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COMPARATIVE ANALYSIS BETWEEN HOMOTOPY GROUP AND HOMOLOGY GROUP

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ABSTRACT

This paper seeks to demonstrate the relation between homology group and homotopy group. The result in this paper is a construction of the homology of a complex torus.

KEYWORDS: continuous, homotopy group, homology group, homotopy extension property, surjective, homeomorphism, homomorphism

INTRODUCTION

The study of Poincar*e* basic groups provided the initial motivation for homology theory in the history of Algebraic Topology. Several works on the configuration of points in higher-dimensional Euclidean space were published during his early years. The number of dimensional holes on a surface is computed using a notion called homology, which is comparable to Betti numbers. The fundamental group also introduced by Poincar*e* was the foundation of topology [Brazas, 2011].

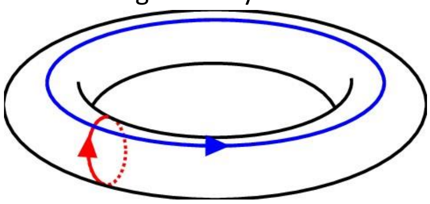

Figure 1: Homology cycles on a torus. With the red line indicating one cycle and blue line indicating the other cycle.

A two-dimensional surface is a torus (that is, the torus itself consists of a 2-dimensional hole with any point consisting of a 0-dimensional hole but has two 1-dimensional holes). Cycles are a term for the holes in the surface. In addition, it was through Emmy Noether that the homology groups were eventually recognized, as well as the assertion that the Betti numbers were essentially numerical invariants of isomorphism. This raises an interesting challenge of how to find a polyhedron's homology groups. To compute the homology of a polyhedron, one must first define the chain complexes and chain mappings before computing the homology group on the chain complex [Hilton, 1988]. But the geometric construction of simplicial complex S with vertices $v_{ii\in I}$ is also as follows. For $V_{ii\in I}$ being points of R^n, such that if $v_i,...,v_n$ is a simplex of S, then the points $V_i,...,V_n$ are linearly independent [Reynaud, 2003]. Also the notion of homotopy invariant seems to have been introduced by Hurewicz [Hurewicz, 1935] and well-studied by Dugindji [Dugundji, 1950]. Transferring topological data to algebraic homotopy groups is through endowing other structure with correlation to the algebraic structure [Brazas, 2011].

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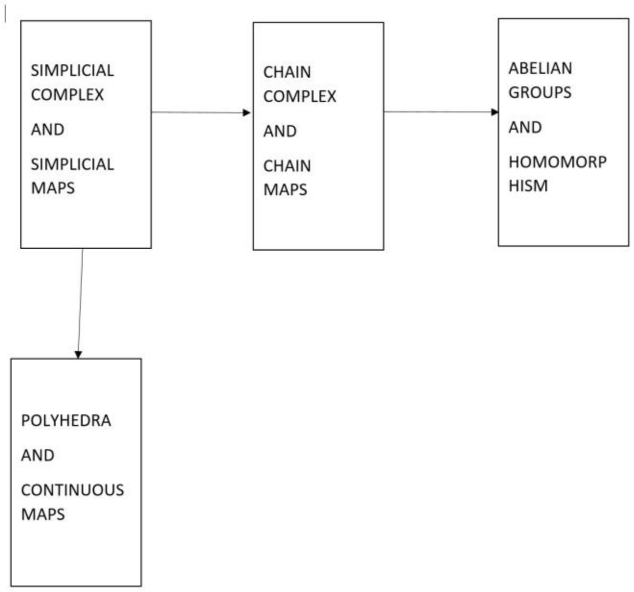

Figure 2: Phase of Homology theory on polyhedra

However, homology groups and homotopy groups do have a relationship. The close relation between $H_n(X)$ and $\pi_n(X)$ arise from a map *f: I −→ X that* can be viewed as a path [Spanier, 1989]. In comparison to homology groups, homotopy groups generalize basic groups but are hard to quantify or compute. Hurewicz contributed significantly to history by establishing a relationship between homotopy and homology groups. Homology theory works with pairings (X, A) and the homotopy extension problem is required in order for such constructions to operate [Whitehead, 2012]. Homotopy groups are the strongest invariant of a topological space [Christensen and Scoccola, 2020]. Homology groups and homotopy groups have other close relation at least for certain class of topological spaces [Adhikari, 2016a].

2 Related works

In Eilenberg and MacLane, (1945) Eilenberg investigated works on the influence of fundamental group on the homology structure of a space X. The paper introduced a relation between homotopy and homology on a finite simplicial connected polytype. J. P. Hlton has also elaborated on two decompositions of a continuous map of a connected space. When the map $f : X \longrightarrow Y$ is retracted to a point, the homology decomposition of f becomes a homology decomposition of Y , while the homotopy decomposition of f becomes a homotopy decomposition of Y [Eckmann and Hilton, 1959].

3 Preliminaries

Definition 1 ([Warner, 2018]Continuity)**.** *Let* (X,τ) *and* (Y,μ) *be topological spaces. A function* $f : X \longrightarrow Y$ *is continuous if for each* $U \in \mu$ *there is* $f^{-1}(U) \in \tau$

Proposition 1. *The function* $f : X \longrightarrow Y$ *is continuous.*

Proof. For if A ⊆Y is closed, then $f^{-1}A = \cup f_i^{-1}(A)$; but $f_i^{-1}(A)$ is closed in X so that f is continuous. □

Definition 2 ([Kinsey, 1997]Homology Group)**.** *The n-dimensional homology group of a polyhedron P is the quotient group* $\ker\partial_n / Im\partial_{n+1}$*.The group* $H_n(P)$ *is the denotion of the total homology group of P.*

Example 1. *The segment I=[0,1] can be represented as a simplicial polyhedron with 1-dimensional simplex α and 2-dimensional simplices α and β [Vasiliev and Vasiliev, 2001].*

Definition 3 ([Hatcher, 2005]Homotopy Groups)**.** *Two continuous maps, f,g :* $X \longrightarrow Y$ *are called homotopic*$(f \sim g)$ *if f can be class of continuous* *maps. F:* $X\times[0,1] \dashrightarrow$ *such that* $F(x,0) = f(x)$ *and* $F(x,1) = g(x)$ *for all* $x \in X$.

Definition 4 (Cycle)**.** *Given S as a topological space, a cycle is a continuous function* $f^1 : [0,1] \leftarrow X$ *such that* $f(0) = f(1)$.

Definition 5 ([Eilenberg and Steenrod, 2015]Retract)**.** *A continuous map* $f : X \dashrightarrow B$ *is called a retraction if f is the identity on B such that* $f(b) = b$ *for all* $b \in B$.

Definition 6 ([Kinsey, 1997]Deformation retract)**.** *f is called a deformation retract if for a continuous map* $f : X \dashrightarrow B$*, f is homotopic to the identity map.*

Definition 7 ([Hurewicz, 1935]Homotopy extension property)**.** *Given a map,* $f_i : X \dashrightarrow Y$ *and a subspace B* $\subset X$ *there exists a homotopy* $F : B \dashrightarrow Y$ *of* $f_i|B$ *that extends to a homotopy* $F : X \dashrightarrow Y$ *of* f_i*. If the pair* (X,B) *is such that the extension problem can always be solved then* (X,B) *has the homotopy extension property.*

Definition 8 ([Hatcher, 2005]Null homotopy)**.** *Given* $f: X \dashrightarrow Y$*, a null homotopy of f is a homotopy of f to a constant map, denoted as* $f \simeq 0$.

Definition 9 ([Eda and Higasikawa, 2001] Loops)**.** *A loop is a path* $f: [x,y] \dashrightarrow P$ *such that* $f(x) = f(y)$.

Theorem 1 (Homotopy extension theorem)**.** *Let P be a complex, Q a subsimplex* $f_1 : P \dashrightarrow Y$ *and* $g_1 : Q \dashrightarrow Y$ *a homotopy such that* $g_1 = f_1Q$*. Also there exists a homotopy* $f_2 : P \dashrightarrow Y$ *such that* $g_2 = f_2Q$.

Proof. From the above theorem, we realized that $g_1 : Q \dashrightarrow Y$ is a homotopy map and can be extended to P. We also extended g_2 to P and define a homotopy map $F : P\times 0 \cup Q\times 1 \dashrightarrow Y$. To extend F to a map of $P\times 1$, we write P^n for the complex $P^n \cup Q$ and also extend F to $F^1 : (P \times 0)\cup(P^1\times 1) \dashrightarrow Y$ for $(u,x)F^1 = uf_1$, where u is a vertex of $P - Q$.

Suppose F extends to $F^n : (P \times 0) \cup (P^n \times 1) \dashrightarrow Y$ and let $T_{n+1} \in P - Q$, then F^n is defined on $(t_1\times 0)\cup(t_2\times 1)$ but we need to prove for the case when F^n is extended to $(t_1 \times 1)$. Then F^n may be extended to $F^{n+1:} (P \times 0) \cup (P^{n+1} \times 1) \dashrightarrow Y$ being continuous on $t_1 \times 1$ for each simplex t_2 of P^{n+1} and hence F^{n+1} is continuous.

We also show that there exists a map $r: t_1\times 1 \dashrightarrow (t_1\times 0) \cup (t_2\times 1)$ which is the retraction map $(t_1 \times 0) \cup (t_2 \times 1)$ from $t_1 \times 1$ and then rF^n is an extension F^n to $t_1 \times 1$. For t embedded in R^{n+1} has dimension n+1 then $t\times 1$ is embedded in R^{n+2}. We let c be the point $(b, 2)$ with first coordinate $(n + 1)$ coordinates and $(n$+2) being the second coordinate. The radial projection from c retracts $t_1 \times 1$ onto $(t_1 \times 0) \cup (t_2 \times 1)$ □

Theorem 2 ([Adams and Franzosa, 2008]Homotopy equivalence)**.** *The relation* $\simeq$ *is an equivalence relation on the set of all continuous functions* $f: X \dashrightarrow Y$

Proof. We show that the relation obeys the following properties;

Reflexive

Symmetric

Transitive

Therefore we show that $\simeq$ is reflexive and define homotopy map $F : X \times I \dashrightarrow Y$ by $F(x,t) = f(x)$, where $t \in [0,1]$. Then $F(x,0) = f(x)$ and $F(x,1) = f(x)$, so $f \simeq f$.
For $\simeq$ to be symmetric, we define a function $g : X \dashrightarrow Y$ such that $G : X \times I \dashrightarrow Y$ such that $G(x,t) = F(x,1 - t)$ is a homotopy between g and f. Therefore $f \simeq g$ and $g \simeq f$
To prove the transitive property, we define another function $h: X \dashrightarrow Y$. We assume that $f \simeq g$ via F and $g \simeq h$ via G and define a homotopic map

$H : X \times I \longrightarrow Y$ by $H(x,t) = (F(x,2t)\ for\ 0 \leq t \leq\ \frac{1}{2}$ an[...] $t \leq\ 1)$.

Since $H(x,0) = f(x)$ and $H(x,1) = h(x)$ and hence $f \simeq g$ and $g \simeq h$ implies that $f \simeq h$ □

Remark. *Homeomorphic spaces are homotopy equivalent but its converse is not true in general [Adhikari, 2016b].*

Definition 10 ([Whitehead, 2012] Homology equivalence)**.** *A map* $f: X \dashrightarrow Y$ *such that* $f_n : H_p(X) \approx H_q(Y)$ *for all p is called a homology equivalence.*

4 Relation between Homology Group and Homotopy Group

In this section, we look at the Hurewicz theorem and the homology of a torus with relation to the torus homotopy.

Definition 11 ([Hatcher, 2005] Chain complex)**.** *A sequence of abelian groups given below;*

$$\cdots \to G_i \xrightarrow{\partial_i} G_{i-1} \longrightarrow \cdots \partial_{i-1} \quad \xrightarrow{\partial_1} G_0$$

iss called a chain complex if $\partial_{i-1} \circ \partial_1 = 0$ for all i. And the above complex is exact if $im\partial_i = \ker\partial_{i-1}$

Proposition 2. *A simply connected space C is homotopy equivalent to a onepoint union of Moore spaces if and only if $h_n(C)$ is a split surjective for all n.*

Definition 13 ([Christensen and Scoccola, 2020]Hurewicz homomorphism)**.** *The Hurewicz homomorphism $h_n(X)$: $\pi_n(X) \longrightarrow H_n(X)$ is defined by $h_n(f) \simeq f$*

Theorem 3 (Hurewicz Theorem)**.** *The homomorphism h_n is an isomorphism if X is path-connected.*

Proof. **Omitted**

Theorem 4. *Given a topological space X and two subspace A and B such that $A \subseteq X$ and $B \subseteq X$. If $f: X \longrightarrow Y$, then*

$$H_n(X, A) \simeq H_n(X, B)$$

Proof. Let X be a torus, $T^2 = S^1 \times S^1$ with $A = S^1 \times \{0\}$ and $B = S^1 \times \{1\}$. We define a homotopy map $H : X \times [0,1] \longrightarrow Y$ and identify each S^1 of the torus with the subspace. If $A \simeq B$, then A is homeomorphic to B. We therefore define an inclusion map $i_\star : A \longrightarrow X$ and retract the X to A. This means that X is homotopic equivalence to A and

$$\therefore H_\star(A) \longrightarrow H_\star(X)$$

is an isomorphism.

Also the map $j_\star : B \longrightarrow X$ is an inclusion map on $B = S^1 \times \{1\}$ such that X can be retracted to B. Finally we can use the exact sequence in homology to show that

$$H_1(X,A) \simeq H_1(X,B)$$

.

Definition 14 ([Lisica, 2010])**.** *Two points on M^n are called homological if they can be connected by a path in M^n.*

4.1 Computing the Homology of a torus

The *n−dimensional* torus is the product of *n−circle* groups S^1 [Dleck, 1982]. The torus can be constructed from the gluing of the opposite sides of the square in the figure below,

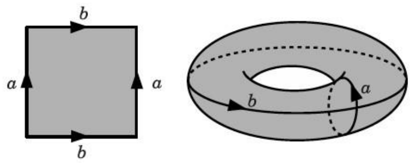

Figure 3: The complex of a torus on the left and the result after construction on the right

Construct a triangulation of the torus. Computing the O-dimensional homology gives Z. Note that the one-skeleton of the torus is a connected graph and its 0-dimensional homology coincides with the graph since only the complex G_0 and G_1 participate in H_0. The 1-dimensional homology of a torus is hardly to compute. To compute the 1-dimensional homology of a torus, we user the Euler characteristics of the torus as basis. Since the Euler characteristics of a torus is 0, therefore we conclude that $H_1 = Z \oplus Z$.

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