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MODEL SETTING FOR OPTIMAL STOPPING AND SINGULAR STOCHASTIC CONTROL FOR OPTIMAL INVESTMENT STRATEGY IN OIL FIELD PROJECT

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Abstract

Investing in projects involving huge financial risks demands great care. Dealing with market uncertainty and taking effective investment decision in oil field project, therefore, requires a reliable guide. The strategy emerged from addressing a problem involving an optimal stopping time with singular stochastic control for jump diffusions. The strategy identified two unique thresholds, one indicating when to apply the control and the other showing when to quit. Optimal strategy for investment in oil field project were obtained. Two particular cases of Brownian motion and Geometric Brownian motion are presented. The model is set to include jumps in the analysis: to obtain better investment strategies in oil field project.

Introduction

The huge financial involvement in oil projects demand that a project manager and investor take great care in decision making. Oil field development projects face market risks largely because the parameter of key importance, the oil price, fluctuates rapidly over time. Moreover, there are other largely technical uncertainties such as the quantity of oil and gas reserves in the ground, as well as geological and reservoir structures. The implication of these uncertainties is high risk for field development and high risk. The problem facing the decision maker is, therefore, a problem of imperfect information. This makes the decision making process a challenging one.

When risk and uncertainty are involved, decisions cannot be taken with a “flip of the coin” strategy. To tackle the problem of "how and when" to invest, this work goes beyond calculations of expected return, and proposes an optimal strategy for investment in the project. A model that delivers this optimality is desired and presented. In the face of uncertainty over future market conditions, a viable model is set to obtain an optimal strategy. There is an attempt to answer the following questions: when should the investor invest and how should the investment be made. Ogbogbo (2016) has determined that the crude oil spot price process is a Jump-diffusion. The aim of the work is to obtain an optimal strategy for investment in an oil field project .The optimal strategy will involve a singular control and an optimal stopping time for the investment. Thus, the work will identify two unique thresholds for the investor; one threshold points out when to apply the control and the other indicates when to quit.

Some models for optimal strategy and control have been obtained for gas storage, number of wells to drill and for oil discovery and extraction. The work by Bringedal (2003) was on gas storage valuation. The gas storage facility was studied because of expanding gas market in Europe. Investing in a gas storage facility is similar to investing in an oil field development project. The objective of the work was to obtain a strategy which would identify a benchmark price level at which to refill the storage facility or sell off gas in it. The value of the storage facility was calculated with constant volatility and mean reversion parameters. Bringedal used a technique called stochastic dual dynamic programming (SDDP). The optimal *strategy* obtained defined a benchmark price level, x, at which one would sell if the spot price P_t is above it, and buy if spot price is below it, i.e. Sell if $P_t > x$ and *Buy* if $P_t < x$. Despite the effort at generating an optimal strategy, the assumption of constant volatility is considered a major simplification of the model. Though a mean reverting process was used in the model, the criticism of the Black-Schole model was based largely on assumption of constant volatility. Benkherouf & Pitts (2005) obtained optimal strategy on the number of oil wells to drill. Their work developed an oil exploration model. They obtained their results analytically, the uncertainty element is in the fact that $n_1$and n_2 are unknown, but are represented by a two-dimensional distribution π fixed a priori (as Euler family of distributions). n_1 is the number of large undiscovered oil wells and n_2 is the number of small undiscovered oil wells. n_1 and n_2 are non-negative integers. The objective of the work was to obtain optimal strategy for drilling that maximizes the total expected return over an infinite horizon, based on the entire history and future prospects.

Maurer & Semmler (2010) worked on an optimal control model of oil discovery and extraction. They obtained the optimal rate of extraction, given the price trajectory for an oil extraction and discovery problem. Using the Hamiltonian and maximum principles they solved the finite horizon optimal control problem which they formulated. They solved the resulting non- linear programming problem numerically using NUDOCCCS, i.e. they used discretization technique to transcribe the optimal control problem into a non-linear programming problem via the code NUDOCCCS. Generally, the Maurer-Semmler model was a finite horizon optimal control model that used two state variables: known stock of resource and cumulated past extraction.

The main similarity and differences between this paper and the existing Literature is briefly explained here to highlight the main contribution of the work.

Optimal strategies have been obtained by researchers in the works mentioned above, but the method and description of the dynamics of the oil price process in this model differs. The model set in this paper is similar to and in line with the general formulation of mixed stochastic optimal control and stopping time problems. However, beyond the setting of the model, this paper went on to obtain optimal strategy for investment in oil field project, in terms of thresholds (for control and stopping time). Basically, a partial differential equation (PDE), associated with the model is given and used along with Ito's lemma in obtaining the strategy. The paper specifically considers a case where there is a running cost function *g (s,x)*, defined as: $(s,x)= e^{-\rho s})$, and (in particular) for, $\rho=0$, where the controller pays a constant running cost that is not discounted. This yields a general solution, which the paper considers for Brownian motion and geometric Brownian motion.

The rest of the paper is presented as follows: the second section presents model basics, model assumption and discusses the mixed optimal stopping and singular stochastic control model.

Experimental

Mixed optimal stopping and singular control model for optimal investment strategy in oil field project

Model basics, assumptions and preliminaries

Definition: Jump diffusion. A Jump- diffusion or Lévy diffusion is the solution of a stochastic differential equation (SDE) driven by Lévy processes.

Consider the following stochastic differential equation in $\mathbb{R}^n: X(0) = X_0 \in \mathbb{R}^n$

$$dX_{(t)} = \mu(t, X(t)dt + \sigma(t, X(t)dW(t) + \int_{\mathbb{R}^n} \gamma(t, X(t^-), Z)\tilde{N}(dt, dZ) \text{...[1]}$$

Where $\mu: [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$,

$\sigma: [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $\gamma: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times l}$

Model basics and assumptions

Let X_t be the price of crude oil at time t. We make the following assumptions:

i. X_t is a jump diffusion.

ii. The system is stochastically controlled, there is a singular control, Γ_t which is the intervention. $\Gamma_i(t)$ is a right continuous increasing adapted process which could be considered as copious production and sales at some benchmark high prices from time i to t, in order to maximize revenue accruing.

iii. $d\Gamma$ may be singular with respect to Lebesque measure dt. [Defition: Given an open set $S=\Sigma_k(a_k,b_k)$ containing disjoint intervals, the Lebesgue measure is defined by $\mu_L(s)\equiv\Sigma_k(b_k-a_k)$].

iv. The investor (also called the controller) is observing a (price) system that is evolving with time, let η be the stopping time of the process.

v. There are costs involved. There is a cost for observing the system (waiting before taking a decision is at a cost, anytime a decision is taken a cost is paid). Let this cost be, $g(t,x_t)$. There is a terminal cost $m(\eta,x_\eta)$, should the controller decide to stop. The control is also applied at a cost given as $n(t, x_t)$.

vi. The jumps are not there at start time $t=0$

vii. Expect jumps at some point in time.

viii. It is assumed that all jumps are positive, $Z>0$ a.s.

Thus, it is a stochastically controlled system,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}} h(t^-, X_{t^-}, \gamma(Z))\tilde{N}(dt, dZ) - d\Gamma_t \quad \ldots [2]$$

The following performance criterion or objective functional for the system is defined,

$$J(t,x) = E^x\left[\int_0^\eta g(t,x_t)dt + m(\eta,x_\eta)|_{\square}\right] + \int_0^\eta n(t,x_t)d\Gamma_t \quad \ldots [3]$$

The challenge is to obtain an optimal strategy for investment that would minimize the performance criterion, $J^{\Gamma,\eta}(s,x)$. We want to find $\phi(s,x)$, the value function, and obtain the optimal strategy:

$(\Gamma^*, \eta^*) \in \Gamma \times \eta$ if (it exists) such that

$$\phi(s,x) = \min_{\Gamma\in\Gamma,\eta\in\eta} J^{\Gamma,\eta}_{(s,x)} = J^{\Gamma*,\eta*}_{(s,x)}$$

where, Γ^*, is the optimal control and η * is the optimal stopping time. $\Gamma\in\boldsymbol{\Gamma}$. $\boldsymbol{\Gamma}$ is the set of admissible controls. Γ is admissible if [2] has a unique strong solution $X^\Gamma(t)$, and [3] is finite.

Cost of Waiting to Invest

In reality irreversibility and the possibility of delay are very important characteristics of most investments, including oil field development projects. The ability to delay irreversible investment expenditures can have a significant effect on the decision to invest. According to Lund (1997) the flexibility of a project is simply a description of the options made available to management as part of the project. It is seen as the possibility to make adjustments or intervention or wait. One type of flexibility is the possibility of waiting for information. Flexibility, such as "Waiting to Invest" is seldom a free good. There is a cost involved. The optimality of decision for a potential investor, will emerge from two choice paths: money now or more money in the future, i.e. there is a choice of selling now, or waiting to watch the market, and then sell crude oil in the future.

The Concept of Optimal control

Two popular approaches central to optimal control theory are, the Hamilton-Jacobi-Bellman method and the Pontryagin Maximum Principle. An optical control problem consists of:

A state process usually denoted *X(t)*. $X(t)\in\mathbb{R}^d$ e.g. a Price Process.

A Control process $\Gamma(t)$. There is a control set Γ, in which $\Gamma(t)$ takes values for every *t*. Application usually dictates the choice of $\Gamma(t)\in\Gamma$. Additional constraints could be placed on $\Gamma(t)$, e.g. in the stochastic setting, Γ could be adapted to a certain filtration.

Admissible controls, Γ.

A control process satisfying the constraints is called an admissible control. The set of all admissible controls is denoted, Γ.

An objective functional J($X(t),\Gamma(t)$) which is the functional to be maximized (or minimized). J has an additive structure, and is given as an integral over t.

The Goal in an optimal control problem is maximize (or minimize) the objective functional J over all admissible controls.

Stopping time

A stopping time with respect to a sequence of random variable $X_1, X_2, \ldots$ is a random variable η, with property that for each t, the occurrence or non-occurrence of the event $\eta = t$ depends only on the values of $X_1, X_2, \ldots X_t$ and furthermore. $Prob(\eta < \infty)=1$. η is almost surely finite.

Optimal stopping has to do with target tracking problems where one has to decide when one has arrived sufficiently close to the target, threshold or benchmark. In Finance the stopping time question is "when do you pay dividend so you don't go bankrupt (i.e. tracking the point when you have accumulated enough wealth). In population dynamics, when do you start the fish harvesting so the fish population does not become extinct. And in this work, when do you sell to obtain maximum revenue or profit, (not invest at a loss). When do you quit.

The concept of singularity

Generally, singularity arises when a process is not well behaved in a particular sense, say, not differentiable. e.g.

$f(x) = \frac{1}{x}$ has singularity at $x = 0$.

With respect to the (controlled) system which is evolving with time, the process is not absolutely continuous over the interval.

A control Γ_t is a *Singular control* if the control is bounded, i.e. $a \leq \Gamma_t \leq b$ and

$$\Gamma_t = \begin{Bmatrix} b, & \phi(x,t) < 0 \\ ?, & \phi(x,t) \leq 0 \\ a, & \phi(x,t) > 0 \end{Bmatrix}$$

Specifically, if ϕ is positive at some times, negative at others and is only zero instantaneously, then the solution is straightforward. The case when ϕ remains at zero for a finite length of time $t_1 \leq t \leq t_2$ is the singular control case.

General formulation of mixed stochastic optimal control and stopping time problems

A probability space, (Ω, F, P) is operated on.

Definition: *A probability space* consists of the triple (Ω, F, P) on the sample space, Ω, where (Ω, F) is a measurable space, F is a collection of subsets of Ω, and P is a measure on F.

Definition: For a given set of returns, the rate at which the price of an asset (or commodity) varies (increases or decreases) is called *volatility* denoted σ. Basically, it is the variation from average over a given period.

Definition: The *drift* of an asset, denoted μ, is a measure of the average rate of growth of the asset(or commodity) price.

There is a controlled system, given as:

$$dX^{\Gamma}(t) = \mu(X(t),\Gamma(t)dt + \sigma(X(t),\Gamma(t))dW(t) + \int_{\mathbb{R}^k} \gamma(X(t^-),\Gamma(t^-),Z)\tilde{N}(dt,dZ)$$

$$X(o) = x \in \mathbb{R}^k \quad \ldots\ldots [4]$$

$\mu: \mathbb{R}^k \times u \to \mathbb{R}^k$, $\sigma: \mathbb{R}^k \times u \to \mathbb{R}^{k\times m}$ and $\gamma: \mathbb{R}^k \times u \times \mathbb{R}^k \to \mathbb{R}^{k\times L}$

μ, σ,and γ are given continuous functions and Γ is the control, assumed to be F_t adapted and with values in the closed, convex set $\Gamma \subset \mathbb{R}$.

Associated to a control $\Gamma = \Gamma(t, \omega)$ and an F_t stopping time $\eta = \eta(\omega) \subset \eta$ (set of admissible stopping times). There is a performance criterion of the form

$$J^{(\Gamma,\eta)}(y) = E^x \left[\int_0^{\eta} f\left(X(t),\Gamma(t)\right)dt + g(X(\eta))\chi_{\{\eta<\infty\}} \right]$$

where $f: \mathbb{R}^k \times \Gamma \to \mathbb{R}$ the profit rate function

$g: \mathbb{R}^k \to \mathbb{R}$ is the bequest function

f and g are giving function

It is assumed that there is a given set $\Gamma=\Gamma(x)$ of admissible controls, which is the set such that a unique strong solution
$X(t)=X^{\Gamma}(t)$ *of* [4] exists and the following [5] holds

$$E^{y}\left[\int_0^{\eta S}|f(X(t),\Gamma(t)|dt\right]<\infty \qquad \forall\, x\in S \ldots\ldots [5]$$

where, $\eta S=\eta S(x,u)=inf\{t>0; X^{(\Gamma)}(t)\notin S\}$.

The family $\{g^{-}(Y^{(\Gamma)}(\eta);\eta\in\eta\}$ is uniformly P^{x}-integrable for all $x\in S$,

where, $g^{-}(x)=\max(0;-g(x))$................[6]

$g(X(\eta(\omega))$ is interpreted as 0, *if* $X(\omega)=\infty$.

$S\subset\mathbb{R}^{k}$ is fixed Borel set such that $S\subset\overline{S^{0}}$.

S is called "Solvency region or solvency set". One may think of S as the "universe" of the system, in the sense that one is interested in the system up to time η which may be interpreted as the time of bankruptcy.
The combined optimal stopping and control-problem is therefore given as follows:
Find $\phi(x)$ and $\Gamma^{*}\in\Gamma,\eta^{*}\in\eta$ such that

$$\phi(x)=\sup\{J^{(\Gamma,\eta)}(y);\ \Gamma\in\Gamma,\eta\in\eta\}=J^{(\Gamma^{*},\eta^{*})}(x)\ldots\ldots [7]$$

where, η is the set of F_t – stopping times $\eta\leq\eta_s$. Oskendal & Sulem (2009) proved a verification theorem for this (and every) problem they discussed.

Optimal strategy for investment in oil field project: mixed optimal stopping and singular stochastic control model

Due to the presence of skewness and kurtosis in the empirical distribution of oil price returns, an adequate model for oil prices would be a jump-diffusion model. Ogbogbo (2016) has shown that crude oil spot price process is a Jump-Diffusion process. Any Lévy process may be decomposed into the sum of a Brownian notion, a linear drift, and a purely discontinuous process composed by superposing independent centered Poisson processes, this result is known as the *Lévy-itô decomposition*. This also means that every *Lévy* process can be approximated with arbitrary precision by a jump-diffusion process. This informs the choice of a Jump diffusion Process in this work.

Definition: Consider the Stochastic differential equation, SDE with jumps (*Lévy* SDE) given in [8] below. Solutions to the Lévy SDE are called *jump-diffusions.*

$$dX_{(t)}=\mu(t,X(t)dt+\sigma(t,X(t)dW(t)+\int_{\mathbb{R}^{n}}\gamma(t,X(t^{-}),Z)\tilde{N}(dt,dZ)\ldots\ldots [8]$$

i.e. solutions in the time homogeneous case, when

$$\mu(t,x)=\mu(x),\sigma(t,x)=\sigma(x)\ and\ \gamma(t,x,Z)=\gamma(x,Z).$$

Dynamics of the system. Let X_t be the stochastic system, describing the oil price process. The SDE of the system has a jump component and is given by [9].

$$dX_t=\mu(X_t)dt+\sigma(X_t)dW_t+\int_{\mathbb{R}}h(t^{-},X_{t^-},\gamma(Z))\,\tilde{N}(dt,dZ)-d\Gamma_t\ \ldots [9]$$

$$X(0)=x_0$$

Where

$$X=X^{\Gamma}=(X_1^{\Gamma}(t),\ \ .\ \ .\ \ .\ \ ,X_n^{\Gamma}(t)$$

W(t) is an n-dimensional Brownian motion independent of .

$\tilde{N}$ is a martingale measure of jumps

$\Gamma(t)$ is the singular control applied to the process X(t)

$X^{\Gamma}(s)=X=(X_1,\ldots\ldots\ldots X_n)\in\mathbb{R},s\leq t$

$\Gamma(t)=(\Gamma_1(t),(t),\Gamma_2(t),\ldots\ldots\ldots,\Gamma_n(t))\subset\mathbb{R}^{n},\ t\geq s$

$\mu(X_t)$ is drift component, $\sigma(X_t)$ is diffusion component.

The controller and the objective functional

The system is evolving with time and being observed by the controller or investor. There are costs involved. There is a cost paid over time for observing the system. Waiting before taking a decision is at a cost, anytime decision is taken, a cost is paid. Should the controller decide to stop, there is a terminal cost. The control is also applied at a cost. Giving rise to the following objective functional:

The objective functional or performance criterion

J=J(s,x) of the form

$$J^{(\Gamma,\eta)}(s,x) = E^{(s,x)} \int_0^{\eta} g(t,x_t)dt + n(t,x_t)d\Gamma_t + m(\eta,x_\eta)|_{\{\eta<\infty\}} \quad \text{.. [10]}$$

$$= J^{(\Gamma,\eta)}(s,x) = E^{(s,x)}\left[\int_0^{\eta} g(t,x_t)dt + m(\eta,x_t)|_{\eta<\infty}\right] + \int_0^{\eta} n(t,x_t)d\Gamma_t \quad \text{... [10b]}$$

where, m and g are continuous functions, $\eta_s=\eta_s(x)=\inf\{t>0: x=x^*\}$

$g(t,x_t)$ is a running cost or observation cost, $m(\eta, x\eta)$ is the terminal cost, and $n(t, x_t)$ is the cost of applying the control. The running cost $g(t, x_t)$ is the cost of waiting to take decision. In the formulation for this particular problem, it is a constant which is not discounted, a sunk cost involved in production. The cost of applying the control, $n(t,x_t)$ is also a constant in this case.
The stopping cost or terminal cost, $m(\eta, x)$, is actually the value of the project at the point the decision is taken. It is the revenue accruing.

Of interest is the function

$$U(s,x) = \inf_{\Gamma,\eta} J^{(\Gamma,\eta)}_{(s,x)}$$

$$\text{i.e. } U(s,x) = \inf_{\Gamma,\eta} E^{(s,x)} \int_0^{\eta} g(t,x_t)dt + n(t,x_t)d\Gamma_t + m(\eta,x_\eta)|_{\{\eta<\infty\}}\Big\} \quad \text{... [11]}$$

Threshold and time
Starting at some point in time and space the interest is in the first time the process hits the threshold,

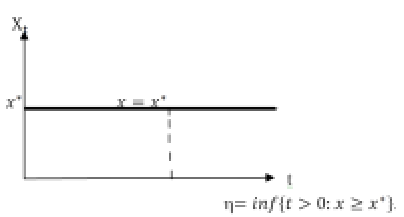

*Fig. 1. Threshold for stopping time, corresponding to price x= x**

The idea of threshold and time raises the question of "how" and "when" with respect to the investment. "When" involves the threshold and time that the investor should call it quits and "how" is concerned with strategy.

The time (Optimal stopping time)
There is a non- empty time set. Therefore
$\eta^*=\{\inf\ t > 0 : x_t \geq x^*\}$

Characterization of the process and domain of operation
There is a PDE associated with this model. The PDE satisfies

a) $L\,U(X,t) = -g(t,X)$ .... [12]
b) $U(X,t) = m(t,X_t)$ .... [13]

$U(X,t)$ is the solution of the PDE.

Domain of operation
The threshold separates the system into two Domains. The non-intervention region is D, connoting "Wait" and B region is "below the threshold". Above the threshold, the process is described by the PDE, below the threshold we have $U(X,t)=m(t,X_t)$

Thus, interest is in the solution, $U(X,t)$, that defines the threshold. This is illustrated in Fig. 2 below.

Fig. 2. Domain of operation

For example, for investor in stock, D region connotes "wait" and the threshold is "invest".
Remark: An important condition in the formulation of the model is that the process must not jump at the threshold, a jump at the threshold implies that the threshold being tracked can be missed. Hence, at the threshold, the process must be *continuous and differentiable*.

Generation of PDE
Definition: Ito's formula for jump processes.
Let X be a diffusion process with jumps, defined as the sum of a drift term, a Brownian motion and a compound Poisson process

$$X_t = X_0 + \int_0^t \mu_s\,ds + \int_0^t \sigma_s\,dW_s + \sum_{k=1}^{N_t} \Delta X_k$$

where, μ_t and σ_t are continuous processes with

$$E\left[\int_0^T \sigma_t^2\, dt\right] < \infty$$

$N_t = N_t\,(w)$ is a compound Poisson process. Consider $Y_t = f(t, X_t), f \in C^{1,2}$, i.e. f is differentiable with respect to time and twice continuously differentiable with respect to the spatial variable X; $f\,[0,T] \times \mathbb{R} \to \mathbb{R}$. Then with X_t defined as above, the process $Y_t = f(t, X_t)$ can be represented in differential form as

$$dY_t = \left(\frac{\partial f(t,X_t)}{\partial t} + \mu_t \frac{\partial f(t,X_t)}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f(t,X_t)}{\partial x^2}\right)dt + \sigma_t \frac{\partial f(t,X_t)}{\partial x} dW_t + [f(X_t + \Delta t) - f(X_t)]$$

The price process X_t satisfies [9] and is a Jump-diffusion.

Let, $Y_t = V(t,X_t)$ …………………. [14]

V is $C^{1,2}$ (C^1 in time, C^2 in space), hence by Itô's lemma

$$dY_t = V_t(t,X_t)dt + V_x(t,X_t)\mu(X_t,t)dt + \frac{1}{2}V_{xx}(t,X_t)\sigma^2(X_t,t)dt + V_x(t,X_t)\sigma(X_t,t)dW_t$$

$$= V_t dt + V_x[\mu(X_t)dt + \sigma(X_t)dW_t)] + \frac{1}{2}V_{xx}\sigma^2(X_t)dt \;\ldots\ldots\ldots\ldots [15]$$

With the singular control, dY_t becomes

$$dY_t = V_t dt + V_x[\mu(X_t)dt + \sigma(x_t)dW_t - d\Gamma_t] + \frac{1}{2}V_{xx}\sigma^2(X_t)dt$$

$$= \left[V_t + \mu(X_t)V_x + \frac{1}{2}V_{xx}\sigma^2(X_t)\right]dt + V_x\sigma(X_t)dW_t - V_x d\Gamma_t \;\ldots\ldots\ldots\ldots [16]$$

[16] describes the dynamics of the process including the control. To have a complete description of the control problem, the costs are added (through the $J(t, x)$ functional) to dY_t process.

Recall the performance criterion $J=J(t,x)$

$$J(t,x) = E^x\left[\int_0^\eta g(t,x_t)dt + m(\eta,x_\eta)\big|_{\{\eta<\infty\}}\right] + \int_0^\eta n(t,x_t)d\Gamma_t \;\ldots\ldots\ldots\ldots [17]$$

Let Z_t denote $J(t,x)$, since $J(t,x)$ is a process. Then the Z_t process is added to dY_t

From [10b/17]

$$dZ_t = g(t, x_t)dt + n(t, x_t)d\Gamma_t$$

then

$$dZ_t + dY_t = \left[V_t + \mu(X_t)V_x + \frac{1}{2}V_{xx}\sigma^2(x,t) + g(t,x_t)\right]dt + V_x\sigma(X_t)dW + (n(t,x_t) - V_x)d\Gamma_t \;\ldots [18]$$

To solve for U(x,t) , g(t,x) is chosen to be of an exponential form

$$g(t,x) = e^{-\rho s} f(x)$$

Since, $U(s,x)$ is the function that solves the PDE, by the principle of optimal control

$$L_x U(s,x) + g(s,x) = 0 \;\ldots\ldots [19]$$

$$U((s,x) = m(s,x)$$

$$L_x U(s,x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} + \mu(x)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial s} + \int_{\mathbb{R}} U(s,x,v) - v(Z)\frac{\partial u}{\partial s} - U(s,x)\Big] v\,dZ \;\ldots [20]$$

[20] includes the jumps

From [19], and excluding the jumps in [20], we have

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} + \mu(x)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial s} + g(s,x) = 0 \;\ldots\ldots\ldots\ldots [21]$$

$U(s,x)$ is desired. $U(s,x)$ solves this time dependent PDE. Consider

$g(s,x) = e^{-ps}$

$U(s,x) = me(s,x) = e^{-ps}x^a$ $a > 0$

$g(s,x) = e^{-ps}$ means that a constant cost is paid for observing the system, which is discounted in time.

By the theory of PDE for optional control let $U(s,x) = e^{-ps}\,y(x)$.

Then

$L_x U(s,x) = e^{-ps}\, Ly(x)$

Recall [21]

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} + \mu(x)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial s} + e^{-\rho s} = 0$$

Since $U(s,x) = e^{-\rho s}\psi(x)$, then

$$\frac{1}{2}\sigma^2(x)U^{II}{}_x + \mu(x)U^{I}{}_x + U_s + e^{-\rho s} =$$

$$\left[\frac{1}{2}e^{-\rho s}\sigma^2(x)\psi^{II}(x) + \mu(x)e^{-\rho s}\psi^{I}(x) - \rho e^{-\rho s}\psi(x) + e^{-\rho s}\right]$$

$$= e^{-\rho s}\left[\frac{1}{2}\sigma^2(x)\psi^{II}(x) + \mu(x)\psi^{I}(x) - \rho\psi(x) + 1\right] = 0$$

Dividing through by e^{-ps}, yields:

$$\frac{1}{2}\sigma^2(x)\psi^{II}(x) + \mu(x)\psi^{I}(x) - \rho\psi(x) = -1 \;\ldots [22]$$

$y(x) = x^\alpha$ $\alpha \geq 0$ is a fixed constant

In particular consider case *when* $p = 0$

Then the controller is paying a constant running cost which is not discounted, then

$$\frac{1}{2}\sigma^2(x)\psi^{II}(x) + \mu(x)\psi^{I}(x) = -1 \quad \ldots\ldots\ [23]$$

$\psi(x) = x^{\alpha}$

By change of variable argument, let

$$\psi^{1}(x) = f(x) \ldots\ldots\ldots\ldots [24]$$

Then [23] reduces to a first order ODE which is solved explicitly for f(x), and $\psi(x)$ is recovered by integration.

$$\frac{1}{2}\sigma^2(x)f^{1}(x) + \mu(x)f(x) = -1 \quad \ldots\ldots\ [25]$$

$$s,x) = \left\{ \begin{array}{ll} x^{\alpha} & 0 < x \le x^* \\ \psi(x) & x^* < x \le x_0 \\ & x_0 \le x < x^* \\ \psi(x) + (x - x^*) & x \ge x^* \end{array} \right\} \ldots\ldots\ldots\ldots [26]$$

Remark
The solution set is a piecewise continuous solution.

- $\psi(x)$ is solution of the PDE when x lies within the interval $x^* < x \le x_0$ (resp $x_0 \le x \le x^*$, depending on which threshold is above the other)
- we have x^{α} (which is the terminal cost), for $0 \le x \le x^*$. This happens, if the controller decides to stop abruptly.
- $\psi(x) + (x - x^*)$ is solution of the PDE in the last interval $x \ge x^*$. This describes points slightly above the threshold. (What happens a little after the solution point is usually observed after solving a PDE).

From [25]

$$f^{1}(x) + 2\frac{\mu(x)}{\sigma^2(x)}f(x) = \frac{-2}{\sigma^2(x)} \qquad \sigma(x) > 0$$

Integrating factor, I.F= $e^{2\int_0^x \frac{\mu(s)}{\sigma^2(s)}ds}$

$$\frac{d}{dx}\left[f(x)\cdot e^{2\int_0^x \frac{\mu(s)}{\sigma^2(s)}ds}\right] = \frac{-2}{\sigma^2(x)}e^{2\int_0^x \frac{\mu(s)}{\sigma^2(s)}ds}$$

$$f(x) = \frac{\int \frac{-2}{\sigma^2(x)} e^{2\int_0^x \frac{\mu(s)}{\sigma^2(s)}ds} dx}{e^{2\int_0^x \frac{\mu(s)}{\sigma^2(s)}ds}} \quad \ldots\ldots\ [27]$$

From [25]. $\psi^{1}(x) = f(x)$

$$\psi(x) = \int_0^x f(s)ds + c \quad \ldots\ldots\ [28]$$

For $\in (0, \infty)$.

Hence

$$U(s,x), = \left\{ \begin{array}{ll} x^{\alpha} & 0 < x \le x^* \\ \int_0^x f(s)ds + c & x^* < x \le x_0 \\ \int_0^x f(s)ds + c + (x - x^*) & x \ge x^* \end{array} \right\} \ldots [29]$$

$$x \ne \infty, \quad x \in (0, \infty)$$

[29] represents the general solution or general case, subsequently particular cases for Brownian motion and Geometric Brownian motion. *f(s)* can be given explicitly when the system is a Brownian motion and Geometric Brownian motion are examined. $y_{(x)}$ must converge. Since we have a mixed stochastic optimal control problem, we desire two thresholds x^* and x_0, which determine the optimal stopping time and when to apply the control respectively. [26] is in this form because the problem is a singular control problem, the function is not absolutely continuous over the interval.

Conditions on $\psi_{(x)}$ will determine which threshold is above the other. If the thresholds coincide i.e. $x^* = x_0$, then we have strictly an optimal control problem or strictly an optimal stopping time problem. Two thresholds are involved in this model notably, threshold for singular control $x = x_0$ which indicates where to apply the control, and threshold for stopping time x=x^* which gives the optimal stopping time. The Control is flat (not applied) while in the D domain. It is applied at the threshold to ensure the system does not fall out of order e.g. A financial institution does not go bankrupt by paying dividend, fish population does not become extinct by over harvesting, an investor investing in an oil field project does not invest at a loss.

Fig. 3 and 4 illustrate position of the thresholds

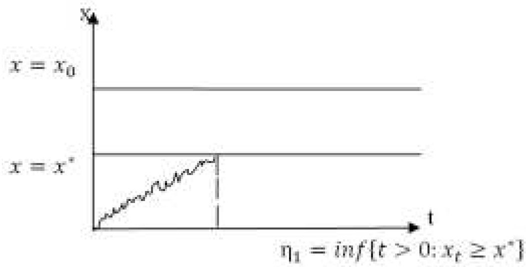

Fig. 3. Threshold (price), that determines stopping time, attained before threshold (price) that determines when to apply control

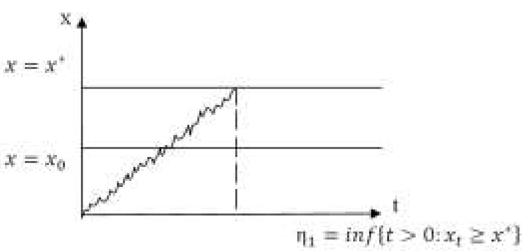

Fig. 4. [Threshold (price), that determines when to apply control, attained before threshold (price) that determines stopping time]

Continuity and Differentiability of $\psi(x)$ at x_0 and x^*

Continuity at $x = x^*$

From [26]

$$\text{(I) } \psi(x) = \psi(x^*) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[30]}$$

$$\text{(ii) } \psi(x^*) = x^{*\alpha}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[31]}$$

Differentiability at $x=x^*$

From [31]

$$\psi^1(x^*) = ax^{*a-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[32]}$$

Dividing [31] by [32], we have

$$\frac{\psi(x^*)}{\psi'(x^*)} = \frac{x^{*\alpha}}{\alpha\, x^{*\alpha-1}}$$

$$x^* = \alpha \frac{\psi(x^*)}{\psi'(x^*)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[33]}$$

Similarly,

For continuity at $x = x_0$

$$\psi(x_0) = x_0^{\alpha} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[34]}$$

For differentiability at $x = x_0$

$$\psi'(x_0) = \alpha x_0^{\alpha-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[35]}$$

[34] ÷ [35] yields

$$x_0 = \alpha \frac{\psi(x_0)}{\psi'(x_0)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[36]}$$

Since the expressions $\psi(x^*)$ and $\psi(x_0)$ are obtained from differentiability and continuity at $x=x^*$ and $x = x_0$ respectively. From [33] and [36] x^* and x_0 can be obtained explicitly for particular cases of Brownian motion, BM and Geometric Brownian motion, GBM. The emerging thresholds must be unique. Since $\psi(x)$is given as an integral, Continuity and differentiability of $\psi(x)$ has been established using Riemann integration, fundamental theorems of Calculus, Order Preserving property of integrals, Leibnitz Integral Rule (for differentiation under the integral sign).Proof is lengthy and not given here.

Uniqueness and position of the thresholds

Uniqueness of x^* and x_0, have been established, proof is lengthy and is not given here

Position of the thresholds

The position of the threshold is determined by the following inequalities,

$\psi(x^*) < \psi(x_0)$ or $\psi(x^*) > \psi(x_0)$

This determines which threshold is above or below the other.

If $\psi(x^*) - \psi(x_0) < 0$, then $\psi(x^*) < \psi(x_0)$ conversely, If $\psi(x^*) - \psi(x_0) > 0$ then $\psi(x^*) > \psi(x_0)$

Existence of the integral; Existence of solution

From solution of the PDE, [27]

$$\psi(x) = \int_0^x f(s)ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{[37]}$$

$x \neq \infty$

Existence of [37] above implies existence of the solution. Then we can obtain $\psi(x_0)$ and $\psi(x^*)$ and by implication x_0 and x^* for this general case (with appropriate boundary and initial conditions). Other conditions that guarantee existence of solution are given below:

(i) f(x) must be continuous on the bounds of the integral,
(ii) solution must converge,
it is given that there are no jumps at the initial process, we expect jumps at some point in time $X(0^-) = x$.

Then it may give the optimal strategy for investment for the general case as follows:
(i) stop immediately if $0 \leq x \leq x^*$; $x = x^*$ or $x = 0$. (This includes stopping abruptly).
(ii) do nothing if $x > x^*$ (ought to have stopped investment already).
(iii) start investing at x_0, if $x_0 \leq x \leq x^*$

Particular cases

Optimal strategy for Brownian motion

For the Brownian motion, σ, μ are constants

$$\frac{1}{2}\sigma^2\psi''(x) + \mu\psi'(x) = -1 \quad \ldots\ldots\ [38]$$

Let $\psi'(x) = P(x)$, $\quad x \in (0, \infty)$.

then [38] becomes

$$\frac{1}{2}\sigma^2 P'(x) + \mu P(x) = -1$$

$$P'(x) + \frac{2}{\sigma^2}\mu P(x) = \frac{-2}{\sigma^2} \quad \ldots\ldots\ [39]$$

$$\text{I.F} = e^{2\int \frac{\mu}{\sigma^2} dx} = e^{\frac{2\mu}{\sigma^2}x}$$

$$\frac{d}{dx}\left[e^{\frac{2\mu}{\sigma^2}x}\, P\right] = \frac{-2}{\sigma^2} \cdot e^{\frac{2\mu}{\sigma^2}x}$$

$$e^{\frac{2\mu}{\sigma^2}}P = \frac{-2}{\sigma^2}\int e^{\frac{2\mu x}{\sigma^2}}dx$$

$$\int e^{\frac{2\mu x}{\sigma^2}}dx = \frac{\sigma^2}{2\mu} e^{\frac{2\mu x}{\sigma^2}} + c$$

$$e^{\frac{2\mu}{\sigma^2}x} \times P(x) = \frac{-1}{\mu} e^{\frac{2\mu}{\sigma^2}x} + c$$

$$P(x) = \frac{\frac{-1}{\mu} e^{\frac{2\mu}{\sigma^2}x}}{e^{\frac{2\mu}{\sigma^2}x}} + \frac{c}{e^{\frac{2\mu}{\sigma^2}x}}$$

$$P(x) = \frac{-1}{\mu} + c e^{\frac{-2\mu}{\sigma^2}x}$$

But $\psi'(x) = P(x)$

Then $\psi(x) = \int P(x)\, dx = \int \frac{-1}{\mu} + c e^{\frac{-2\mu}{\sigma^2}x}$

$$= \int \frac{-1}{\mu} dx + c \int e^{\frac{-2\mu x}{\sigma^2}} dx$$

$$= \frac{-x}{\mu} + c.\frac{\sigma^2}{-2\mu} \cdot e^{\frac{-2\mu x}{\sigma^2}} + k_2.$$

Since c, σ and, μ are constants, let $c.\frac{\sigma^2}{-2\mu}$ be k_1, then

$$\psi(x) = \frac{-x}{\mu} + k_1 e^{\frac{-2\mu x}{\sigma^2}} + k_2 \quad \ldots\ldots\ldots\ldots\ldots\ [40]$$

This explicit solution, [40] is the same result obtained, when σ, μ are substituted as constants in [27].

Hence, for the Brownian motion case

$$U(s,x) = \begin{cases} x^{\alpha} & 0 < x \leq x^* \\ \frac{-x}{\mu} + k_1 e^{\frac{-2\mu x}{\sigma^2}} + k_2 & x^* < x < x_0 \\ & x_0 \leq x < x^* \\ \frac{-x}{\mu} + k_1 e^{\frac{-2\mu}{\sigma^2}x} + k_2 + (x - x^*) & x \geq x^* \end{cases} \quad \ldots\ldots\ldots\ldots\ [41]$$

Optimal Strategy is specified as follows:

- Stop immediately if $0 < x \le x^*$ i.e. $x = x^*$ or $x = 0$
- Do nothing if $x > x^*$
- Start producing and selling copiously at $x = x_0$, if $x_0 \le x < x^*$

Recall [33] $x^* = \alpha \frac{\psi(x^*)}{\psi'(x^*)}$ and [36] $x_0 = \alpha \frac{\psi(x_0)}{\psi'(x_0)}$

For a Brownian motion, x^* and x_0 are obtained explicitly from [33] and [36], since α is a fixed constant and $\alpha > 0$ given. k_2 is to be obtained by extra initial conditions.

Optimal Strategy for Geometric Brownian Motion

For the Geometric Brownian motion, let $\mu = x, \sigma = x^2$

$$\frac{1}{2}x^2\psi''(x) + x\psi'(x) = -1$$

$$\psi'' + \frac{2}{x}\psi' = -2x^{-2}. \text{ Let } m = \psi' = \frac{d\psi}{dx}. \text{ Then } \psi'' = \frac{d^2\psi}{dx^2} = \frac{dm}{dx} \Rightarrow \frac{dm}{dx} + \frac{2}{x}m = -2x^{-2}$$

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2. \ \frac{d}{dx}(m.x^2) = -2x^{-2} \cdot x^2 = -2. \text{ Then } m.x^2 = \int -2dx$$

$$= -2x + C.$$

$$\text{Hence, } m = \frac{1}{x^2}(-2x + C) = x^{-2}(-2x + C).$$

$$\text{Therefore } m = \frac{-2}{x} + \frac{C}{x^2}$$

$$\text{But } m = \frac{d\psi}{dx} \quad \therefore \frac{d\psi}{dx} = \frac{-2}{x} + \frac{C}{x^2}. \text{ Hence } \quad \psi = -2\ln x - \frac{C}{x} + D$$

C and D are constants to be determined using initial conditions. For given initial conditions, the constants C and D can be obtained leading to explicit solution for $\psi(x)$, x^* and x_0.

These initial conditions could be project-specific information, in addition to crude oil price data for particular oil fields during a given period. We now have the optimal optimal strategy for investment for the GBM case.

$$U(s,x) = \begin{cases} x^{\alpha} & 0 < x \le x^* \\ -2\ln x - \frac{C}{x} + D & x^* < x < x_0 \\ & x_0 \le x < x^* \\ -2\ln x - \frac{C}{x} + D + (x - x^*) & x \ge x^* \end{cases} \quad \text{.............} [42]$$

To obtain optimal strategy for the Jump-diffusion case, [20] will be used, jumps will be included in [21]. Thus the analysis and method of solution will include jumps in obtaining $\psi(x)$, x^* and x_0. Work continues to obtain precise optimal strategy in this case.

Conclusion

In the work a model has been presented for obtaining an optimal strategy for investment in oil field project. Oil price, which is the main profit-determining parameter, is given as a Jump -diffusion process. Running cost is given as a constant cost which is not discounted in time. The results obtained so far and some particular cases have been presented. Conditions for existence of solution were given. The Brownian motion process and Geometric Brownian Motion process were used as particular cases, for which the work obtained explicit solutions, and distinct optimal strategy. BM and GBM cases provide strategy for investment which is good enough. However, it is expected that the best strategy to invest will be attained when jumps are included in the result, since oil price process is not exactly Gaussian. Work therefore continues to obtain possibly explicit optimal strategy for the Jump-diffusion case as stated, and to validate the optimal strategy using empirical data and project information from fields in the Niger-Delta. Also running cost may be considered as a constant cost which is discounted in time.

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