\]

(3)

where \(x > 0, \alpha > 0, \beta > 0\).

The Inverted Weibull (IW) distribution as proposed by [26] has the cdf and pdf given respectively as follows:

\[
F(x) = e^{-\left(\frac{x}{\beta}\right)^k}
\]

(4)

\[
f(x) = k\theta x^{-(k+1)} e^{-\left(\frac{x}{\beta}\right)^k}
\]

(5)

where \(x > 0, \theta > 0, k > 0\); \(\theta\) is the quality or scale parameter while \(k\) is the shape parameter.

The standard two-parameter Exponentiated Inverted Weibull (EIW) distribution was also proposed by [20] with the cdf and pdf defined respectively as:

\[
F(x; \theta, \beta) = \left(e^{-\frac{x}{\beta}}\right)^\theta; x > 0, \theta > 0, \beta > 0
\]

(6)

\[
f(x; \theta, \beta) = \theta\beta x^{-(\beta+1)} \left(e^{-\frac{x}{\beta}}\right)^\theta
\]

(7)

\[x > 0, \theta > 0, \beta > 2.22\]

THE WEIBULL EXPONENTIATED INVERTED WEIBULL (WEIW) DISTRIBUTION

From equation (4), the cdf of Exponentiated Inverted Weibull (EIW) distribution is defined for this study as follow:

\[
G(x; \theta, k, \nu) = \left(e^{-\left(\frac{x}{\beta}\right)^k}\right)^\nu
\]

(8)

The density function obtained from the derivative of equation (8) with respect to \(x\) is given as:

\[
g(x) = \nu k\theta x^{-(k+1)} \left(e^{-\left(\frac{x}{\beta}\right)^k}\right)^\nu
\]

\[g(x) = b k \theta x^{-(k+1)} e^{-\left(\frac{x}{\beta}\right)^k}\]

(9)

The proposed (WEIW) distribution has the cdf derived, using equations (2) and (8) as follow:

\[
F(x) = 1 - \exp\left(-\alpha \left[\frac{e^{-\left(\frac{x}{\beta}\right)^k}}{1-e^{-\left(\frac{x}{\beta}\right)^k}}\right]^{\theta}\right)
\]

(10)
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The associated pdf of WEIW is obtained from the first derivative of \( F(x) \) in equation (10) and is given as:

\[
  f(x) = a\beta b\theta k^k x^{-(k+1)} \left( 1 - \frac{e^{-\left(\frac{\theta}{x}\right)}}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta x + 1} \exp \left(-\alpha \left( \frac{e^{-\left(\frac{\theta}{x}\right)}}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta} \right)
\]

where \( x > 0; \alpha, \beta > 0 \) and \( b, k, \theta > 0 \).

The WEIW distribution has \( \theta \) as scale parameter with \( \alpha, \beta, \nu, k > 0 \).

**DEVELOPMENT OF THE PROBABILITY FUNCTION OF WEIW DISTRIBUTION**

The probability density function of WIEW is written as:

\[
  f(x) = a\beta b\theta k^k x^{-(k+1)} \left( \frac{1}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta x + 1} R(x)
\]

where \( R(x) = \text{reliability function}, x > 0; \alpha, \beta > 0 \) and \( b, k, \theta > 0 \).

On applying the power series expansion on the reliability function, the resulted function became:

\[
  R(x) = \exp \left( -\alpha \left( \frac{e^{-\left(\frac{\theta}{x}\right)}}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta} \right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left( \frac{e^{-\left(\frac{\theta}{x}\right)}}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta i}
\]

On substituting \( R(x) \) into \( f(x) \) and performing some arithmetic operations, we obtain:

\[
  f(x) = a\beta b\theta k^k x^{-(k+1)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \alpha \left( \frac{e^{-\left(\frac{\theta}{x}\right)}}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta i}
\]

\[
  f(x) = \nu \beta k^k x^{-(k+1)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \alpha \left( \frac{e^{-\left(\frac{\theta}{x}\right)}}{1 - e^{-\left(\frac{\theta}{x}\right)}} \right)^{\beta i}
\]

**APPLICATION OF THE BINOMIAL SERIES EXPANSION ON EQUATION (13)**

\[
  [1 - Z]^{-\alpha} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\alpha + j)}{\Gamma(\alpha)} Z^j
\]

\[
  \left( 1 - e^{-\left(\frac{\theta}{x}\right)} \right)^{-\beta (i+1)} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\beta (i+1) + j)}{\beta \Gamma(\beta (i+1) + 1)} e^{-\left(\frac{\theta}{x}\right)^j}
\]
The survival (reliability) function is derived and given as:

\[
R(x) = 1 - F(x) = \exp\left(-\alpha \left[\left(\frac{e^{-\theta (x)^k}}{\phi(x)^k}\right)^\beta \right]\right)
\]

(17)

where \(x > 0, \alpha, \beta > 0, \) and \(\nu, k, \theta > 0\)

The hazard/failure rate function is obtained using:

\[
h(x) = \frac{f(x)}{1 - R(x)} = \alpha \beta \nu k \theta x^{-k-1} \left[\frac{e^{-\theta (x)^k}}{\phi(x)^k}\right]^\beta \left[\frac{1 - e^{-\theta (x)^k}}{1 - e^{-\theta (x)^{k+1}}}\right]^\nu
\]

(18)

where \(x > 0, \alpha, \beta > 0, \) and \(\nu, k, \theta > 0\)

In equation (16). Also, some statistical properties of the WEIW distribution can be derived from those of Exponentiated Inverse Weibull distribution.

**PROPERTIES OF THE WEI DISTRIBUTION**

Some of the properties of the new distribution are conferred in this section.

**RELIABILITY AND HAZARD/FAILURE FUNCTIONS OF WEI DISTRIBUTION**

The survival (reliability) function is derived and given as:

\[
R(x) = 1 - F(x) = \exp\left(-\alpha \left[\left(\frac{e^{-\theta (x)^k}}{\phi(x)^k}\right)^\beta \right]\right)
\]

(17)

where \(x > 0, \alpha, \beta > 0, \) and \(\nu, k, \theta > 0\)

The hazard/failure rate function is obtained using:

\[
h(x) = \frac{f(x)}{1 - R(x)} = \alpha \beta \nu k \theta x^{-k-1} \left[\frac{e^{-\theta (x)^k}}{\phi(x)^k}\right]^\beta \left[\frac{1 - e^{-\theta (x)^k}}{1 - e^{-\theta (x)^{k+1}}}\right]^\nu
\]

(18)

where \(x > 0, \alpha, \beta > 0, \) and \(\nu, k, \theta > 0\)
The reversed hazard/failure rate function is obtained as:

\[
rh(x) = \frac{f(x)}{F(x)} = \frac{a\beta b\theta^k x^{-k-1} \left( \frac{e^{-\theta x}}{1-e^{-\theta x}} \right)^{\beta-1}}{1-\exp{-a \left( \frac{e^{-\theta x}}{1-e^{-\theta x}} \right)^k}}
\]

(19)

\[x > 0; \alpha, \beta > 0 \text{ and } v, k, \theta > 0\]

\[\text{Figure 2: The Survival functions and Failure (Hazard) rate Plots}\]

The failure/hazard rate function is described by different shapes such as decreasing, increasing, reversed J-shape and inverted bathtub depending on parameter values.

**Theorem 1:** If \(X\) is a random variable from WEIW distribution, then the hazard rate function is of the form represented as:

\[h_{WEIW}(x) = g_{EIV}(x) \frac{[G_{EIV}(x)]^{\beta-1}}{[R(x)]^{\beta+1}}\]

(20)

**Proof:**

\[h(x) = \alpha \beta v \theta^k x^{-k-1} \left( \frac{e^{-\theta x}}{1-e^{-\theta x}} \right)^{\beta-1} \left( \frac{e^{-\theta x}}{1-e^{-\theta x}} \right)^{\beta-1} = g(x) \frac{[G(x)]^{\beta-1}}{[R(x)]^{\beta+1}}\]

where \(R(x) = 1-G(x), x > 0; \alpha, \beta > 0 \text{ and } v, k, \theta > 0\)

**QUANTILE FUNCTION AND MEDIAN OF WEIW DISTRIBUTION**

**Theorem 2:** If \(X\) is a random variable from WEIW distribution, then the quantile function is given as:

\[Q(u) = \theta \left[ -\log \left( \frac{1-\log(1-u)^{1/\beta}}{1+\frac{1-\log(1-u)^{1/\beta}}{v}} \right)^{1/\beta} \right]^{-1/k}\]

(21)

The quantile function can be proved using the relation \(Q(u) = F^{-1}(u)\), where \(\mu\) is a random variable from uniform distribution on interval \((0, 1)\)

The median is obtained for the middle observation at \(u = 0.5\) and is given by;

\[x(\text{median}) = \theta (22)\]

Let \(X\) be a random variable from the WEIW distribution, simulation can be done through the inverse transformation of a variable using uniform interval \(\mu(0, 1)\) and the random variable \(X\) is taken as,

\[X = \theta \left[ -\log \left( \frac{1-\log(1-u)^{1/\beta}}{1+\frac{1-\log(1-u)^{1/\beta}}{v}} \right)^{1/\beta} \right]^{-1/k}\]

(23)
MOMENT OF WEIW DISTRIBUTION

Moments are vital statistical measure for characterizing distributions; the \( r \)-th moment for the WEIW distribution is derived as follows:

**Theorem 3:** Let \( X \) be a random variable from the WEIW distribution, then the \( r \)-th moment of \( X \) is given by the following expression.

\[
\mu_r = E(X^r) = \int_0^\infty x^r f(x, \{\alpha, \beta\}) \, dx
\]

**Proof:**

From equation (14),

\[
f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} x^{i+j} g(x) [G(x)]^j \frac{G(x)^{j-i}}{G(x)^j}
\]

\[
f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

\[
f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

The \( r \)-th ordinary moment of the WEIW distribution is given by:

\[
\mu_r = E(X^r) = \int_0^\infty x^r f(x, \{\alpha, \beta\}) \, dx
\]

**The Mean and Variance of WEIW Distribution**

The mean of WEIW random variable \( X \) is given by:

\[
E(X) = \mu_1 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

\[
E(X) = \mu_1 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

\[
E(X) = \mu_1 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

**Theorem (4):** If \( X \) has WEIW distribution, then the moment generating function has the form expressed as;

\[
w_{i,j,\theta} = t^i \theta^j \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)}
\]

**Proof:**

The moment generating function for a continuous random variable is defined by:

\[
M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x) \, dx
\]

\[
M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)
\]

\[
E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)
\]

\[
E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

\[
M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} w_{i,j,\theta} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)} \sum_{k=0}^{\infty} \frac{v^k x^{i+j+k}}{k!}
\]

where

\[
w_{i,j,\theta} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)}
\]

\[
E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)}
\]

\[
w_{i,j,\theta} = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^i}{i! j!} \frac{\Gamma(\beta(i+1)+j)}{\Gamma(\beta(i+1)+1)}
\]
CHARACTERISTIC FUNCTION OF WEIW DISTRIBUTION

The characteristic function for a continuous random variable $X$ from WEIW distribution is given as follows

$$
\phi_X(t) = E(e^{itX}) = \int_0^\infty e^{itx} f(x) dx
$$

$$
= \sum_{r=0}^\infty \frac{(it)^r}{r!} \int_0^\infty x^r f(x) dx = \sum_{r=0}^\infty \frac{(it)^r}{r!} \phi_X^r
$$

$$
\phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} w_{r,j} \theta^r v [\beta(i + 1) + j]^r (1 - \frac{r}{k})
$$

$$
w_{r,j} = \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j}}{i! j!} \Gamma(\beta(i + 1) + 1 + j)
$$

DISTRIBUTION OF ORDER STATISTICS OF WEIW DISTRIBUTION

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with associated order statistics $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ of size $n$ from a distribution with density and cumulative functions denoted as $f(x)$ and $F(x)$ respectively, then the probability density function of the $r^{th}$ order statistics $X_{(r)}$ is given by:

$$
f_{X_{(r:n)}}(x) = \frac{1}{\bar{y}^{(r, n - r + 1)}} F(x)^{r-1}(1 - F(x))^{n-r} f(x)
$$

The $r^{th}$ order statistics for a WEIW random variable $X$ is derived using equation (12) and (13) in equation (31) as follows:

$$
f_{X_{(r:n)}}(x) = \left\{ \begin{array}{cl}
\frac{1}{\bar{y}^{(r, n - r + 1)}} & \{1 - \exp\left(-\alpha \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\frac{1}{\gamma} \right]\}^{r-1} \\
& \left\{ \exp\left(-\alpha \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\frac{1}{\gamma} \right] \right\}^{n-r} \\
& \ast \alpha \nu \beta k \theta x^{-\gamma(k+1)} \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\beta \exp\left(-\alpha \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\frac{1}{\gamma} \right) \right\}
\end{array} \right.
$$

Using Binomial expansion, it can be reexpressed as:

$$
f_{X_{(r:n)}}(x) = \left\{ \frac{1}{\bar{y}^{(r, n - r + 1)}} \sum_{i=0}^{n-r} \frac{(-1)^i (n-r)! f(x)^i F(x)^{r+i-1}}{i! (r+i-1)!} \right\}^{r-1}
$$

$$
f_{X_{(r:n)}}(x) = \left\{ \frac{1}{\bar{y}^{(r, n - r + 1)}} \sum_{i=0}^{n-r} \frac{(-1)^i (n-r)! \alpha \nu \beta k \theta x^{-\gamma(k+1)} \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\beta \exp\left(-\alpha \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\frac{1}{\gamma} \right) \right\}^{r+i-1}
$$

This minimum order statistics from equation (33) when $r = 1$ is given by:

$$
f_{X_{1:n}}(x) = n \alpha \nu \beta k \theta x^{-\gamma(k+1)} \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\beta \exp\left(-\alpha \left[ \frac{\left(\frac{\theta}{\beta} \right)^\gamma}{1 - \left(\frac{\theta}{\beta} \right)^\gamma} \right]^\frac{1}{\gamma} \right)
$$
This maximum order statistics from equation (26) when \( r = n \) is given as:

\[
f_{x_{n:n}}(x) = \begin{cases} 
\frac{1}{\theta^{r(n-r+1)}} \left( 1 - \exp \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right) \right) \\
* \alpha v \beta k x^{-(k+1)} \left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{v \beta} \frac{1 - \exp \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right)}{v \beta + 1 + \sum_{i=1}^{n} e^{-\left( \frac{y}{\theta} \right) k}} \end{cases} 
\]

(MAXIMUM LIKELIHOOD ESTIMATE (MLE))

Let \( X_1, X_2, ..., X_n \) be random sample of size \( n \) from the WEIW distribution, the method of maximum likelihood for estimating the unknown parameters is applied to the density function to obtain the likelihood and log-likelihood functions presented respectively as:

\[
Lik = \prod_{i=1}^{n} \alpha v \beta k x^{-(k+1)} \left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{v \beta} \frac{1 - \exp \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right)}{v \beta + 1 + \sum_{i=1}^{n} e^{-\left( \frac{y}{\theta} \right) k}}
\]

\[
logLik = \begin{cases} 
- \left( \beta + 1 \right) \sum_{i=1}^{n} \log \left[ 1 - \left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{v} \right] + \sum_{i=1}^{n} \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right) \\
\end{cases}
\]

\[
\frac{dLik}{d\alpha} = \frac{n}{a} - \sum_{i=1}^{n} \left( \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{v \beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right)
\]

\[
\frac{dLik}{d\theta} = \frac{v \beta k}{\theta} \left( \sum_{i=1}^{n} \frac{1}{1 - \left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{v \beta}} \right) - \sum_{i=1}^{n} e^{-\left( \frac{y}{\theta} \right) k} \log\left( \frac{1}{\theta} \right) \left( \beta + 1 \right) \sum_{i=1}^{n} \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right)
\]

\[
\frac{dLik}{d\beta} = \frac{nk}{\theta} \left( \sum_{i=1}^{n} e^{-\left( \frac{y}{\theta} \right) k} \right) - \sum_{i=1}^{n} e^{-\left( \frac{y}{\theta} \right) k} \log\left( \frac{1}{\theta} \right) \left( \beta + 1 \right) \sum_{i=1}^{n} \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right)
\]

\[
\frac{dLik}{dv} = \frac{n v \beta k}{\theta} \left( \sum_{i=1}^{n} \frac{1}{1 - \left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{v \beta}} \right) - \frac{v \beta k}{\theta} \left( \sum_{i=1}^{n} e^{-\left( \frac{y}{\theta} \right) k} \right) \left( \beta + 1 \right) \sum_{i=1}^{n} \left( -\alpha \frac{\left( e^{-\left( \frac{y}{\theta} \right) k} \right)^{\beta}}{1 - e^{-\left( \frac{y}{\theta} \right) k}} \right)
\]
The derivatives of the log-likelihood function with respect to the unknown parameters when equated to zero for simultaneous solutions can provide us with the estimates which is analytically difficult owing to the complex nature of the function, hence the Newton Raphson Algorithm in R-software could be employed for obtaining numerical solutions for the estimates. The confidence intervals are obtainable from the inverse dispersion matrix \( I^{-1}(\hat{\theta}) \) which can be generated from second derivatives of log Likelihood function.

The 100(1 – p)% two sided confidence intervals for the parameters are of the form:

\[
\hat{\beta} \pm Z_{p/2} \sqrt{I^{-1}_{\beta\beta}(\hat{\theta})}, \quad \hat{\kappa} \pm Z_{p/2} \sqrt{I^{-1}_{\kappa\kappa}(\hat{\theta})};
\]

\[
\hat{\alpha} \pm Z_{p/2} \sqrt{I^{-1}_{\alpha\alpha}(\hat{\theta})} \quad \text{and} \quad \hat{\theta} \pm Z_{p/2} \sqrt{I^{-1}_{\theta\theta}(\hat{\theta})}
\]

where \( Z_{p/2} \) is used as the 100 \( (1 – p) \)% upper percentile of the standard normal distribution.

**DISCUSSION OF RESULTS / APPLICATION**

The findings from the study and application to real life examples are presented in this section.

**APPLICATION TO REAL DATASET**

In this section, we demonstrate the superiority of the proposed distribution WEIW using real data sets in reliability engineering to select the best model among the competing models. A statistical software \((R-4.0.3)\) is used to obtain the MLE of the model parameters. For the model selection, the criteria used are Akaike information criterion (AIC) and Bayesian information criterion (BIC) as:

\[
AIC = 2k - 2\ln(L) \quad \text{and} \quad BIC = 2\ln(L) + k\ln(n)
\]

where \( k \) is the number of parameters in the model and \( L \) is the maximized value of the likelihood function for the model. The AIC is the measure of the relative quality of a statistical model for a given set of data. BIC is also a criterion for model selection among a finite set of models and is closely related to the AIC. In Table 2, the IEW model is more appropriate in terms of AIC and BIC since the values of AIC and BIC of the WEIW model were smallest among the competing models. Therefore, we can conclude that the performance of the WEIW model is better.

The output values from goodness of fit for making decisions were also generated for \( LL \) P-values and Kolgomorov Smirnoff (K-S) statistics which are also choice of better model selection criteria. The model with best fit to the data is expected to have the smallest estimated value of estimated model criteria. The dataset on tests of the endurance of deep groove ball bearings from [28] was adopted and applied to the WEIW distribution and other competing family-related models. The results are presented in Table 1 and Figure 3.

**Table 1:** Maximum Likelihood Estimates and Criteria for Model Selection for drug-resistant tuberculosis on Ball Bearings Data

<table>
<thead>
<tr>
<th>Models</th>
<th>Parameter Estimates</th>
<th>AIC</th>
<th>BIC</th>
<th>LL</th>
<th>K-S</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>WEIW</td>
<td>( \hat{\alpha} = 0.6712, \hat{\beta} = 0.0771, \hat{\theta} = 16.1518, \hat{b} = 3.0251, \hat{k} = 0.9797 )</td>
<td>778.430</td>
<td>789.741</td>
<td>384.377</td>
<td>0.1343</td>
<td>0.1344</td>
</tr>
<tr>
<td>IW</td>
<td>( \hat{\theta} = 53.0868, \hat{k} = 1.4208 )</td>
<td>779.600</td>
<td>784.123</td>
<td>387.798</td>
<td>0.1617</td>
<td>0.1618</td>
</tr>
<tr>
<td>EIW</td>
<td>( \hat{\theta} = 53.4332, \hat{b} = 1.0032, \hat{k} = 1.7858 )</td>
<td>797.072</td>
<td>803.858</td>
<td>395.534</td>
<td>0.1672</td>
<td>0.1673</td>
</tr>
<tr>
<td>St.EIW</td>
<td>( \hat{\theta} = 12.9671, \hat{k} = 0.6857 )</td>
<td>835.268</td>
<td>839.791</td>
<td>415.632</td>
<td>0.2313</td>
<td>0.2314</td>
</tr>
</tbody>
</table>

**Figure 3:** Plots of the pdf and cdf of distributions on Ball Bearings Data

From Table 1, it is revealed that the WEIW has the smallest values of chosen model criteria and the smallest Kolmogorov Smirnoff test statistics; it has also revealed that the WEIW has the largest p-value and the lowest values of model chosen decision criteria which is an indication of a better performance over other models in Table 2. The
three-parameter EIW and the standard two-parameter st. EIW gave a worse fit than IW. Figure 3 which characterizes the density and estimated cdf plots with the other related models. Figure 3 which characterizes the density and estimated cdf plots with the other related models, also reinforced the conclusions from the computations in Table 3 that WEIW is the best choice for modeling these dataset.

Figure 4: The TTT plot for the endurance of deep groove ball bearings data. TTT = total time on test. Figure 4 provides a total time on test (TTT) plot for the durability of deep groove ball bearings data. Since the plot is concave and lying above the line, it means that its distribution may have an increasing hazard rate. Therefore it can be properly accommodated by a WEIW \((\alpha, \beta, \lambda)\) model with increasing failure rate.

CONCLUSIONS

The Exponentiated Inverted Weibull (EIW) model has been extended to obtain the WeibullExponentiated Inverted Weibull distribution (WEIW). The WEIW addresses some of the limitations identified with the EIW and also provides better flexibility than the EIW and the IW models. The statistical properties studied show that the distribution is positively skewed; the shape can be unimodal, approximately symmetric and is suitable for modeling right-shaped (positively skewed) data. The parameters of distribution were estimated using maximum likelihood estimation. A real-life dataset was used to examine its performance, and results from data analysis revealed that the WEIW distribution has the capacity to provide a better fit for modeling the real-life data.

AUTHORS’ CONTRIBUTIONS

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

REFERENCES


Bjerkedal T. Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli, American Journal of Epidemiology 1960; 72(1): 130-140.
