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A NEUMANN PROBLEM FOR AN ELASTIC CYLINDER UNDER OUT-OF-PLANE LOADING

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ABSTRACT

The deformation fields for a solid cylinder subjected to self-equilibrating out-of-plane shear loads are studied by analysing the general functional form of the displacement field, which were derived as a solution of a Neumann problem. The shear stress states along the segments $\theta = \pm\pi$, $0 < r < a$ and at the origin are determined. The stress component at the surface of the cylinder, which is not immediately predictable from applied load is also derived.

Key words: out-of-plane shear; solid cylinder; right-half plane; stress concentration.

INTRODUCTION

A homogenous and isotropic solid cylindrical material occupying the region $-\infty < z < \infty, r \leq a$, $-\pi \leq \theta \leq \pi$, is subjected to uniformly distributed self-equilibrating shear loads of magnitude τ along the segments $r = a$, $0 < \theta \leq \pi$ and $-\pi \leq \theta \leq 0$ (Fig. 1). All components of displacement vanish except $w(r, \theta)$, the one perpendicular to the plane z = constant, which satisfies the Laplace equation.

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w(r, \theta)=0, \ -\pi \leq \theta \leq \pi, \quad r \leq a \qquad (1)$$

The loading induces the boundary conditions

$$\frac{\partial w}{\partial r}(a,\theta)=\begin{cases}\dfrac{\tau}{\mu} & 0<\theta<\pi\\ -\dfrac{\tau}{\mu} & -\pi<\theta<0\end{cases} \qquad (2)$$

where μ is shear modulus of elasticity. Equations (1) - (2) will be solved to get the general form of the displacement field everywhere in the cylinder.

Analytic Solution By Transform Technique

Theocaris and Defermos, 1964, have analysed a rectangular strip under plane stress using a different technique. Here, the task of solving (1) - (2) in a cylinderical region is transformed into a right-half plane problem using the holomorphic function

$$\zeta(z)=\frac{a+z}{a-z}, \quad z=x+iy$$

Let $\zeta(r, \theta) = u(r, \theta) + iv(r, \theta)$

and denote a polar coordinate (ρ, ϕ) for the right-half plane (Fig. II) by

$$u = \rho\cos\phi \qquad \text{and} \qquad v = \rho\sin\phi$$

Then

$$u(r, \theta)=\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}$$

$$v(r, \theta)=\frac{2ar\sin\theta}{a^2-2ar\cos\theta+r^2}$$

and

$$\rho(r, \theta)=\left(\frac{a^2+2ar\cos\theta+r^2}{a^2-2ar\cos\theta+r^2}\right)^{1/2} \qquad (3)$$

$$\tan\phi(r, \theta)=\frac{2ar\sin\theta}{a^2-r^2} \qquad (4)$$

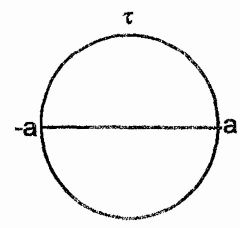

Fig. 1: Geometry of the problem

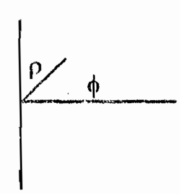

Fig. II: The right half plane

$$e^{i\theta} = \frac{\rho^2-1}{\rho^2+1}+\frac{2i\rho}{\rho^2+1}, \ \phi = \pi/2 \ (0< \quad)$$

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Therefore, for $-\pi \leq \theta \leq \pi$, we have

$$\frac{\partial \rho}{\partial r}(a,\ \theta) = 0 \qquad \frac{\partial \phi}{\partial r}(a,\ \theta) = \frac{1}{a \sin\theta}$$

where the relation between $\sin\theta$ and $\rho(a,\ \theta)$ can be derived using Fig. III.

Denoting the right-half plane displacement by $W(\rho,\ \phi) = w(r,\ \theta)$ we obtain

$$\frac{\partial w}{\partial r}(a,\ \theta) = \frac{\partial W}{\partial \phi}\left(\rho,\ \pm\frac{\pi}{2}\right)\frac{\partial \phi}{\partial r}(a,\ \theta)$$

The shear stresses that are non-zero are

$$\sigma_{\theta z}(r,\ \theta) = \frac{\mu}{r}\frac{\partial w}{\partial \theta}(r,\ \theta) \qquad (5)$$

$$\sigma_{\gamma z}(r,\ \theta) = \mu\frac{\partial w}{\partial r}(r,\ \theta) \qquad (6)$$

Problem (1) - (2) is thus transformed into the Neumann problem

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right)W(\rho,\ \phi) = 0\ , \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \qquad (7)$$

$$\rho > 0$$

$$\frac{\partial W}{\partial \phi}\left(\rho,\ \pm\frac{\pi}{2}\right) = \frac{2a\tau}{\mu}\rho\left(1+\rho^2\right)^{-1} \qquad (8)$$

The asymptotic relations as $\rho \to 0$ and as $\rho \to \infty$ are obtained by assuming a solution of the form

$$W(\rho,\ \phi) = O\left(\rho^k\right) \qquad (9)$$

and noting that the stresses have the behaviour (Earmme et al 1992)

$$\sigma_{\rho z}(\rho,\ \phi) = \sigma_{\phi z}(\rho,\ \phi) = O\left(\rho^{k-1}\right)$$

From (7) with $\rho < 1$, we have

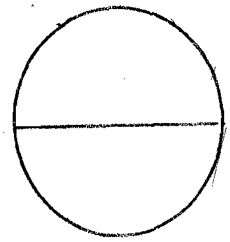
$$\sigma_{\phi z}\left(\rho,\ \pm\frac{\pi}{2}\right) = 2a\tau\left(1-\rho^2+\rho^4 - \ .\ .\ .\ \right)$$

implies $k = 1$ and yields $W((\rho,\ \phi) = O(\rho)$ as $\rho \to 0$

From (7) with $\rho > 1$, we have

$$\sigma_{\phi z}(\rho,\ \phi) = 2a\tau\left(\rho^{-2} - \rho^{-4} + \rho^{-6} - \ .\ .\ .\ \right)$$

implies $k = -1$ and gives $W(\rho,\ \phi) = O\left(\rho^{-1}\right)$ as $\rho \to \infty$



$$e^{i\theta} = \frac{\rho^2-1}{\rho^2+1} + \frac{2i\rho}{\rho^2+1}\ ,\ \phi = -\pi/2\ (-\pi\ 0<\theta<\pi)$$

Fig. III: Corresponding coordinates at the boundary of the cylinder

Let $\overline{W}(s,\ \phi)$ denote the Mellin transform of $W(\rho,\ \phi)$ defined by

$$\overline{W}(s,\ \phi) = \int_0^\infty W(\rho,\ \phi)\rho^{s-1}\,d\rho\ ,\quad -1 < \mathrm{Re}\,s < 1 \qquad (10)$$

Taking the Mellin transform of (7) and (8) gives

$$\left(\frac{d^2}{d\phi^2} + s^2\right)\overline{W}(s,\ \phi) = 0 \qquad (11)$$

$$\frac{d\overline{W}}{d\phi}\left(s,\ \pm\frac{\pi}{2}\right) = \frac{2a\tau}{\mu}f(s) \qquad (12)$$

where

$$f(s) = \int_0^\infty \left(1+\rho^2\right)^{-1}\rho^s\,d\rho$$

Using formular 3.241 2 (Gradshyteyn and Ryzhik 1965) we get

$$f(s) = \frac{\pi}{2\cos\frac{\pi}{2}s} \qquad -1 < \mathrm{Re}\,s < 1$$

The solution

$$\overline{W}(s,\ \phi) = A\sin s\phi + B\cos s\phi \qquad (13)$$

together with (12) give

$$sA\cos\tfrac{\pi}{2}s - sB\sin\tfrac{\pi}{2}s = \frac{2a\tau}{\mu}f(s)$$

$$sA\cos\tfrac{\pi}{2}s + sB\sin\tfrac{\pi}{2}s = \frac{2a\tau}{\mu}f(s)$$

Therefore $B = 0$ and (13) becomes

$$\overline{W}(s, \phi) = \frac{\pi a \tau \sin s\phi}{\mu s \cos^2 \frac{\pi}{2} s}$$

to which we apply the inverse Mellin transform

denoted by

$$W(\rho, \phi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{W}(s, \phi) \rho^{-s} \, ds$$

to get the displacement as

$$W(\rho, \phi) = \frac{\pi a \tau}{\mu 2 \pi i} \int_{c-i\infty}^{c+i\infty} \frac{\sin s\phi \; \rho^{-s}}{s \cos^2 \frac{\pi}{2} s} ds \,, \quad -1 < c < 1 \qquad (14)$$

In (14) the integrand has double poles at

$s = \pm(2n - 1)$, $n = 1, 2, 3, \ldots$ Residue theory is used to evaluate the integral. The contours are then closed appropriately, $\text{Re}\,s < 0$ for $\rho < 1$ and $\text{Re}\,s > 0$ for $\rho > 1$, in accordance with Jordan's lemma (Whittaker and Watson 1962). The displacement can then be written as:

$$w(r, \theta) = W(\rho, \phi)$$

$$= \frac{4a\tau}{\pi\mu} \begin{cases} -\ln\rho \sum_{n=1}^{\infty} \frac{\rho^{2n-1}}{2n-1} \sin(2n-1)\phi \\ \quad + \sum_{n=1}^{\infty} \frac{\rho^{2n-1}}{(2n-1)^2} \sin(2n-1)\phi \\ \quad\quad -\phi \sum_{n=1}^{\infty} \frac{\rho^{2n-1}}{2n-1} \cos(2n-1)\phi \qquad \rho < 1 \\ \qquad\qquad |\phi| \le \frac{\pi}{2} \end{cases} \qquad (15)$$

$$= \frac{4a\tau}{\pi\mu} \begin{cases} \ln\rho \sum_{n=1}^{\infty} \frac{\rho^{1-2n}}{2n-1} \sin(2n-1)\phi \\ \quad + \sum_{n=1}^{\infty} \frac{\rho^{1-2n}}{(2n-1)^2} \sin(2n-1)\phi \\ \quad\quad -\phi \sum_{n=1}^{\infty} \frac{\rho^{1-2n}}{2n-1} \cos(2n-1)\phi \qquad \rho > 1 \\ \qquad\qquad |\phi| \le \frac{\pi}{2} \end{cases}$$

[illegible]
displacement.

3. Stress Distribution

The general expression for obtaining the stresses anywhere in the cylinder is got from (15) by chain rule together with formulas (5) and (6).

The form of the stress along the segments $\theta = \pm\pi$, $\theta = 0$, $0 < r < a$ which correspond to the form with $\phi = 0$, $\rho < 1$ and $\rho > 1$ respectively, can now be derived.

For $\theta = \pm\pi$, $0 \le r < a$, we have

$$\frac{a-r}{a+r} = \rho < 1 \,, \quad \frac{\partial\rho}{\partial\theta}(r, \pm\pi) = 0$$

Then

$$\frac{\partial w}{\partial\theta}(r, \pm\pi) = \frac{\partial W}{\partial\phi}(\rho, 0)\frac{\partial\phi}{\partial\theta}(r, \pm\pi) \,, \quad \rho < 1$$

leads to

$$\sigma_{\theta z}(r, \pm\pi) = \frac{2a\tau}{\pi r} \ln\left(\frac{a-r}{a+r}\right)$$

since $\frac{\partial\phi}{\partial r}(r, \pm\pi) = 0$ and $\frac{\partial W}{\partial\rho}(\rho, 0) = 0$ it follows that $\frac{\partial w}{\partial r}(r, \pm\pi) = 0$. Hence $\sigma_{rz}(r, \pm\pi) = 0$. When $\theta = 0$, $0 \le r < a$ we have $\frac{a+r}{a-r} = \rho > 1$. Thus

$$\frac{\partial w}{\partial\theta}(r, 0) = \frac{\partial W}{\partial\phi}(\rho, 0)\frac{\partial\phi}{\partial\theta}(r, 0) \,, \quad \rho > 1$$

leads to

$$\sigma_{\theta z}(r, 0) = \frac{2a\tau}{\pi r} \ln\left(\frac{a+r}{a-r}\right)$$

In this case as well $\frac{\partial w}{\partial r}(r, 0) = 0$, together with (6) imply $\sigma_{rz}(r, 0) = 0$.

Since the stresses are continuous we apply L'hopital's rule to get the stress at the origin. The result is $\sigma_{\theta z}(0, 0) = \lim_{r \to 0} \sigma_{\theta z}(r, 0) = \frac{2a\tau}{\pi}$.

On the surface of the cylinder, $r = a$, $-\pi \le \theta \le \pi$ we have $\frac{\partial\rho}{\partial\theta}(a, \theta) = -\frac{(1+\rho^2)}{2a}$, $\frac{\partial\phi}{\partial\theta}(a, \theta) = 0$ and $\rho(a, \theta) = \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2}$.

Note that $\phi = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$, $\rho = 1$ map onto $\frac{\pi}{2} \le \theta < \pi$, $-\pi < \theta \le -\frac{\pi}{2}$, $r = a$ respectively, and that $\phi=\frac{\pi}{2}$, $\phi=-\frac{\pi}{2}$, $\rho>1$ map onto

$0<\theta\le\frac{\pi}{2}$, $-\frac{\pi}{2}\le\theta<0$, $r=a$ respectively.

Therefore

$$\frac{\partial w}{\partial\theta}(a,\theta)=\frac{\partial W}{\partial\rho}\left(\rho,\pm\frac{\pi}{2}\right)\frac{\partial\rho}{\partial\theta}(a,\theta)$$

together with (5) lead to

$$\sigma_{\theta z}(a,\theta)=\frac{\tau}{\pi a}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right),\quad 0<\theta<\pi$$

$$=-\frac{\tau}{\pi a}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right),\quad -\pi<\theta<0\ .$$

Conclusion

Though the solution of the Neumann problem (1) - (2) is not unique we obtained a solution that possesses standard characters relative to shearing.

The stress states indicate high concentration at the origin. $\sigma_{\theta z}$ (a, θ), $\theta \neq -\frac{\pi}{2}$, becomes smaller for larger cylinders (a > 1).

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