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THE COMPUTATION OF SYSTEM MATRICES FOR BILINEAR SQUARE FINITE ELEMENTS

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ABSTRACT

An algorithm for generating the system matrices corresponding to bilinear square finite elements is given. Also a relationship is established which enables the system matrices for a larger number of elements to be generated from system matrices of a smaller set of elements of the same type.

KEY WORDS: Bilinear square elements, system matrices, finite element method.

INTRODUCTION

This paper provides an algorithm for generating the system matrices corresponding to bilinear square finite elements on a square domain for the solution of second order partial differential equations. Also a relationship is established which enables the system matrices for a larger set of bilinear square elements to be obtained from the system matrices of a smaller set of elements of the same type.

In this paper the second order partial differential equations are those whose key integral (obtained from the variational form) is of the form.

$$\int_R\int \left(u_t v + u_x v_x + u_y v_y\right) dx d_y = 0$$

where R is a square domain (Strang & Fix, 1973; Norrie & Vries, 1973).

If we let ϕ_j be the basic function, for the trial space on R, associated with node j of the finite element method subdivisions, then

$$M = \left(m_{ij}\right), \qquad m_{ij} = \int_R\int \phi_i \phi_j \, dxdy \qquad and$$

$$K = \left(k_{ij}\right), \qquad k_{ij} = \int_R\int \left(\phi_{i,x}\phi_{j,x} + \phi_{i.y}\phi_{j,y}\right) dxdy$$

with $\phi_{\ell,s} = \dfrac{\partial}{\partial s}\phi_\ell$, are the finite element system matrices (Norrie & de Vries, 1973).

M is the mass matrix while K is the stiffness matrix (Barry, 1974; Norrie & de Vries, 1973; Strang & Fix, 1973). The solution of the key integral is reduced by Finite Element Method to the solution of the system of equations:

$$MP(t) + KP(t) = 0$$

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where P is the vector of nodal parameters (Olayi, 1977; Norrie & de Vries, 1973; Strang & Fix, 1973).

WHY SQUARE DOMAIN/ELEMENTS

The author's use of square rather than rectangular domain or elements is based on: (a) any domain that admits rectangular elements can also admit square elements; (b) computation is greatly simplified if square elements are used instead of rectangular elements; (Barry, 1974; Olayi, 1977).

Any rectangular domain, with vertices (x_1,y_1), (x_2,y_1), (x_2,y_2) and (x_1,y_2) in the (x, y) - coordinate system can be transformed into a unit square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1) in the (s, t) coordinate system, by the transformation:

$$x = x_1 + (x_2 - x_1)s$$

$$y = y_1 + (y_2 - y_1)t$$

This transformation may not even be necessary as any rectangle can be subdivided into squares.

GENERATING THE SYSTEM MATRICES

Divide the square domain R, into N^2 square subdivisions. Associate with each node j the basic function ϕ_j which is made up of bilinear elements. ϕ_j is nonzero on those elements which share node j and zero elsewhere (see any of the references for details on ϕ_j).

Theorem 1

If R is divided into N^2 equal squares of size h and $\phi_j(x,y)$, $j = 1,2,3, \ldots, n^2$, where $n = N + 1$, are the basis functions of the trial space composed of bilinear square elements, then the mass and stiffness matrices have only six distinct entries each, i.e.

a) $$\left\{\iint \phi_i \phi_j \, dxdy \,\middle|\, i,j = 1,2,\ldots n^2\right\} = \{0,\ m,\ 2m,\ 4m,\ 8m,\ 16m\}$$

b) $$\left\{\int_R\int \left(\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right)dxdy \,\middle|\, i,j = 1,2,\ \ldots\ldots\ n^2\right\}$$

$$= \{-k,\ -2k,\ 0,\ 4k,\ 8k,\ 16k\}$$

where $m = h^2/36$; $k = 1/6$.

Proof

The $(i, j)^{th}$ elements of M and K depend on the positions of the nodes i and j, and those in the same class have the same value as follows.

A. Diagonal elements, i.e. i = j

D-1 If i is a node at a vertex (corner note) then $m_{ij} = \int_R\int \phi_i^2 \, dx\, dy = 4m$ and

$$k_{ij} = \int_R\int \left[\phi_{i,x}\phi_{i,x} + \phi_{i,y}\phi_{i,y}\right]dxdy = 4k$$

D-2 If i is a node on the boundary of R other than a corner node

$m_{ij} = \int_R \int \phi_i^2 dx\, dy = 8m$ and $k_{ji} = \int\int \left(\phi_{i,x}^2 + \phi_{i,y}^2\right) dxdy = 8k$

D-3 If i is a node in the interior of R, $m_{ij} = \int_R \int \phi_i^2 dx\, dy = 18m$ and

$k_{ij} = \int\int \left(\phi_{i,x}^2 + \phi_{i,y}^2\right) dxdy = 16k$

B. Off diagonal elements i.e. $1 \neq j$

0-1 If i and j are adjacent nodes on the boundary of R, then

$$m_{ij} = \int\int \phi_i \phi_j \, dxdy = 2m \ , \ k_{ij} = \int\int \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy = -k$$

0-2 If i and j are adjacent nodes at least one of which is an interior node, then

$$m_{ij} = \int\int \phi_i \phi_j \, dxdy = 4m \text{ and } k_{ij} = \int\int \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy = -2k$$

0-3 If i and j are nodes on the same diagonal of the same element,

$$m_{ij} = \int\int \phi_i \phi_j \, dxdy = m \text{ and } k_{ij} = \int\int \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy = -2k$$

0-4 If i and j are at least 2h apart, $m_{ij} = \int\int \phi_i \phi_j \, dxdy = 0$,

$$k_{ij} = \int\int \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy = 0 .$$

The above can be easily established once the ϕ_j are constructed. Remember ϕ_j is a piecewise bilinear function, nonzero only on the elements on which it is defined; and so the limits of integration depend on the associated elements (Strang and Fix, 1973; Olayi, 2000). If each element e is transformed into the square,

$S = \{(x,y) \mid 0 \le x,y \le h\}$, then only integrals of the following form:

(i) $\int_0^h \left(1 - \frac{1}{h}\right)^2 d\omega$, (ii) $\int_0^h \omega \left(1 - \frac{\omega}{h}\right) d\omega$

(iii) $\int_0^h \omega^2 d\omega$ and (iv) $\int_0^h d\omega$.

are involved in the computation of m_{ij} and k_{ij}. This completes the proof.

It is clear from the proof that the system matrices can be generated using D-1 to D-3 and 0-1 to 0-4, by constructing only a few selected basis functions and the elements fall into six classes only regardless of the size of the system matrices. This will save a lot of time, labour and space.

Theorem 2

Suppose $M^{(1)}$ and $K^{(1)}$ are the system matrices corresponding to N_1^2 bilinear square finite elements of R of size h_1 while $M^{(2)}$ and $K^{(2)}$ are the system matrices for N_2^2 bilinear squares finite elements of size h_2, where $N_2^2 > N_1^2$.

Then

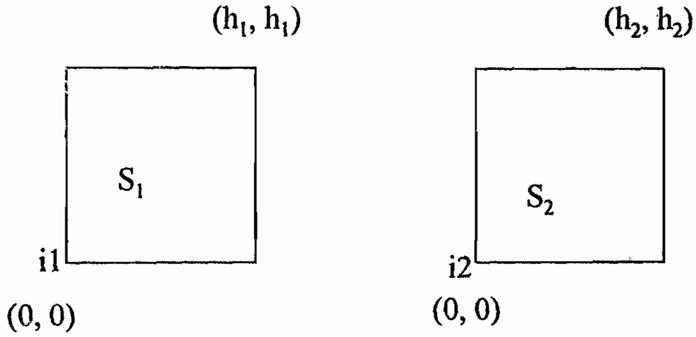
(a) $m_j^{(2)} = (1/\beta^2)m_j^{(1)}$, $j = 1, 2, \ldots\ldots 6$

(b) $k_\ell^{(2)} = k_\ell^{(1)}$, $\ell = 1, 2, \ldots\ldots 6$

where $\beta = h_1/h_2$.

Proof

For each of the two subdivisions, choose one element from each which falls into the same classification as given and transform them into S_1 and S_2. Note that $N_2^2 > N_1^2$ implies $h_1 > h_2$.



Let ϕ_{i1}, ϕ_{j1} be the basis function for node i1, j1 respectively of S_1 and ϕ_{i2}, ϕ_{j2} the basis functions for the corresponding nodes i2, j2 of S_2.

$$\phi_{i2}(t,\mu) = \phi_{i1}(h_1 t/h_2, h_1\mu/h_2)$$
$$= \phi_{i1}(\beta t, \beta\mu) \qquad \text{(i)}$$

Similarly $\phi_{j2}(t, \mu) = \phi_{j1}(\beta t, \beta\mu)$ (ii)

$$m_{i2,j2}^{(2)} = \int_{S_2}\int \phi_{i2}(t,\mu)\,\phi_{j2}(t,\mu)\,dt d\mu$$

$$= \int_{S_2}\int \phi_{i1}(\beta t,\beta\mu)\,\phi_{j1}(\beta t,\beta\mu)\,d\mu dt$$

$$= h_2^2/h_1^2 \int_{S_1}\int \phi_{i1}(v,\omega)\,\phi_{j1}(v,\omega)\,dv d\omega$$

$$= \left(\frac{1}{\beta^2}\right) m_{i1,j1}^{(1)}$$

And

$$k_{i2,j2}^{(2)} = \int_{S_2}\int \left(\phi_{i2,t}(t,\mu)\ \phi_{j2,t}(t,u) + \phi_{i2,\mu}(t,\mu)\,\phi_{j2,\mu}(t,\mu)\right) dt dv$$

$$= \beta^2 \int_{S_2}\int \left(\phi_{i1,t}(\beta t, \beta\mu)\,\phi_{j1,t}(\beta t, \beta\mu) + \phi_{i1,\mu}(\beta t, \beta\mu)\,\phi_{j1,\mu}(\beta t, \beta\mu)\right) dt d\mu$$

$$= \int_{S_2}\int \left(\phi_{i1,v}(v, \omega)\,\phi_{j1,v}(v, \omega) + \phi_{i1,\omega}(v, \omega)\,\phi_{j1,\omega}(v, \omega)\right) dv d\omega$$

$$= k^{(1)}_{i1,j1}$$

This completes the proof.

This theorem is important since the accuracy of the Finite Element Method depends on the size of the element. When the number of subdivisions is increased to achieve a better result, it is no longer necessary to start all over from the beginning to compute the corresponding system matrices.

CONCLUSION

Theorems 1 and 2 provide the second (the first was for system matrices of linear triangular elements; Olayi, 2000) algorithm in automating the generation of system matrices in Finite Element Method. Several algorithms have been published and are being used for automatic generation of finite elements (Bellingeri and co., 1995).

It is hoped that this will encourage others to seek for algorithms that can be used to automatically generate system matrices corresponding to other types of finite elements. This will not only ease the complexity in finite element programming but also will bring the most powerful technique for solving a large class of partial differential equations, (Strang & Fix, 1973), within the reach of more people.

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