GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL. 17, NO.3, 2011: 295-298
 ISSN 1118-0579
www.globaljournalseries.com, Email: info@globaljournalseries.com

A KINEMATIC MODEL FOR CALCULATING THE MAGNITUDE OF ANGULAR MOMENTUM TRANSFER IN AN ACCRETION DISK: I-TIME-INDEPENDENT CASE

J. A. OBU AND F. O. FAITHPRAISE

(Received 21 September 2010; Revision Accepted 2 May 2011)

ABSTRACT

Keplerian velocity laws imply the existence of velocity shear and shear viscosity within an accretion disk. Due to this viscosity, angular momentum is transferred from the faster moving inner regions to the slower-moving outer regions of the disk. Here we have formulated a model for calculating the magnitude of angular momentum transfer in a steady-state accretion disk using only two parameters; the transport coefficient of vorticity, ω and the rate of change of angular velocity with radial distance, $d\Omega / dR$. With this model, the mass accretion rate in an accretion disk $\dot{M}$, can be determined without necessarily making use of the observed value of the luminosity of the accreting system.

KEYWORDS: Keplerian Velocity, Shear viscosity, Accretion Disk, Coefficient of Vorticity

1. INTRODUCTION

An accretion disk is a material-gas disk found around numerous types of astrophysical objects ranging from protostars to massive black hole candidates at the center of galaxies. The disk structure is formed by material falling into a gravitational source at their centers. Depending on the system, the material gas is pulled into the accretion disk from either the interstellar medium or from another star. This gas undergoes a differential rotation, with the inner portions completing an orbit faster than the outer portions.

One basic idea behind the accretion disk theory is that viscosity in the gas disk converts the free energy of differential rotation into thermal energy which is then radiated away. As this potential energy is released, the gas slowly spirals inwards into the central object.

The shear viscosity associated with differential rotation of the disk is generally accepted as the cause of transportation of angular momentum to outer portion of the accretion disk. In this work we derived an expression to determine the magnitude of angular momentum transferred during the accretion process and use it to formulate an equation for the mass accretion rate, making use of the transport coefficient of vorticity (kinematic viscosity), ν and change of angular velocity with radial distance $(d\Omega/dr)$.

The motivation for this work stems from the fact that one can analytically determine the total energy emitted (i.e. its luminosity) from a disk from the rate of viscous dissipation.

2. The energy minimization model

To illustrate the basic issues involved in disk accretion we consider an idealized situation with two bodies orbiting a central point M. Suppose that masses m_1 and m_2 are in circular Keplerian orbits around the central mass M>> m_1 , m_2 (see Figure 1). The total energy, E, and angular momentum $\vec{J}$, of this system are

$$E = -\frac{GM}{2}\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right), \qquad 1$$

and

$$J = (GM)^{\frac{1}{2}}\left(m_1 r_1^{\frac{1}{2}} + m_2 r_2^{\frac{1}{2}}\right) \qquad 2$$

where r_1 and r_2 are the radial coordinates of the bodies with respect to the central mass. Now suppose that the orbits are perturbed by small amounts while conserving $\vec{J}$. Then the relation between the perturbations is

$$m_1 r_1^{-\frac{1}{2}} \Delta r_1 = m_2 r_2^{-\frac{1}{2}} \Delta r_2 \qquad 3$$

and the corresponding change in *E*, in terms of change Δr_1 of the first body is

$$\Delta E = -\frac{GMm_1 \Delta r_1}{2 r_1^2}\left[\left(\frac{r_1}{r_2}\right)^{\frac{3}{2}} - 1\right] \qquad 4$$

Suppose we wish to reduce the energy *E* of the system (e.g. because it is radiating). If $r_1 > r_2$ we can reduce the energy of the system if Δr_1>0, i.e. moving body 1 further away from the center. If body 1 is originally closer to the center, then the energy can be reduced by a negative displacement, i.e. moving body 1 closer in. Thus energy can be reduced while conserving orbital angular momentum by moving the initially closer body in and moving the initially outer body further out. This is the basic action of the accretion disk; energy is released as material both accretes and spreads to larger distances. If mass can be transferred between bodies, as shown by Lynden-Bell and Pringle (1974), then the system's energy can be minimized by transferring mass from the outer body to the inner body.

J. A. Obu, Department of Physics, University of Calabar, Calabar
F. O. Faithpraise, Department of Physics, University of Calabar, Calabar

This process needs a way to connect different particles in the disk, and one way of visualizing this is to consider two neighbouring annuli of width ℓ on both sides of a surface **r** = constant as shown schematically in Figure 1. The particles in the annuli rub against each other while undergoing differential rotation. If there is friction between adjacent annuli, the resulting torques will attempt to bring the two annuli into co-rotation. A net torque will then be exerted on the outer annulus, so that it spins up and gains angular momentum. Thus, the torque or transport of angular momentum is caused by the exchange of parcels of particles between each annulus, which bring differing angular momenta to the annuli to which they are transferred.

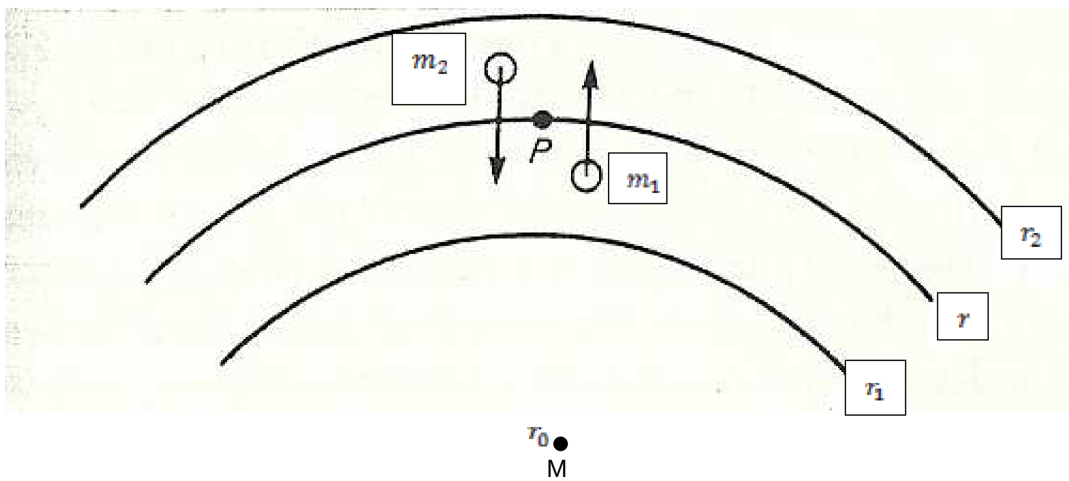

Figure 1: Schematic angular momentum transfer between two annuli in a disk undergoing differential rotation.

3. Calculation of angular momentum transfer in terms of coefficient of vorticity (kinematic viscosity)

In gaseous accretion disks, because the material diffuses in both directions at all radii, the kinematics of the system is more complicated than as modeled above. Various models of angular momentum transfer have been formulated (Balbus and Hawley, 1991; 2002). Here we have formulated the magnitude of angular momentum transfer in terms of kinematic viscosity, using the picture of a viscous torque g, which supposes that the gas exhibits small random or turbulent motions which cause radial mixing. This means that material between adjacent annuli will be exchanged, and since the material originating in the two annuli will have different specific angular momentum, this will cause a transfer of angular momentum.

In this model we assumed that, turbulent elements of the gas moving at a typical turbulent random velocity $\tilde{v}$ travel a mean free path ℓ before mixing with other materials on the other side of **r =** constant. Because the chaotic motion takes place in an equilibrium flow, this process cannot result in the net transfer of any matter between the two annuli, so that mass crosses the r = constant surface at equal rates in both directions (Bradshaw, 1971; Shakura and Sunyaev, 1973). This mass is of the order $H\rho\tilde{v}$, per unit arc length, where $\rho(r)$ is the mass density. And as earlier stated, since the two mass fluxes nonetheless carry different angular momenta, there is transport of angular momentum due to the chaotic process. A net torque or angular momentum transfer is thus produced at **r**, by the differing angular momenta of the two streams of material; one from material originating at **r** - ℓ/2 and moving outward across **r** to mix with annular material centered at **r**+ ℓ/2; and the other starting at **r** + ℓ/2 and moving inward across **r** to mix with the inner annulus at **r** - ℓ/2 (see Figure 1). This turbulent exchange drives the accretion mechanism envisioned by Lynden-Bell and Pringle (1974). Similar treatment is given by Hartmann (1998).

Noting that in this kinematic viscosity model, no net angular momentum is transported unless there is shearing orbital motion (i.e. $d\Omega / dr \neq 0$), the net angular momentum flux can be calculated by considering that material originating at **r** - ℓ/2 which is diffusing outwards has (as seen by a co-rotating observer), an orbital velocity *u*, so that

$$u\left(r - \tfrac{l}{2}\right) = \left(r - \tfrac{l}{2}\right)\Omega\left(r - \tfrac{l}{2}\right)$$

$$= \left(r - \tfrac{l}{2}\right)\left[\Omega(r) - \tfrac{l}{2}\tfrac{d\Omega}{dr}\right], \quad 5$$

where we have approximated the difference in angular velocities to first order using a Taylor series.

Thus the net angular momentum flux per unit length through **r** = constant in the inward direction is

$$\rho\tilde{v}H\left(r - \tfrac{l}{2}\right)\left[\Omega(r) - \tfrac{l}{2}\tfrac{d\Omega}{dr}\right] \quad 6$$

A similar expression

$$u\left(r + \tfrac{l}{2}\right) = \left(r + \tfrac{l}{2}\right)\left[\Omega(r) + \tfrac{l}{2}\tfrac{d\Omega}{dr}\right], \quad 7$$

and

$$\rho\tilde{v}H\left(r + \tfrac{l}{2}\right)\left[\Omega(r) + \tfrac{l}{2}\tfrac{d\Omega}{dr}\right] \quad 8$$

applies to the material at **r** + ℓ/2 for the orbital velocity, and angular momentum in the outward direction, respectively. The inner material diffuses outwards at a speed $\tilde{v}$ and the outer material diffuses inwards at $\tilde{v}$ across **r**. The net radial motion of material in the accretion disk is assumed to be small compared to the turbulent motion $\tilde{v}$.

We then integrate over the z-direction to average over disk structure perpendicular to the disk plane. Because this chaotic motion takes place in equilibrium flow, to first order, the process cannot result in any net transfer of mass between the two rings. Therefore, the mass crosses the surface **r** = constant at equal rates in both directions, of the order $\Sigma\tilde{v}$ per unit arc length for a disk with mass density per unit area $\Sigma = \int_{-\infty}^{+\infty} \rho dz$ .This gives the net outward transfer of angular momentum across **r** per unit length according to a co-rotating observer at **r** to be

$$\Sigma\tilde{v}\left[\left(r-\frac{l}{2}\right)^2 \times \frac{-l}{2}\frac{d\Omega}{dr} - \left(r+\frac{l}{2}\right)^2 \times \frac{l}{2}\frac{d\Omega}{dr}\right] = -\Sigma\tilde{v}lr^2\frac{d\Omega}{dr} \qquad 9$$

where we have assumed that ℓ is small compared to the scale over which Ω varies significantly. The product $\tilde{v}l$ is reminiscent of the transport coefficient of vorticity, (or kinematic viscosity) ν, (defined as $= \nabla \times \tilde{v}$), so that we can write

$$\nu = \tilde{v}l \qquad 10$$

The torque exerted by the inner annulus on the outer annulus gives the net outward angular momentum flux. With this result for the total outward angular momentum we can find the total torque, g of the inner annulus on the outer annulus simply by multiplying by 2π**r** to have

$$g(r) = -2\pi r\Sigma\nu r^2 \frac{d\Omega}{dr} \qquad 11$$

Note that a negative gradient of angular velocity, i.e. Ω decreasing outward leads to a positive outward flux of angular momentum.

4. Formulation of the mass accretion rate in terms of vorticity

We start by writing down the fluid equation for a thin disk in cylindrical coordinates. We expect u_θ to be the main component of velocity with small radial flow, u_r caused by the effect of viscosity. If we put $u_z = 0$ and $\partial/\partial\theta = 0$, then the continuity equation and the θ-component of the Navier-Stokes equation for viscous fluid flow (Nakayama,and Boucher, (1999) in cylindrical geometry are

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) = 0 \qquad 12$$

and

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r}\right) = \mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) \qquad 13$$

Note that we are using the Navier-Stokes equation obtained by assuming that the coefficient of viscosity μ , is constant. And the equation as given here is only valid when $\mu = \rho\nu$, is constant, which is not the case in an accretion disk. So we figure out the appropriate expression for viscous stress from first principles. In order for us to ignore the contribution of the pressure gradient in the radial force balance, we have supposed that the orbital speed $r\Omega$, much exceeds the isothermal sound speed $c^2 = kT/m$ in the midplane of the disk. Since the disk has a characteristic vertical height $H = c/\Omega$ we see that the approximation $r\Omega >> c$ holds to the extent that the disk is partially thin i.e. $H << r$.

Integrating eqns (12 and 13) over z and writing $\Sigma = \int_{-\infty}^{\infty} \rho dz$ for surface density, we have the final form of both the continuity equation given by

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma u_r) = 0 \qquad 14$$

and a preliminary expression for the momentum conservation given by

$$\Sigma\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r}\right) = \text{viscous terms} \qquad 15$$

where we have neglected the variation of u_r and u_θ with z (since, although ρ varies with z, we do not expect the components of velocity to vary very much).

We now add together, the equation of continuity (eqn. 14) multiplied by **r** u_θ and the Navier-Stokes equation (15) multiplied by **r.** Using the angular velocity, $\Omega = u_\theta/r$, this gives the angular momentum equation in standard conservation form:

$$\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma r^3\Omega u_r\right) = G \qquad 16$$

Here $\Sigma r^2\Omega 2\pi r dr$ is the angular momentum associated with an annular ring from between **r** and **r** + d**r,** and the second term on the left hand side is the divergence of angular momentum flux $\Sigma r^2\Omega u_r \hat{n}_r$, due to radial flow. The right hand side is the *sources-sink* term for angular momentum, which in this case is the viscous torque.

If we were to multiply eqn. 16 by $2\pi r dr$, then we get an equation telling us how the angular momentum of the annulus changes and we can write the net torque as

$$2\pi r\, G dr = g(r) - g(r+dr), \qquad 17$$

recalling that we defined g as the viscous torque in eqn. 11. Hence,

$$G = -\frac{1}{2\pi r}\frac{\partial g}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{d\Omega}{dr}\right) \qquad 18$$

Finally, by using the relation for viscous torque we can eliminate G from the angular momentum conservation equation to obtain

$$\frac{\partial}{\partial t}\left(\Sigma r^2\Omega\right) + \frac{1}{2}\frac{\partial}{\partial r}\left(\Sigma r^3\Omega u_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{d\Omega}{dr}\right). \qquad 19$$

We now consider the possibility of a steady disk, which means that we have to look for time-independent solutions of mass and momentum conservation. Putting the time derivative terms equal to zero, the mass and momentum conservation (eqns. 12 and 13) can be integrated to give

$$r\Sigma u_r = C_1 \qquad 20$$

and

$$\Sigma r^3 \Omega u_r - \nu \Sigma r^3 \frac{d\Omega}{dr} = C_2 \qquad 21$$

where C_1 and C_2, are constants of integration. For a steady disk, the mass outflow rate is $\dot{M}_{out} = 2\pi r \Sigma u_r$. This is a constant for a steady-state disk and so

$$C_1 = \frac{\dot{M}_{out}}{2\pi}. \qquad 22$$

To calculate C_2, we note that the matter at the surface of the gravitating body at $r = r_0$ must be dragged into a rigid rotation so that $d\Omega/dr = 0$ there. Then, from eqns. (20 - 22)

$$C_2 = \frac{\dot{M}_{out}}{2\pi} r^2{}_0 \Omega = \frac{\dot{M}_{out}}{2\pi}\left(GMr_0\right)^{\frac{1}{2}} \qquad 23$$

where we have assumed Keplerian rotation in the final step. On substituting this constant of integration back into the solution for the momentum equation (eqn. 21), and again making use of Keplerian motion, we get

$$\dot{M}_{out} = -\frac{3\pi\nu\Sigma}{\left[1-\left(\frac{r_0}{r}\right)^{\frac{1}{2}}\right]}. \qquad 24$$

This shows that the mass outflow rate ($\dot{M}_{out} < 0$) depends linearly on kinematic viscosity (or the vorticity) ν. The mass outflow rate equals the inflow (accretion) rate but in opposite direction, so that the accretion rate in terms of kinematic viscosity is thus,

$$\dot{M}_{acc} = \frac{3\pi\nu\Sigma}{\left[1-\left(\frac{r_0}{r}\right)^{\frac{1}{2}}\right]} \qquad 25$$

6. CONCLUSION

We have derived an expression to determine the magnitude of angular momentum transferred during the accretion process and use it to formulate a semi-analytical model of an accretion process. The result shows that in the case where the viscosity in an accretion disk is due to random molecular motions, a kinematic viscosity prescription is clearly appropriate for use to determine the mass accretion rate. This formulation is inherently very useful, since we don't need to know the luminosity of the accreting system to determine its mass flow rate. However, it is worthy to note that the model as described above would work well if the material in the accretion disk has a value of viscosity sufficient to cause enough mass inflow. The result obtained above will be extended in a future work to time-dependent accretion disks.

REFERENCES

Balbus, S. A. and Hawley, J. F., 1991. Powerful local shear instability in weakly magnetized disks.I. Linear Analysis. ApJ 376: 214.

Balbus, S. A. and Hawley, J. F., 2002. On the nature of angular momentum transport in nonradiative accretion flows. ApJ 573: 749-753.

Bradshaw, P., 1971. An introduction to turbulence and its measurements, Pergamon Press, Oxford.

Hartmann, L., 1998. Accretion power in star formation, Cambridge University Press, Cambridge.

Lynden-Bell, D. and Pringle, J. E., 1974. MNRAS 168:603.

Nakayama, Y. and Boucher, R.F., 1999. Introduction to fluid dynamics, John Wiley and Sons Inc., New York. 84-101.

Shakura, N. I. and Sunyaev, R. A., 1973. Astronomy & Astrophysics, 24:337.