

\textbf{REGULARITY THEORY FOR FULLY NONLINEAR UNIFORMLY ELLIPTIC EQUATIONS}

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\textbf{Abstract}

With the aim of obtaining at least Cordes-Nirenberg, Schauder and Calderon-Zygmund estimates for solutions of Fully Nonlinear Uniformly Elliptic Equations, we arrive at $W^{2, p}$, $C^1$, $\alpha$ regularity estimates for those equations, improving the existing estimates.

\textbf{Introduction}

Consider fully nonlinear second order uniformly elliptic equations of the form

$$F(D^2u, x) = f(x)$$

where $x \in \Omega$ and $u$ and $f$ are functions defined in a bounded domain $\Omega$ of $\mathbb{R}^n$, and $F(M, x)$ is a real valued function defined on $S \times \Omega$, where $S$ is the space of real $n \times n$ symmetric matrices. Assume $F$ is uniformly elliptic in the sense that there are positive constants $\lambda < \Lambda$ such that for any $M \in S$ and $x \in \Omega$,

$$\lambda \|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda \|N\| \quad \forall N \in S, x \in \Omega.$$  \hfill (2)

When $N$ is a symmetric matrix, $N > 0$ means, i.e. is non-negative definite. $\|M\|$ denotes the $(L^2, L^2)$-norm of $M$ (i.e. $\|M\| = \sup_{x\in\Omega} |Mx|$; therefore $\|N\|$ is equal to the maximum eigenvalue of $N$ whenever $N \geq 0$).

Recalling that any $N \in S$ can be uniquely decomposed as $N = N^+ - N^-$, where $N^+, N^- > 0$ and $N^+ N^- = 0$, it follows that $F$ is uniformly elliptic if and only if

$$F(M + N \leq F(M), x + \|N^+\| - \lambda \|N^-\| \quad M, N \in S, x \in \Omega.$$  \hfill (3)

It also follows from (3) that if $F$ is uniformly elliptic, then

$$\lambda |\hat{e}| \leq |F(M, x) - F(0, x)| \quad M \in S, x \in \Omega.$$ \hfill (4)

where $|\hat{e}| = \max\{|e_1|, \ldots, |e_n|\}$, the $e_j$ ($1 \leq j \leq n$) being the eigenvalues of $M$. 


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In [1], rather detailed regularity estimates were obtained for solutions of (1), where \( \Omega \) was the unit ball and \( F(M, x) \) was convex or concave in \( M \). As is normal, it is natural to ask whether we can remove the convexity conditions on \( F \) and make \( \Omega \) a general bounded domain in \( \mathbb{R}^n \). We show here that we can, without even assuming that \( \Omega \) is \( F \)-convex in the sense of [2].

Our methods are not direct generalization of those of [1]. Rather we use only the philosophy that the most useful square matrix is a diagonal one, the approach being frontal. We obtain sharp Hölder and Sobolev regularity results, and from the Hölder estimates, show that once \( f \) in (1) is continuous and \( F(0, \cdot) \) is locally integrable in \( (L^\infty(\Omega)) \), every solution of (1) is a viscosity solution.

We consider in this paper only those solutions of \( u \) of (1) such that the distributional derivatives \( \partial^2 u \) are actual functions on \( \Omega \), and we also assume that the boundary of \( \Omega \) has Lebesgue measure zero.

Our results are as follow:

**Theorem 1.** If \( F(0, \cdot) \) and \( f \) are in \( L^p(\Omega) \), \( 1 < p < \infty \), then \( u \in W^{2,p}(\Omega) \) and there is a constant \( K \) independent of \( F \) such that

\[
\|u\|_{W^{2,p}(\Omega)} \leq K \|F(0, \cdot)\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}.
\]

**Theorem 2.** If \( F(0, \cdot) \) and \( f \) are in \( L^{n/1-\alpha}(\Omega) \), \( 0 < \alpha < 1 \), then \( u \in C^{1,\alpha}(\Omega) \) and there is constant \( K \) independent of \( F \) such that

\[
\|u\|_{C^{1,\alpha}(\Omega)} \leq K \|F(0, \cdot)\|_{L^{n/1-\alpha}(\Omega)} + \|f\|_{L^{n/1-\alpha}(\Omega)}.
\]

**Theorem 3.** If \( F(0, \cdot) \) and \( f \) are in \( L^\infty(\Omega) \), then \( u \in C^{2,\alpha}(\Omega) \) for every domain \( \Omega_0 \subset \subset \Omega \) and there is a constant \( K = K(\Omega_0, \Omega) \) such that

\[
\|u\|_{C^{2,\alpha}(\Omega)} \leq K \|F(0, \cdot)\|_{L^\infty(\Omega_0)} + \|f\|_{L^\infty(\Omega)}.
\]

**Theorem 4.** If \( f \) is continuous and \( F(0, \cdot) \in L^\infty_{1oc}(\Omega) \), then every solution of (1) is a viscosity solution.

**Proof of Theorems**

First if \( M = (M_{ij}) \) is in \( S \), we define \( |M| := \sqrt{\sum_{i,j} M_{ij}^2} \). It then follows, using the fact that \( M = \text{ODO}^t \), where \( D_{ij} = e_i^\top D e_j \) (\( e_i \) being the eigenvalues of \( M \)) and \( O \) is an orthogonal matrix,
and Cauchy-Schwartz Inequality that

\[ |M| \leq n \epsilon_j^{1/2}. \]

Therefore, since \( D^2 u = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \), we have from (4) and (5) that

\[ D^2 u \leq K \left( |F(0, .)| + |f| \right), \]

for some \( K > 0 \).

Putting \( u = 0 \) outside \( n \) and using Poincaré Inequality (noting that the boundary of \( \Omega \) has Lebesgue measure zero) we get Theorem 1 from (6).

To prove Theorem 2 we use (from [3] p. 123) Lemma 5. Let \( u \in \mathcal{D}'(\mathbb{R}^n) \) and assume that \( \partial_j u \in L^p(\mathbb{R}^n), j = 1, \ldots, n \), where \( p > n \). Then \( u \) is continuous and with \( \gamma = 1 - n/p \), we have

\[ \sum_{j=1}^{n} \sup_{x \neq y} \left| \frac{|u(x) - u(y)|}{|x - y|} \right| \leq C \left\| \partial_j u \right\|_{L^p(\mathbb{R}^n)} \]

for some \( c > 0 \).

Putting again \( u = 0 \) outside \( \Omega \) and noting that the boundary of \( \Omega \) has Lebesgue measure zero, we get Theorem 2 from Lemma 5, Theorem 1 and Poincaré Inequality.

To prove Theorem 3, we note that there is a constant \( K = K(\Omega_{\Omega^c}, \Omega) \) such that

\[ \sup_{x, y \in \Omega_{\Omega^c}} \left| \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \frac{\partial^2 u(y)}{\partial x_i \partial x_j} \right| \leq \left\| \partial_j u \right\|_{L^\infty(\Omega)} \]

for \( 1 \leq i \leq n \), and then use Theorem 1.

To prove Theorem 4, we note that from the hypothesis of Theorem 4, Theorem 2 holds on any domain \( \Omega_{\Omega^c} \subset \subset \Omega \).

References


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