ON THE BARDEEN-COOPER-SCHRIEFFER (BCS) HAMILTONIAN

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ABSTRACT

We present a detailed diagonalization calculation of the Bardeen-Cooper-Schrieffer (BCS) Hamiltonian via the unitary transformation method which allows for the Bogolyubov-Valatin transformation to be arrived at naturally. The energy gap equation is derived and solved, and averages of some representative thermodynamic quantities are obtained.

KEYWORDS: Hamiltonian, unitary matrix, superconductivity, interaction, ground state.

1. INTRODUCTION

Superconductivity is the flow of electric current without resistance. The phenomenon was discovered by Kammerling Onnes in Leiden in 1911 and was given a microscopic quantum explanation by Bardeen, Cooper and Schrieffer (Bardeen, Cooper, Schrieffer, 1957). The BCS theory was soon found to apply to superconductors such as tin, lead, titanium and zinc besides many other elements in the periodic table. The critical temperatures ($T_c$) of these elemental superconductors fall below 10.0K. Superconductivity was later discovered in compounds such as Nb$_3$Sn, V$_3$Si, Nb$_3$Al and Nb$_3$Ge. However elemental and compound superconductors all have a serious drawback: their transition temperatures are well below 30K. This circumstance has proved to be a constraint in the exploitation of these superconductors in technology. Therefore there has been a great interest in finding materials that would superconduct at higher temperatures.

Theoretical microscopic explanation of superconductivity was not forthcoming before 1957, however, in 1950 Frohlich conjectured that electron-phonon interaction was the main process involved in superconductivity (Frohlich, 1950). Frohlich's work generated optimism about superconductivity, enough to make Kittel write in 1953 (Kittel, 1953) that a period of progress was at hand.

In 1986, IBM researchers Bednorz and Muller discovered high temperature superconductivity (HTSC) in ceramics (Bednorz, Muller, 1986). The Bednorz-Muller superconductor has $T_c=40K$.

Soon after a still higher temperature $T_c=90K$ was found in Yttrium compound (Wu et al, 1987). In 2009 $T_c=135K$ was seen in mercury barium copper oxide by Chu (Chu et al, 2009); the transition temperatures of recent cuprates are expected to increase further. Thus it is now possible to anticipate room temperature superconductors in the near future (Maksimov, 2008). As for the theory behind the Hi-$T_c$, the situation is that even twenty five years after the discovery of superconductivity in the cuprates, there is still no consensus about the mechanism of electron pairing in these substances (Sadovskii, 2004; Ashkanazi and Johnson, 2012). However many researchers favor the interpretation of experimental data (especially, Nuclear Magnetic Resonance (NMR) and Angle Resolved Photoemission Spectroscopy (ARPES)) in terms of the electron-phonon interactions (EPI) mechanism (Kuli, 2004; Wysokinski, 1998). One main reason for the revival of interest in EPI is the discovery of superconductivity in MgB$_2$ in 2001 with an unexpectedly high $T_c=40K$ (Muranaka, Akimitsu, 2011). All experimental data on MgB$_2$ satisfy the EPI mechanism (Kotegawa, et al, 2001). In this paper we shall study the BCS Hamiltonian, which is concretely based on the EPI.

A technically convenient way to derive the equations of superconductivity theory is by using a unitary matrix to diagonalize the BCS Hamiltonian. The unitary matrix method (UMM) allows us to obtain the Bogolyubov-Valatin transformations as a matter of course, making the gap equation to follow naturally. By making use of the grand potential difference with respect to the normal and superconducting states, thermodynamic quantities can be found.

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2. The BCS Hamiltonian

Let us write the BCS Hamiltonian in terms of the chemical potential $\mu$ and particle number $N$ as

$$H' = H'_e - \mu N + H'_i = H'_e - \mu N + \frac{1}{V} \sum_{p'p} U(p-p') a^+_p a_{p'}$$

where $U(p-p')$ is the pair potential, $a^+_p, a_p$ are the spin ($\sigma = \uparrow, \downarrow$) creation and destruction operators of electrons respectively and $V$ is the volume. The kinetic part of the Hamiltonian is

$$H_e = -\mu N = \sum_{p\sigma} (\epsilon_p - \mu) a^+_p \sigma a_p$$

Combining the two expressions, we have

$$H' = \frac{1}{V} \sum_{p\sigma} \xi(p) a^+_p \sigma a_p + \frac{1}{V} \sum_{p'p} U(p-p') a^+_p a_{p'} a_{p'}$$

where $\xi(p) = \epsilon_p - \mu$ is the quasiparticle energy. Let us transform the interaction part of the Hamiltonian (3) in the mean field approximation:

$$H_i = \frac{1}{V} \sum_{p} U(p-p')(a^+_p a_{p} a^+_p a_{p})$$

Making the notation

$$\Delta = \frac{1}{V} \sum_{p} U(p-p')(a^+_p a_{p})$$

then

$$H_i = \sum_{p} (\Delta a^+_p a_{p} - a^+_p a_{p})$$

The Hamiltonian (1) takes the form,

$$H' = \sum_{p} \xi(p) a^+_p a_p + \sum_{p} \Delta a^+_p a_{p} + \sum_{p} \Delta a^+_p a_{p} = \sum_{p}(a^+_p a_{p} - \Delta a^+_p a_{p})$$

3. Diagonalization of $H'$

We seek a canonical transformation that will diagonalize the Hamiltonian (7): the unitary matrix method (UMM) is best suited for this programme. Consider the unitary matrix $U_p$ and its adjoint $U^*_p$, in the form of 2x2 matrix.

$$U_p = \begin{pmatrix} u_p & -v_p \\ v_p & u_p \end{pmatrix}, \quad U^*_p = U_p^* = \begin{pmatrix} u_p^* & v_p \\ -v_p & u_p^* \end{pmatrix}$$

Now for the 2x2 matrix in (7), we have (ignoring the subscripts for convenience),

$$H'_m = U_p \left( \xi_p \Delta - \xi_p \Delta \right) U^*_p = \left( \begin{array}{cc} u^* & v^* \\ -v & u \end{array} \right) \left( \begin{array}{cc} \xi & \Delta \\ \Delta & -\xi \end{array} \right) \left( \begin{array}{cc} u & -v^* \\ v^* & u^* \end{array} \right) = \left( \begin{array}{cc} (u^2 - v^2)\xi + 2uv\Delta & (u^2 - v^2)\Delta - 2uv\xi \\ (u^2 - v^2)\Delta - 2uv\xi & (v^2 - u^2)\xi - 2uv\Delta \end{array} \right) = \left( \begin{array}{cc} E & 0 \\ 0 & -E \end{array} \right)$$

In forming (9) we used the reality of $u^*, v^*$ and $\Delta$. The two equations obtained from (9) are

$$(u^2 - v^2)\xi + 2uv\Delta = E, \quad (u^2 - v^2)\Delta - 2uv\xi = 0$$

These equations are solved to give: $E\xi = (\xi^2 + \Delta^2)a$, $E\Delta = (\xi^2 + \Delta^2)b$, where $a = u^2 - v^2, b = 2uv$.

Therefore we have the eigenvalue $E_p = \sqrt{\xi_p^2 + \Delta_p^2}$ and as a result

$$\text{diag} \left( \frac{\xi_p}{\Delta_p}, -\frac{\xi_p}{\Delta_p} \right) = \begin{pmatrix} \sqrt{\xi_p^2 + \Delta_p^2} & 0 \\ 0 & -\sqrt{\xi_p^2 + \Delta_p^2} \end{pmatrix}$$
Now for the sake of clarity we write the diagonalization of the entire Hamiltonian (7) in the form
\[ H_{\text{diag}} = \sum_p (\hat{a}_{p1} \hat{a}_{-p1}) U U^\dagger \left( \frac{\xi_p}{\Delta_p} \right) \left( \frac{\Delta_p}{\xi_p} \right) U U^\dagger \left( \frac{a_{p1}^\dagger}{a_{-p1}^\dagger} \right) \]

Here
\[ U^\dagger \left( \begin{array}{c} \hat{a}_{p1} \\ \hat{a}_{-p1} \end{array} \right) = \left( \begin{array}{c} \gamma_{p1} \\ \gamma_{-p1} \end{array} \right), \quad \left( \begin{array}{c} a_{p1}^\dagger \\ a_{-p1}^\dagger \end{array} \right) U = \left( \begin{array}{c} \gamma_{p1}^\dagger \gamma_{-p1} \end{array} \right) \]

where \( \gamma_{p1}, \gamma_{p} \) are the new creation and destruction operators that have replaced \( a_{p1}^\dagger a_p \) respectively. The new operators also satisfy the commutation rule \( \gamma_{p1} \gamma_{p}^\dagger + \gamma_{p} \gamma_{p1}^\dagger = 1 \). Then
\[ H_{\text{diag}} = \sum_p \left( \gamma_{p1} \gamma_{p} \right) \left( \begin{array}{cc} E_p & 0 \\ 0 & -E_p \end{array} \right) \left( \begin{array}{c} \gamma_{p1}^\dagger \gamma_{-p1} \end{array} \right) = \sum_p E_p \gamma_{p1}^\dagger \gamma_{p} \]
The operators \( a, a^\dagger \) describe the state of the non-interacting individual particles of the Fermi gas and can no more be used when the gas particles are interacting. Thus when the interaction is switched on the Fermi gas will now be described in terms of \( \gamma_{p1} \) and \( \gamma_{p} \) (Lifshitz, Pitaevskii, 1978).

### 4. Bogolyubov-Valatin transformation

The ‘old’ operators \( a^\dagger, a \) can be written in terms of the \( \gamma_{p1}^\dagger, \gamma_{p} \) as follows:
\[ \left( \begin{array}{c} a_{p1} \\ a_{-p1} \end{array} \right) = \text{inv} \left( \begin{array}{cc} u_{p}^\dagger & v_p \\ -v_p & u_{p} \end{array} \right) \left( \begin{array}{c} \gamma_{p1}^\dagger \\ \gamma_{-p1} \end{array} \right) = \left( \begin{array}{c} u_{p} \gamma_{p1} + v_p \gamma_{-p1}^\dagger \\ -v_p \gamma_{p1}^\dagger + u_{p} \gamma_{-p1} \end{array} \right) \]
where inv indicates the taking of the inverse of the 2x2 matrix. Therefore
\[ \alpha_{p1} = u_{p} \gamma_{p1} + v_p \gamma_{-p1}^\dagger, \quad \alpha_{-p1} = u_{p} \gamma_{-p1}^\dagger + v_p \gamma_{p1} \]
Equations (16) are the Bogolyubov-Valatin transformations. The row matrix of \( a^\dagger, a \) can also be written in terms of \( \gamma_{p1}, \gamma_{-p1} \) as follows
\[ \left( \begin{array}{c} a_{p1}^\dagger \\ a_{-p1} \end{array} \right) = \left( \begin{array}{c} \gamma_{p1}^\dagger \gamma_{-p1} \end{array} \right) U_p^{-1} = \left( \begin{array}{cc} u_{p} \gamma_{p1} + v_p \gamma_{-p1}^\dagger \\ -v_p \gamma_{p1}^\dagger + u_{p} \gamma_{-p1} \end{array} \right) \]
This last expression can also be obtained directly from the Bogolyubov-Valatin transformation (16). We may now be able to calculate the mean of a pair of operators such as
\[ \left( \begin{array}{c} \alpha_{p1} \alpha_{-p1}^\dagger \end{array} \right) = u_{p} \gamma_{p1} \left( -1 + 2 \left( \gamma_{p1}^\dagger, \gamma_{p1} \right) \right) = - \frac{u_{p} \gamma_{p1} \gamma_{p} \tanh E_p}{2T} \]
From equation (10) and the eigenvalue \( E_p \), we find
\[ u_{p} \gamma_{p1} \gamma_{p} = \frac{1}{2} \left( 1 - \frac{\xi_p^2}{E_p} \right) \frac{1}{\tilde{f}_2}, \quad u_{p}^2 + v_p^2 = 1 \]
where
\[ u_{p}^2 = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{\xi_p^2 + \Delta_p^2}} \right), \quad v_p^2 = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{\xi_p^2 + \Delta_p^2}} \right) \]
are the Bogolyubov coefficients. Substituting (18) and (19) in (5) we obtain
\[ \Delta_p = \frac{1}{2T} \sum_{p'} U(p - p') \Delta_{p'} \frac{\gamma_{p1} \gamma_{p} \tanh E_{p'}}{E_{p'}} \]
The expression (21) is the BCS gap equation.
5. Solution of the gap equation

In eqn.(21) the summation can be replaced by an integration at the thermodynamic limit $N/V$ as $N \to \infty, V \to \infty$. That is

$$\frac{1}{2V} \sum_p \to \int \frac{d^3 p}{(2\pi)^3} = \int \frac{\xi^2 d\xi d\Omega_p}{2\pi^2} = \frac{N(0)}{4} \int d\xi \int d\Omega_p$$

where $N(0)$ is the density of states at the Fermi surface. In the BCS Fermi system the summary spin of the pair is zero, therefore the gap function $\Delta_p$ is isotropic. The solid angle integration helps to write

$$\int U(p - p') d\Omega_p/4\pi = V_{\text{eff}} = g$$

and the gap equation becomes

$$1 = N(0)g \int_0^\omega_D \frac{d\xi}{\xi^2 + \Delta^2} \tanh(\beta \sqrt{\xi^2 + \Delta^2})$$

where $\omega_D$ is the Debye cut-off frequency, and $\beta = \frac{1}{kT}$, at zero temperature,

$$\int_0^\omega_D d\xi = \frac{\omega_D}{2} \ln \left( \frac{2\omega_D}{\Delta_0} \right)$$

Eqn.(23) will now take the following form

$$1 = N(0)g \ln \left( \frac{2\omega_D}{\Delta_0} \right).$$

This yields the BCS gap function

$$\Delta_e = 2\omega_D \exp \left( -\frac{1}{N(0)g} \right), \quad \Delta_e = \Delta(T = 0)$$

From expression (23) the equation for the transition temperature $T_c$ can also be determined if the integrand is represented as

$$\int_0^{\omega_D} \frac{dE}{2kT} \tanh \frac{E}{2kT} = \int_0^{\omega_D/kT} \frac{dx}{x} \tanh x = \ln \left( \frac{\omega_D}{\Delta_0} \right)$$

here $\ln y \approx 0.577$ is the Euler constant. Equation (23) can now be written as

$$1 = N(0)g \ln \left( \frac{2\omega_D}{\pi T_c} \right)$$

Solving equation (28) for $T_c$ yields the BCS formula for transition temperature (Bardeen, Cooper, Schrieffer, 1957)

$$T_c = 1.14 \omega_D \exp \left( -\frac{1}{N(0)g} \right)$$
6. The mean of some thermodynamic quantities

The analyses given above can enable averages of pertinent thermodynamic quantities to be calculated (Fetter, Walecka, 1971). From the eqn. (2), we write the mean of the kinetic energy as

$$\mu_{p}\gamma_{-s}$$

Let us calculate the mean potential or pairing energy by defining it as follows

$$\langle E_{pot} \rangle = \frac{1}{\mathcal{V}} \sum_{p}(pp_{p}) \langle U(pp_{p}) \rangle \langle \alpha_{p}^{+} \alpha_{p} \rangle \langle \alpha_{-p}^{+} \alpha_{p} \rangle$$

then using the Bogolyubov-Valatin transformations (16) we get

$$\langle E_{pot} \rangle = -\frac{\sum_{p} \Delta_{p}^{2}}{2E_{p}} \tanh E_{p}$$

The total average energy of the Fermi system is thus equal to

$$\langle E - \mu N \rangle = \sum_{p} \left( \frac{\xi_{p}^{2}}{2E_{p}} - \frac{\Delta_{p}^{2}}{2E_{p}} \tanh E_{p} \right)$$

The difference in energy between the normal and superconducting states at zero temperature is known as the condensation energy; this can be determined if it is noticed that for the normal state at T=0K, $$\Delta_{p} = 0, E_{p} = \xi_{p}$$. Thus

$$\langle E - \mu N \rangle_{n} - \langle E - \mu N \rangle_{s} = \sum_{p} \left( \frac{\xi_{p}^{2}}{2E_{p}} - \frac{\Delta_{p}^{2}}{2E_{p}} \right) = \frac{\mathcal{V}N(0)}{4\pi} \int_{0}^{\xi_{F}} \left( \frac{\xi_{p}^{2}}{\xi_{F}^{2} + \Delta_{p}^{2}} \right) d\xi$$

$$= \frac{\mathcal{V}N(0)\Delta_{F}^{2}}{4\pi}$$

7. Calculations in terms of the grand potential

The mean value of the interaction energy (4) can be represented in the form

$$\langle H_{I} \rangle = \frac{1}{\mathcal{V}} \sum_{p} U(p, p') \langle \alpha_{p}^{+} \alpha_{p} \rangle \langle \alpha_{-p}^{+} \alpha_{p} \rangle = -\frac{\Delta_{p}^{2}}{g}$$

where $$\Delta_{p} = g \langle \alpha_{p}^{+} \alpha_{p} \rangle = -g \langle \alpha_{p}^{+} \alpha_{p} \rangle$$. From statistical physics, we have for the grand potential (Abrikosov et al, 1975) $$\frac{d\Omega}{d\mathcal{V}} = \frac{1}{\mathcal{V}} \langle H_{I} \rangle$$, and integrating this, with respect to the superconducting and normal states, we get the result (Abrikosov et al, 1975)

$$\Omega_{s} - \Omega_{n} = \mathcal{V} \int d\mathcal{G} \langle H_{I} \rangle = -\mathcal{V} \int \frac{d\Delta^{2}}{\mathcal{G}^{2}} d\mathcal{G} = \mathcal{V} \int_{0}^{\Delta_{F}} \left( \frac{1}{\mathcal{G}} \right) d\mathcal{G} d\Delta$$

Taking 1/\mathcal{G} from (23) and substituting it in (36), one obtains the expression

$$\Omega_{s} - \Omega_{n} = \mathcal{V}N(0) \int_{0}^{\Delta_{F}} d\xi \int_{0}^{2\Delta_{F}} \frac{2\Delta_{F}^{2}}{E_{p}} \tanh E_{p} \frac{2\Delta_{F}^{2}}{E_{p}} d\Delta$$

which is integrated by parts and after a little algebra yields

$$\Omega_{s} - \Omega_{n} = \frac{\mathcal{V}N(0)\Delta_{F}^{2}}{2} - \mathcal{V}N(0)\Delta_{F} \ln \frac{\Delta_{F}}{\beta} + \frac{4\mathcal{V}N(0)}{\beta} \int_{0}^{\Delta_{F}} \ln(1 + e^{-\beta E}) d\xi + \frac{1}{\beta^{2}} \frac{\mathcal{V}N(0)\pi^{2}}{3}$$
According to the general principles of statistical physics (see for example, Lifshitz and Pitayevskii, 1978), the expression (38) is true for all thermodynamic potentials. As a result we can write (37) for the internal energy as

\[ E_x - E_n = -VN(0)\frac{\Delta^2}{2} - VN(0)\Delta^2 \ln\frac{\Delta}{\Delta_0} + \frac{1}{3} VN(0)\pi^2(kT)^2 - 4VN(0)kT \int_0^\infty \ln(1 + e^{-\beta\xi}) d\xi \]

On the strength of (39) for the normal state of the electronic system, the energy and heat capacity are

\[ E_n = \frac{1}{3} VN(0)\pi^2k^2T^2, \quad \frac{C_n}{V} = \frac{2}{3} VN(0)\pi^2k^2T \]

The integral in the last term of eqn. (39) yields

\[ e^{-\beta\Delta} \left( \frac{2\pi\Delta}{\beta} \right)^{3/2} \]

therefore when all temperature independent terms are dropped in that equation, the electronic energy and specific heat in the superconducting state are respectively

\[ E_x = 2VN(0)(kT)^{1/2}(2\pi\Delta_0)^{1/2}e^{-\beta\Delta_0}/kT, \quad \frac{C_x}{V} = \left( \frac{2\pi}{1/2} N(0) \right) e^{-\beta\Delta_0}/kT \]

The calculations presented here are valid at low temperatures where \( \Delta \gg T \), but for \( T \) close to \( T_c \) the size of the gap \( \Delta \) is small and an approximate formula for the gap and \( T_c \) can be found. For this let us rewrite the gap equation (21) in the form

\[ \Delta_p = \frac{g}{4\pi\beta} \int_0^\infty \Delta_p \sum_{n\neq 0} \frac{1}{\omega_n^2 + \frac{\Delta^2}{(2\pi\beta)^2}} d^2p = \frac{2VN(0)}{4\pi\beta} \int_0^\infty d\xi \sum_{n\neq 0} \frac{\Delta_p}{\omega_n^2 + \xi^2 + \Delta^2} \]

Using the expansion

\[ \frac{1}{(\omega_n^2 + \xi^2 + \Delta^2)} = \frac{1}{\omega_n^2 + \xi^2} - \frac{\Delta^2}{(\omega_n^2 + \xi^2)^2} + \frac{\Delta^4}{(\omega_n^2 + \xi^2)^3} \]

we have from (42) after some integration:

\[ \frac{1}{g} = \frac{2VN(0)}{4\pi\beta} \int_0^\infty \frac{2\xi \tanh\xi}{2\xi} d\xi - \frac{2VN(0)}{4\pi\beta} \left( \frac{\beta}{\pi} \right)^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{3}{16\pi\beta} V\gamma(0)\Delta^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \]

In the last expression, we have used for fermions \( \omega_n = \pi(2n+1)/\beta \), and now applying the prescription

\[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = (1 - 2^{-3})\zeta(3) \]

to eqn. (43), and replacing its first term by eqn. (27) we obtain the result

\[ \ln\left( \frac{2\omega_0 \nu}{\pi T_c} \right) = \ln\left( \frac{2\omega_0 \nu}{\pi T} \right) - \frac{7}{8} \zeta(3) \frac{\Delta^4}{(\pi T)^2} + \frac{93}{128} \zeta(5) \frac{\Delta^4}{(\pi T)^4} \]

where \( \zeta(p) \) is the zeta function. Then,

\[ \ln\left( \frac{T}{T_c} \right) = -\frac{7}{8} \zeta(3) \frac{\Delta^4}{(\pi T)^2} + \frac{93}{128} \zeta(5) \frac{\Delta^4}{(\pi T)^4} \]

Retaining the first term on the right hand side and using the expansion

\[ \ln\left( \frac{T}{T_c} \right) = 1 - \frac{T}{T_c} + \frac{1}{2} \frac{\zeta(3) \Delta^2}{(\pi T)^2} \]

we get

\[ \left( 1 - \frac{T}{T_c} \right) = \frac{7}{8} \zeta(3) \Delta^2 \]
Therefore the equation of the gap at temperature $T_c-T \ll T_c$ is
\[
\Delta = \pi T_c \cdot \frac{8}{7 \zeta(3)} \sqrt{1 - \frac{T}{T_c}} \approx 3.08 T_c \sqrt{1 - \frac{T}{T_c}}
\]

8. **Relationship between $\Delta(T=0)$ and $\Delta(T)$**.

Consider the general gap equation (21) which can be written in the form
\[
g^2 \int \frac{1}{\sqrt{\xi^2 + \Delta^2}} d^3p = 1
\]
where $n_p = (1 + e^{-\frac{E}{T}})^{-1}$.

Equation (48) can be represented as (Lifshitz and Pitayevskii, 1978)
\[
-1 + \frac{g}{2} \int \frac{d^3p}{\sqrt{\xi^2 + \Delta^2}} = g \int \frac{n_p d^3p}{\sqrt{\xi^2 + \Delta^2}}
\]

At $T=0K$,
\[
\frac{1}{(2\pi)^3} \int \frac{d^3p}{\sqrt{\xi^2 + \Delta^2}} = N(0) \int \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} = N(0) \ln \left( \frac{2\omega_D}{\Delta_0} \right)
\]
and at $T \neq 0$,
\[
\frac{1}{(2\pi)^3} \int \frac{d^3p}{\sqrt{\xi^2 + \Delta^2}} = N(0) \ln \left( \frac{2\omega_D}{\Delta} \right)
\]

Now substituting eqn.(25),(51), and (52) in (50) yields for the left hand side of that equation
\[
-\frac{N(0)g}{2} \ln \frac{2\omega_D}{\Delta_0} + \frac{N(0)g}{2} \ln \frac{2\omega_D}{\Delta} = N(0)g \ln \frac{\Delta}{\Delta_0}
\]
The integral on the right hand side of eqn. (50) is evaluated (Lifshitz and Pitayevskii, 1978) to give
\[
N(0)g \exp \left( -\frac{\Delta_0}{T} \right) \sqrt{\frac{2\pi T}{\Delta_0}}.
\]
Finally eqn.(50) can be written as
\[
\ln \frac{\Delta}{\Delta_0} = e^{-\frac{\Delta_0}{T}} \sqrt{\frac{2\pi T}{\Delta_0}}
\]

In the usual way the left hand side of (54) is expanded to give
\[
\frac{(\Delta_0 - \Delta)}{\Delta_0} \quad \text{as a result of which eqn (54) becomes}
\]
\[
\Delta = \Delta_0 \left( 1 - e^{-\frac{\Delta_0}{T}} \sqrt{\frac{2\pi T}{\Delta_0}} \right)
\]

We may write also
\[
\ln \left( \frac{2\omega_D}{\Delta} \right) = \frac{1}{N(0)g} + \ln \left( \frac{\Delta_0}{\Delta} \right)
\]
then
\[
\ln \left( \frac{\Delta}{\Delta_0} \right) = \ln \left( \frac{2\omega_D}{\Delta} \right) - \ln \left( \frac{2\omega_D}{\pi T} \right) + \frac{\Delta^2}{(\pi T)^2} \frac{7\zeta(3)}{8} - \frac{\Delta^4}{(\pi T)^4} \frac{93}{128} \zeta(5)...
\]

From here in the first order approximation the expression for $T_c$ in terms of $\Delta_0$ is obtained
\[
T_c = \frac{\gamma \Delta_0}{\pi}
\]
A special property of superconductors is the $\lambda$-shape jump in the specific heat capacity at the critical point. The jump is sensed by the thermodynamic quantities such as the grand potential, the Helmholtz free energy and the entropy. Substituting $1/g$ (eqn (44)) in eqn (36) yields the Helmholtz free energy difference at $T \to T_c$ as

$$F_s - F_n = -\frac{4VN[0][T_c^2]}{7\zeta(3)}\left(1 - \frac{T}{T_c}\right)^2$$

If note is taken that $\Omega_s - \Omega_n = \Delta \Omega$, then the entropy change of the electron gas at $T_c$ is given as

$$S_s - S_n = -\frac{4VN[0]T_c^2}{7\zeta(3)}\left(1 - \frac{T}{T_c}\right)$$

The jump in specific heat capacity at $T \to T_c$ is given by

$$C_s - C_n = \Delta C_v = \frac{T}{\Omega} \left(\frac{\partial^2 \Omega}{\partial T^2}\right) = \frac{8VN[0]}{7\zeta(3)} T_c$$

Everywhere in this work the density of states is taken at the Fermi surface, its explicit value being $N(0) = \frac{m^*}{g_F^2} \pi^2$, where $m^*$ is the electron effective mass and $p_F$ is the Fermi momentum.

**CONCLUSION**

The relevance of the BCS theory consists of its methodology, its physical, and rich mathematical foundation. It serves as a testing ground for statistical physics and thermodynamics of many-body systems and provides a variety of techniques for handling new physical problems (Barlas and Varma, 2012). In recent times one of such problems is finding a correct mechanism for high temperature superconductivity in the cuprates and pnictides (Choi, 2012). It is our opinion that such a mechanism when found will receive an explanation that will not radically deviate from the BCS formalism.

**REFERENCES**


superconductors from realistic Frohlich and Coulomb interactions. Cond-mat/12080240.
