ON THE HOMOGENEOUS COMPLEX MONGE-AMPERE EQUATION

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Abstract

Harmonic functions are used to construct nonzero solutions of the homogeneous Complex Monge-Ampere equation which particularize to results of Lempert and Bracci-Patrizio. Mathematics Subject Classification:$35J15, 35J60$

Introduction

In Darko(2002), we constructed viscosity solutions of the inhomogeneous complex Monge-Ampere equation. The constructed solutions turned out to be zero in the homogeneous case. Meanwhile Lempert(1981), has the following remarkable result.

Let $D \subset C^n$ be a bounded strongly convex domain with smooth boundary and let $z_0 \in D$, then there is a solution to the homogeneous Monge-Ampere equation:

$$u \quad \text{plurisubharmonic in } D$$

$$M_c(u) := \det\left(\frac{\partial^2 u}{\partial z_j \partial z_k}\right) = 0, \text{ in } D\backslash\{z_0\}$$

$$du \neq 0$$

$$u(z) = 0 \text{ for } z \in \partial D \tag{1}$$

$$u(z) - \log||z_0 - z|| = O(1) \text{ as } z \to z_0$$

This result turned out to be very useful in several Complex Variables as pointed out in Bracci & Patrizio(2005), where we have equally remarkable result:

Let $D \subset C^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$, then the Monge-Ampere equation with singularity at the boundary point p :

$$u \text{ plurisubharmonic in } D$$

$$M_c(u) := \det\left(\frac{\partial^2 u}{\partial z_j \partial z_k}\right) = 0$$

$$du \neq 0$$

$$u(z) = 0 \text{ for } z \in \partial D \setminus \{p\}$$

$$u(z) \approx ||p-z||^{-1} \text{ as} z \to p \text{ non-tangentially} \tag{2}$$

has a solution. Because of the importance of the above results, it is natural to seek to generalise to other domains and other singularities. We therefore have the following.

Theorem 1

Let Ω be any bounded domain in C^n. Then the homogeneous complex Monge-Ampere equation

$$M_c(u) := \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = 0$$

has a non-zero solution u in Ω such that u is plurisubharmonic $du \neq 0$

$$u(z) = 0 \quad \text{for } z \in \partial\Omega \qquad (3)$$

Smooth Solutions of $M_c(u) = 0$

Let h be a real harmonic function on C such that h is never zero on C and $dh \neq 0$.

Define the distribution H_j in C^n by

$$H_j(\varphi) = h(\varphi(0, 0, ..., j, ..., 0, 0)) \qquad (4)$$

the action of h being in the jth coordinate. $\varphi \in D(C^n)$−a test function and let f be a nonzero function in $L^1_{loc}(C^n)$, such that $df \neq 0$.

Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset ...$, with $U^{\infty}_{\nu=1}\Omega_\nu = \Omega$, be an exhaustion of Ω. Let $\{\varphi_\nu\}^{\infty}_{\nu=1}$ be a sequence of functions with $\varphi_\nu \in C^{\infty}_0(\Omega_{\nu+1}), \varphi_\nu \equiv 1$ on Ω_ν and $0 \leq \varphi_\nu \leq 1$.

Define $V_\nu \in C^{\infty}(C^n)$ by

$$V_\nu = \{(H_1 + H_2 + ... + H_n) * (\varphi_\nu f)\}\, \varphi_\nu \qquad (5)$$

where $*$ is convolution.

Then it is clear that $M_c(u) = 0$ in Ω_ν and $\{v_\nu\}$ tends locally uniformly to a C^∞ function u on Ω such that

$$M_c(u) = 0 \text{ on } \Omega \qquad (6)$$

and (3) is satisfied.

Conclusion

By choosing the locally integrable function f in (5) appropriately, we can specialise our theorem to the cases of Lempert and Bracci-Patrizio.

References

Bracci, E & Patrizio, G (2005) Monge-Ampere foliations with singularities at the boundary of strongly convex domains. Math.Ann **332**, 499-522

Darko .P. W.(2002) Continuos and L^P Estimates for the Complex Monge-Ampere Equation on Bounded Domains in C^nIJMMS **30** 11 705-707.

Lempert, L.(1981) La me'trique de Kobayashi et la representation de domains sur la boule. Bull. Soc.Math. Fr. **109** 427-474.

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