HÖLDER ESTIMATES FOR THE $\overline{\partial}$-OPERATOR ON BOUNDED DOMAINS IN $\mathbb{C}^n$

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Abstract
Holder estimates are obtained for the $\overline{\partial}$ operator on bounded domains in $\mathbb{C}^n$ with boundaries of Lebesgue zero.

Introduction
The pioneering work on the type of Hölder estimates for the $\delta$-operator that we consider was done by AL Y [1] and SIU [5] for the (0,1)-forms and for (0,q)-forms by Lieb Range [4]. Since then various Hölder estimates for the $\delta$-operator have appeared (see references) [2] and [3]. Most of the results for Hölder estimates for the $\delta$-operator mentioned above have been for strongly pseudoconvex domains or pseudoconvex domains of finite type.

Working with the Bochner-Martinelli-Koppelman kernel it dawned on us that we could get a generalization of Alt-Sui-Lieb-Range results to all bounded domains in $\mathbb{C}^n$ with boundaries of Lebesgue measure zero (at least for the range of Hölder estimates we consider here). This short paper shows that we are right.

Preliminaries
Let $U$ be open in $\mathbb{R}^n$, $0 < \alpha < 1$, $k \geq 0$ an integer. We define $C^{k,\alpha}(U)$ to be the space of functions on $U$ such that

$$|f|_{C^{k,\alpha}(U)} := \sup_{\Omega} |f| + \sup_{x \neq y, x, y \in \Omega} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x - y|^\alpha}$$

is finite, where $D^\gamma$ is a derivative or order $|\gamma|, \gamma_1, \ldots, \gamma_n, \gamma_j \geq 0$. If $U \subset \mathbb{C}^n$ is open, we use the real underlying coordinates of $\mathbb{C}^n$ considered as $\mathbb{R}^n$ to define $C^{k,\alpha}(U)$. $C^{k,\alpha}(U)$ is defined similarly.

If $f = \sum_{(0,q)} f_{i_1,\ldots,i_q} d_{i_1} \wedge \ldots \wedge d_{i_q}$ is an (0,q)-form on $U$, where means the summation is over increasing multi-indices, we write $f$ as $\sum f_{I} d^I$ for short, $I = (i_1, \ldots, i_q)$ and set

$$|f|_{C^{k,\alpha}(U)} := \max_{I} |f_I|_{C^{k,\alpha}(U)}$$
Our result is then

**Theorem 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with boundary of Lebesgue measure zero and \( 0 < \alpha < 1 \) and \( k = 1, 2, \ldots \), and let \( f \in C^{\alpha}_{k, \mathbb{C}}(\Omega) \) be \( \bar{\partial} \)-closed then there is \( u \in C^{\alpha}_{k, \mathbb{C}}(\Omega) \) such that

\[
\bar{\partial} u = f \quad \text{in the sense of distributions and}
\]

\[
|u|_{C^{\alpha}_{k, \mathbb{C}}(\Omega)} \leq |f|_{C^{\alpha}_{k, \mathbb{C}}(\Omega)}
\]

where \( \bar{\partial} \) does not depend on \( f \)

**Bochner-Martinelli-Koppelman Formula and \(-u = f\)**

**Theorem 2** (Bochner-Martinelli-Koppelman). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^1 \) boundary. For \( f \in C^{\alpha}_{k, \mathbb{C}}(\Omega) \), \( 0 \leq q \leq n \), we have

\[
f(z) = Bq(\cdot, z) \wedge f + \int_{\Omega} Bq(\cdot, z) \wedge \bar{\partial} \delta f
\]

\[
+ \bar{\partial} Bq - 1(\cdot, z) \wedge f, z \in \Omega
\]

(1)

where \( Bq(\xi, z) \) is the Bochner-Martinelli-Koppelman kernel of degree \((0, q)\) in \( z \) and of degree \((n, n - q - 1)\) in \( \xi \). Recall, with \( \beta = |z - \xi|^2 \),

\[
Bq(\xi, z) = \frac{(-1)^{q(n - 1)/2}}{(2\pi)^n} \frac{n - 1}{q} \beta \wedge \partial_{\xi} \beta \wedge (\partial_{\xi} \partial_{\bar{\xi}} \beta)^{n-q-1} \wedge (\partial_{\xi} \partial_{\bar{\xi}} \beta)^q
\]

(2)

**Lemma 3.** With \( f \) as in Theorem 1, if

\[
u(z) = B_q(\cdot, z) \wedge f, z \in \Omega,
\]

then \( \bar{\partial} \nu = f \).

Proof With \( f = \int f \bar{\partial} \delta \bar{\partial} \) defined as zero outside \( \Omega \), regularize \( f \) coefficientwise: \( f_m = (f_m)_n \wedge (\bar{\partial} \delta) \) where

\[
(f_m)_n(\cdot, \xi) = \int f(z - \xi/m) \varphi(\xi) d\lambda(\xi)
\]

(4)

\[
m^2 \varphi \in C^\infty(\Omega^0, \mathbb{C}), f \varphi d\lambda = 1, \varphi \geq 0, \sup \varphi = \{ z \in C^\infty : |z| \leq 1 \} \text{ and } \lambda \text{ is Lebesgue measure.}

Then \( \|f_m\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)} \), \( f, f_m \to f \) in \( L^1(\Omega) \) as \( m \to \infty \), and \( f_m \) is \( \bar{\partial} \)-closed in \( \mathbb{R}^n \) in the sense of distributions.
Now let
\[ u_m(z) = B_q(\cdot, z)^\alpha f_m, \] (5)
then from Theorem 2, 
\[ \partial u_m = f_m \]
and since \( f_m \to f \) in \( L^1_{(0,q+1)}(\Omega) \), we have \( u_m \to u \) in \( L^1_{(0,q)}(\mathbb{C}^n) \) and \( u = f \).

Note that in the proof of Lemma 3, since \( f \in C^{1,a}_{(\Omega)} \) it follows that \( f \) belongs to the Sobolev space \( W^{1,c}_{(\Omega)}(\mathbb{C}^n) \), even though it man not belong to \( C^{(\mathbb{C}^n)} \), and since all derivatives are taken in the distribution sense, that is all we need!

**Holder Estimates**
In this section we finish the proof of Theorem 1:

From (5), we get
\[ \partial u_m(z) = B_q(\cdot, z)^\alpha f_m \] (6)
where
\[ \partial u_m = \left| \partial \right|^{\alpha} \]
\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}), z = (x_1 + iy_1, \ldots, x_n + iy_n), i = \sqrt{-1}, \]
and the derivatives are taken coefficientwise. From (4), \( \alpha < k \), as \( m \to \infty \) \( \partial^\alpha u_m \to \partial^\alpha u \) in \( L^1_{(0,q+1)}(\Omega) \) and so from (6) \( \partial^\alpha u_m \to \partial^\alpha u \) in \( L^1_{(0,q)}(\mathbb{C}^n) \) and 
\[ \partial^\alpha u(z) = B_q(\cdot, z)^\alpha f, z \in \Omega. \] (7)

Now from known properties of \( B_q(\xi, z) \) (see for example [2], page 269), we get the estimate from (7),
\[ u^{k,\alpha}_{(0,q)}(\Omega) \leq |f|^{k,\alpha}_{(0,q+1)}(\Omega) \]
\[ (0 < \alpha < 1, k \geq 1). \]

**Reference**


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