International Journal of Basic, Applied and Innovative Research

ISSN: 2315 - 5388 IJBAIR, 2018, 7(3): 68-82 E-ISSN: 2384 - 681X

www.arpjournals.com; www.antrescentpub.com

RESEARCH PAPER

A FIFTH ORDER COMPOSITE INTEGRATOR FORMULA FOR THE SOLUTION OF INITIAL-VALUE PROBLEM

[1]AASHIKPELOKHAI, U.S.U and [2]AKEREJOLA, R. F.

[1]Department of mathematics, Ambrose Alli University, Ekpoma, Edo, Nigeria; [2]Department of Basic Sciences, Auchi Polytechnic, Auchi, Edo, Nigeria.

Correspondence: Email: rfakerejola@gmail.com

Published: 30th September, 2018

***Endorsed By**: Innovative Science Research Foundation (**ISREF**) and International Society of Science Researchers (**ISSCIR**).*

***Indexed By**: African Journal Online (AJOL); Texila American University; Genamics; Scholarsteer; EIJASR; CAS-American Chemical Society; and IRMS Informatics India (J-Gate)*

ABSTRACT

In this research work, we employed cramer's rule to develop a fifth order composite integrator scheme capable of solving initial value problems in ordinary differential equation of the form:

$$y^{(1)} = f(x,y), \quad y(x_0) = y_0 \quad \forall \quad a \le x \le b$$

We examined the convergence and consistency nature of our integrator and it is found to be consistent. We equally implemented our composite integrator formula on an initial value problem in ordinary differential equations. Our results compared favorably with the existing method. We therefore recommend the method for use by ODE solvers and for researchers currently working in this area.

Keywords: Differential Equation, Rational, Polynomial, Integrator Error

INTRODUCTION

This research work is centered on the solution of initial – valued problem in ordinary differential equation of the form:

$$y^{(1)} = f(x,y) \ , \qquad y(x_0) = y_0 \ , \qquad a \le x \le b \qquad (1.1)$$

Initial-valued problems in ordinary differential equations (ODEs) can be seen in such diverse and fascinating problems from physical situations, chemical kinetics, (Abhulimen and Otunta, 2007), biological simulations (Ademiluyi and Kayode, 2001), engineering construction works, nuclear reactors (Elakhe, 2010), the diagnosis of diabetes, the spread of gonorrhea (Braun, 1993) and practical realities.

Euler's rule is the simplest among all numerical methods in ordinary differential equations because of its explicit and one-step nature. It requires no additional starting values and readily permits a change of step length during computation.

In an attempt to extend the approximation method of Euler, Runge in 1895 worked on Euler method to give a more elaborate scheme which was capable of greater accuracy. According to Agbeboh (2006), the Runge –Kutta method which is one of the methods of solving numerical problems, represents an important family of implicit and explicit iterative methods for approximation of ordinary differential equations in numerical analysis. The general explicit one-step method is of the form:

$$y_{n+1} - y_n = h\emptyset(x_n, y_n; h) \tag{1.2}$$

Exponential integrators have become active area of research, according to Fatunla (1982), exponential integrators form the class of numerical methods for solutions of stiff differential equations and also partial differential equations which include hyperbolic as well as parabolic problems such as heat. This class of integrators can be constructed to be explicit or implicit for numerical ordinary differential equations or serve as the time integrator for numerical partial differential equations. Examples of published works in this area include the works of Fatunla (1978, 1980).

Various scholars have worked extensively on the area of rational integrators, providing encouraging results in the solution of problems arising from mathematical formulation of physical solutions in population models, mechanical oscillations, process control and electrical circuit theory which often lead to initial valued problems (IVPs) in ordinary differential equation.

Among these scholars include, the work of Aashikpelokhai (1991), who developed a class of rational integrator that handles singular, stiff and oscillatory initial valued problems in ordinary differential equations.

Following closely the work of Aashikpelokhai (1991), Otunta and Ikhile (1997), developed a new class of rational integrator for stiff and singular initial valued problem in ODEs based on the rational interpolants of Fatunla (1980) and Lambert (1973).

Aashikpelokhai and Momodu (2008) designed a quadratic base integration scheme for the solution of singulo – stiff differential equation.

Elakhe (2011) developed a cubic base (polynomial of degree 3) singulo oscillatory – stiff rational integrator.

Still on rational integrator, Ukpebor (2016), analyze the Region of Absolute Stability of an order 19 Rational Integrator. The list is endless.

The modification of old composite formulae have been made to suite modern trend, for example, Agbeboh (2006), Momodu (2006) and Elakhe (2011), Abhulimen (2014), were extension of Lambert and Shaw (1965).

However, Fatunla (1982) developed a class of k-steps method; this class is not composite as it is the case with Lambert and Shaw (1965), Momodu (2006) and Elakhe (2011). The k-step methods at each stage consist of solving Simultaneous Linear Algebraic Equations (SLAE).

Aashikpelokhai (1991) followed the steps of Fatunla (1982) by developing a class of one-step rational integrator of order $2k-1$, where k is any arbitrary positive integer.

It is easy to find the composite function for $k=1$ from Aashikpelokhai (1991), however for composite formula with$k=2$ and above, in as much as composite function may be desired in algebraic approach, derivation in this class of integrators involved solving Linear Algebraic Equations (LAE) at each stage from another transformation of the initial valued problem (IVP) into matrix of coefficients.

The composite integrator formula for $k=1$ from Aashikpelokhai (1991) is given by:

$$y_{n+1} = \frac{y_n{}^2}{y_n - y_n{}^{(1)}h} \tag{1.4}$$

(1.3.1)

Which is the same as Lambert and Shaw (1965) with s = 0 and Fatunla (1982) with k =1. Aashikpelokhai (1991) also obtained result for k = 2 given as follows:

$$y_{n+1} = \frac{6ay_n+2\left(by_n+3ay_n^{(1)}\right)h}{6a+2bh+ch^2} \qquad (1.5)$$

Where: $a = y_n y_n^{(2)} - 2y_n^{(1)} y_n^{(2)} b = 3y_n^{(1)} y_n^{(2)} - y_n y_n^{(3)} c = 2y_n^{(1)} y_n^{(3)} - 3y_n^{(2)} y_n^{(2)}$

Our concern here is to choose $k = 3$, derive the composite integrator formula using Cramer's rule and compare our results with the existing method. We shall also examine the convergence and consistency of the new integrator.

It should be noted here that earlier formulae in computational ODE were mainly in composite forms as exemplified by Euler rule, Modified Euler rule, Trapezoidal rule, Runge – kutta method and linear multi – step method. This is as a result of its simplicity in implementation of initial valued problems.

2.0 EXISTENCE AND UNIQUENESS OF A SOLUTION

From a practical point of view of scientific modeling, it is very important to examine whether there exists a solution to an initial value problem and if it exist, whether it is unique.

Theorem 2.1: Aashikpelokhai *et al* (2011)

Consider the initial value first – order linear differential equation:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = h(x), \qquad a_1(x) \neq 0 y(x_0) = y_0, \;\; a \leq x \leq b$$

It has a unique solution in the interval $[a, b]$ in which it is defined on the real line.

Proof:

By the method of integrating factor, the general solution is given as;

$$y = \left[\exp\left(-\int \frac{a_0(x)}{a_1(x)}dx\right)\right]\left[A + \int \left\{\frac{h(x)}{a_1(x)} exp\left(\int \frac{a_0(x)}{a_1(x)}dx\right)\right\}dx\right] \qquad (2.1)$$

where A is the integration constant.

Existence:

Select any point $x = x_0$in $[a, b]$ and any value $y = y_0$ along the Y – axis. Substitute the pair (x_0, y_0) into (2.1), solve for the constant A.

This value of A yields a particular solution $y = y(x)$ obtained from (2.1). For every choice of arbitrary $x = x_0$ in the interval $[a, b]$ and any $y = y_0$ values chosen along the Y – axis, when the pair (x_0, y_0) is substituted into the result (2.1) we obtain a new particular A which in turn yields a corresponding new solution. Hence every initial value problem above has at least one solution in the interval$[a, b]$.

Uniqueness:

To prove uniqueness, we wish to prove that if any two solutions are given, then they must be identical. Let y_1, y_2 be such solutions of the given initial value problem. In this case we have for each $i = 1,2$

$$a_1(x)\frac{dy_i}{dx} + a_0(x)y_i = h(x), \qquad a_1(x) \neq 0$$

Implying, by linearity of the differential operator

$$a_1(x)\frac{d(y_2 - y_1)}{dx} + a_0(x)(y_2 - y_1) = 0, \qquad a_1(x) \neq 0$$

and $(y_2 - y_1)(x_0) = y_2(x_0) - y_1(x_0) = 0$

Hence $y_2 - y_1$ is a solution of the homogenous initial value problem (ivp)

$$a_1(x)\frac{dy}{dx} + a_0(x)y = h(x), \qquad a_1(x) \neq 0 y(x_0) = y_0$$

However, the solution to the homogenous ivp is given by:

$$\text{y} = \text{Aexp}\left(-\int \frac{a_0(x)}{a_1(x)}dx\right) \qquad (2.2)$$

Where A is our arbitrary constant of integration.

Hence,

$$y_2 - y_1 = \text{A exp}\left(-\int \frac{a_0(x)}{a_1(x)}dx\right)$$

i.e $\quad y = y_2 - y_1 \qquad (2.3)$

Substituting $y(x_0) = 0$ into equation (2.2), we obtain

$$\text{A exp}\left(-\int \frac{a_0(x)}{a_1(x)}dx\right) = 0$$

but then

$$\text{A exp}\left(-\int \frac{a_0(x)}{a_1(x)}dx\right) \not\equiv 0$$

for every value of x on the real line.

Hence, $A = 0$, meaning that equation (2.2) we now have $y = 0$ as the solution to the ivp. But by equation (2.3), $y = y_2 - y_1$, hence $y_2 - y_1 = 0$ and so $y_2 = y_1$.

3.0 DERIVATION OF OUR METHOD

Preliminaries

The theoretical basis on which rational integrators work is the operator transform. Consider the general rational operator

$$U: \mathbb{R} \to \mathbb{R} \tag{3.1}$$

Defined by the identity

$$U(x)Q_n(x) \equiv p_m(x) \tag{3.2}$$

Where $Q_n(x)$ and $P_m(x)$ are real polynomials defined by

$$P_m(x) = \sum_{r=0}^{m} p_r x^r \tag{3.3}$$

$$Q_n(x) = \sum_{r=0}^{n} q_r x^r \tag{3.4}$$

The definition U(x) is by Pade′ and it is used by Pade′Approximants which then give rise to Pade′Integrators. Lambert (1973)

The approximation using the infinite series of the function U is given by

$$U(x_{n+1}) = \sum_{r=0}^{\infty} c_r x^r{}_{n+1}, \qquad c_r = \frac{h^r y_n{}^{(r)}}{r!} \tag{3.5}$$

We can therefore write

$$y(x_{n+1}) \cong y_{n+1} = \frac{p_m(x_{n+1})}{Q_n(x_{n+1})}, \qquad Q_n(0) \equiv 1 \tag{3.6}$$

Main derivation

Employing the rational interpolating function (3.6) where $m = k - 1$ and $n = k$, we have

$$y_{n+1} = \frac{P_{k-1}(x_{n+1})}{Q_k(x_{n+1})} \tag{3.7}$$

$P_{k-1}(x_{n+1}) = \sum_{i=0}^{k-1} P_i x^i{}_{n+1}$ and $Q_k(x_{n+1}) = 1 + \sum_{i=1}^{k} q_i x^i{}_{n+1}$

Following the work of Aashikpelokhai (1991), we obtain the parameters$q_i, i = 1(1)3$, by solving $Sq = b$ where:

$s_{ij} = \frac{h^{2k-i-j} y_n{}^{(2k-i-j)}}{(2k-i-j)!x_{n+1}{}^{2k-i-j}}$ and $b_i = \frac{-h^{2k-i} y_n{}^{(2k-i)}}{(2k-i)!x_{n+1}{}^{2k-i}}$, $i, j = 1(1)3$ and $k = 3$

Therefore;

$$s_{11} = \frac{h^4 y_n{}^{(4)}}{4!x^4{}_{n+1}}, \qquad s_{12} = \frac{h^3 y_n{}^{(3)}}{3!x^3{}_{n+1}}, \qquad s_{13} = \frac{h^2 y_n{}^{(2)}}{2!x^2{}_{n+1}}$$

$$s_{21} = \frac{h^3 y_n{}^{(3)}}{3!x^3{}_{n+1}} \qquad s_{22} = \frac{h^2 y_n{}^{(2)}}{2!x^2{}_{n+1}} \qquad s_{23} = \frac{h y_n{}^{(1)}}{1!x_{n+1}}$$

$$s_{31} = \frac{h^2 y_n{}^{(2)}}{2!x^2{}_{n+1}} \qquad s_{32} = \frac{h y_n{}^{(1)}}{1!x_{n+1}} \qquad s_{33} = y_n$$

$$b_1 = \frac{-h^5 y_n{}^{(5)}}{5!x^5{}_{n+1}} \qquad b_2 = \frac{-h^4 y_n{}^{(4)}}{4!x^4{}_{n+1}} \qquad b_3 = \frac{-h^3 y_n{}^{(3)}}{3!x^3{}_{n+1}}$$

In matrix form:

$$\begin{bmatrix} \frac{h^4 y_n^{(4)}}{4!x^4{}_{n+1}} & \frac{h^3 y_n^{(3)}}{3!x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} \\ \frac{h^3 y_n^{(3)}}{3!x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!x_{n+1}} \\ \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!x_{n+1}} & y_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \frac{-h^5 y_n^{(5)}}{5!x^5{}_{n+1}} \\ \frac{-h^4 y_n^{(4)}}{4!x^4{}_{n+1}} \\ \frac{-h^3 y_n^{(3)}}{3!x^3{}_{n+1}} \end{bmatrix} \tag{3.8}$$

We move now to find solutions to $q_i, \quad i = 1(1)3$ through the use of crammer's rule.
From equation (3.8), we let;

$$\text{(i)} \quad A = \begin{bmatrix} \frac{h^4 y_n^{(4)}}{4!x^4{}_{n+1}} & \frac{h^3 y_n^{(3)}}{3!x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} \\ \frac{h^3 y_n^{(3)}}{3!x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!x_{n+1}} \\ \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!x_{n+1}} & y_n \end{bmatrix}, \qquad \text{(ii)} \quad b = \begin{bmatrix} \frac{-h^5 y_n^{(5)}}{5!x^5{}_{n+1}} \\ \frac{-h^4 y_n^{(4)}}{4!x^4{}_{n+1}} \\ \frac{-h^3 y_n^{(3)}}{3!x^3{}_{n+1}} \end{bmatrix} \tag{3.9}$$

Thus

$$|A| = \begin{vmatrix} \frac{h^4 y_n^{(4)}}{4!x^4{}_{n+1}} & \frac{h^3 y_n^{(3)}}{3!x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} \\ \frac{h^3 y_n^{(3)}}{3!x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!x_{n+1}} \\ \frac{h^2 y_n^{(2)}}{2!x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!x_{n+1}} & y_n \end{vmatrix}$$

$$= \frac{h^4 y_n^{(4)}}{4!\,x^4{}_{n+1}} \begin{vmatrix} \frac{h^2 y_n^{(2)}}{2!\,x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!\,x_{n+1}} \\ \frac{h y_n^{(1)}}{1!\,x_{n+1}} & y_n \end{vmatrix} - \frac{h^3 y_n^{(3)}}{3!\,x^3{}_{n+1}} \begin{vmatrix} \frac{h^3 y_n^{(3)}}{3!\,x^3{}_{n+1}} & \frac{h y_n^{(1)}}{1!\,x_{n+1}} \\ \frac{h^2 y_n^{(3)}}{2!\,x^2{}_{n+1}} & y_n \end{vmatrix} - \frac{h^2 y_n^{(2)}}{2!\,x^2{}_{n+1}} \begin{vmatrix} \frac{h^3 y_n^{(3)}}{3!\,x^3{}_{n+1}} & \frac{h^2 y_n^{(2)}}{2!\,x^2{}_{n+1}} \\ \frac{h^2 y_n^{(2)}}{2!\,x^2{}_{n+1}} & \frac{h y_n^{(1)}}{1!\,x_{n+1}} \end{vmatrix}$$

Simplifying:

$$= \frac{h^4 y_n^{(4)}}{24x^4{}_{n+1}} \left(\frac{h^2 y_n^{(2)} y_n - 2h^2 \left(y_n^{(1)}\right)^2}{2x^2{}_{n+1}} \right) - \frac{h^3 y_n^{(3)}}{6x^3{}_{n+1}} \left(\frac{h^3 y_n^{(3)} y_n - 3h^3 y_n^{(1)} y_n^{(2)}}{6x^3{}_{n+1}} \right) + \frac{h^2 y_n^{(2)}}{2x^2{}_{n+1}} \left(\frac{2h^4 y_n^{(1)} y_n^{(3)} - 3h^4 \left(y_n^{(2)}\right)^2}{12x^4{}_{n+1}} \right)$$

Further simplification gives,

$$|A| = \frac{h^6}{144x^6{}_{n+1}} \begin{bmatrix} 3y_n y_n^{(2)} y_n^{(4)} - 6\left(y_n^{(1)}\right)^2 y_n^{(4)} - 4y_n\left(y_n^{(3)}\right)^2 \\ +24 y_n^{(1)} y_n^{(2)} y_n^{(3)} - 18\left(y_n^{(2)}\right)^3 \end{bmatrix} \tag{3.10}$$

Replacing the first column of (3.9) (i) by the column vector (3.9) (ii);

$$|A_1| = \begin{vmatrix} \frac{-h^5 y_n{}^{(5)}}{5!\, x^5{}_{n+1}} & \frac{h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} & \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} \\ \frac{-h^4 y_n{}^{(4)}}{4!\, x^4{}_{n+1}} & \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} & \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} \\ \frac{-h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} & \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} & y_n \end{vmatrix}$$

Leading us to:

$$= -\frac{h^5 y_n{}^{(5)}}{5!\, x^5{}_{n+1}} \begin{vmatrix} \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} & \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} \\ \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} & y_n \end{vmatrix} - \frac{h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} \begin{vmatrix} \frac{-h^4 y_n{}^{(4)}}{4!\, x^4{}_{n+1}} & \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} \\ \frac{-h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} & y_n \end{vmatrix} + \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} \begin{vmatrix} \frac{-h^4 y_n{}^{(4)}}{4!\, x^4{}_{n+1}} & \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} \\ \frac{-h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} & \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} \end{vmatrix}$$

Simplifying;

$$= -\frac{h^5 y_n{}^{(5)}}{120 x^5{}_{n+1}} \left(\frac{h^2 y_n{}^{(2)} y_n - 2h^2 \left(y_n{}^{(1)}\right)^2}{2x^2{}_{n+1}} \right) - \frac{h^3 y_n{}^{(3)}}{6x^3{}_{n+1}} \left(\frac{-h^3 y_n{}^{(4)} y_n + 4h^4 y_n{}^{(1)} y_n{}^{(3)}}{24x^4{}_{n+1}} \right) + \frac{h^2 y_n{}^{(2)}}{2x^2{}_{n+1}} \left(\frac{-h^5 y_n{}^{(1)} y_n{}^{(4)} + 2h^5 y_n{}^{(2)} y_n{}^{(3)}}{24x^5{}_{n+1}} \right)$$

This leads to:

$$|A_1| = \frac{h^7}{720x^7{}_{n+1}} \left[\begin{matrix} 6\left(y_n{}^{(1)}\right)^2 y_n{}^{(5)} - 3y_n y_n{}^{(2)} y_n{}^{(5)} - 20 y_n{}^{(1)} \left(y_n{}^{(3)}\right)^2 + 5 y_n y_n{}^{(3)} y_n{}^{(4)} \\ +30\left(y_n{}^{(2)}\right)^2 y_n{}^{(3)} - 15 y_n{}^{(1)} y_n{}^{(2)} y_n{}^{(4)} \end{matrix} \right] \tag{3.11}$$

Next, we replace the second column of (3.9) (i) by the column vector (3.9) (ii);

$$|A_2| = \begin{bmatrix} \frac{h^4 y_n{}^{(4)}}{4!\, x^4{}_{n+1}} & \frac{-h^5 y_n{}^{(5)}}{5!\, x^5{}_{n+1}} & \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} \\ \frac{h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} & \frac{-h^4 y_n{}^{(4)}}{4!\, x^4{}_{n+1}} & \frac{h y_n{}^{(1)}}{1!\, x_{n+1}} \\ \frac{h^2 y_n{}^{(2)}}{2!\, x^2{}_{n+1}} & \frac{-h^3 y_n{}^{(3)}}{3!\, x^3{}_{n+1}} & y_n \end{bmatrix}$$

$$= \frac{h^4 y_n{}^{(4)}}{4! x^4{}_{n+1}} \begin{vmatrix} \frac{-h^4 y_n{}^{(4)}}{4! x^4{}_{n+1}} & \frac{h y_n{}^{(1)}}{1! x_{n+1}} \\ \frac{-h^3 y_n{}^{(3)}}{3! x^3{}_{n+1}} & y_n \end{vmatrix} + \frac{h^5 y_n{}^{(5)}}{5! x^5{}_{n+1}} \begin{vmatrix} \frac{h^3 y_n{}^{(3)}}{3! x^3{}_{n+1}} & \frac{h y_n{}^{(1)}}{1! x_{n+1}} \\ \frac{h^2 y_n{}^{(3)}}{2! x^2{}_{n+1}} & y_n \end{vmatrix} + \frac{h^2 y_n{}^{(2)}}{2! x^2{}_{n+1}} \begin{vmatrix} \frac{h^3 y_n{}^{(3)}}{3! x^3{}_{n+1}} & \frac{-h^4 y_n{}^{(4)}}{4! x^4{}_{n+1}} \\ \frac{h^2 y_n{}^{(2)}}{2! x^2{}_{n+1}} & \frac{-h^3 y_n{}^{(3)}}{3! x^3{}_{n+1}} \end{vmatrix}$$

Evaluating, we have;

$$= \frac{h^4 y_n{}^{(4)}}{24x^4{}_{n+1}} \left(\frac{4h^4 y_n{}^{(1)} y_n{}^{(3)} - h^4 y_n y_n{}^{(4)}}{24x^4{}_{n+1}} \right) + \frac{h^5 y_n{}^{(5)}}{120x^5{}_{n+1}} \left(\frac{h^3 y_n{}^{(3)} y_n - 3h^3 y_n{}^{(1)} y_n{}^{(2)}}{6x^3{}_{n+1}} \right) + \frac{h^2 y_n{}^{(2)}}{2x^2{}_{n+1}} \left(\frac{3h^6 y_n{}^{(2)} y_n{}^{(4)} - 4h^6 \left(y_n{}^{(3)}\right)^2}{144x^6{}_{n+1}} \right)$$

This leads to;

$$|A_2| = \frac{h^8}{2880{x^8}_{n+1}}\left[\begin{matrix}20{y_n}^{(1)}{y_n}^{(3)}{y_n}^{(4)} - 5y_n\left({y_n}^{(4)}\right)^2 - 12{y_n}^{(1)}{y_n}^{(2)}{y_n}^{(5)} + 4y_ny_n{}^{(3)}{y_n}^{(5)} \\ +30\left({y_n}^{(2)}\right)^2{y_n}^{(4)} - 40{y_n}^{(2)}\left({y_n}^{(3)}\right)^2\end{matrix}\right] \quad (3.12)$$

To compute for $|A_3|$, we replace the third column of (3.9) (i) by the column vector (3.9)(ii);

$$|A_3| = \begin{vmatrix} \frac{h^4{y_n}^{(4)}}{4!\,{x^4}_{n+1}} & \frac{h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} & \frac{-h^5{y_n}^{(5)}}{5!\,{x^5}_{n+1}} \\ \frac{h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} & \frac{h^2{y_n}^{(2)}}{2!\,{x^2}_{n+1}} & \frac{-h^4{y_n}^{(4)}}{4!\,{x^4}_{n+1}} \\ \frac{h^2{y_n}^{(2)}}{2!\,{x^2}_{n+1}} & \frac{h{y_n}^{(1)}}{1!\,x_{n+1}} & \frac{-h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} \end{vmatrix}$$

$$= \frac{h^4{y_n}^{(4)}}{4!\,{x^4}_{n+1}}\begin{vmatrix} \frac{h^2{y_n}^{(2)}}{2!\,{x^2}_{n+1}} & \frac{-h^4{y_n}^{(4)}}{4!\,{x^4}_{n+1}} \\ \frac{h{y_n}^{(1)}}{1!\,x_{n+1}} & \frac{-h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} \end{vmatrix} - \frac{h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}}\begin{vmatrix} \frac{h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} & \frac{-h^4{y_n}^{(4)}}{4!\,{x^4}_{n+1}} \\ \frac{h^2{y_n}^{(3)}}{2!\,{x^2}_{n+1}} & \frac{-h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} \end{vmatrix} - \frac{h^5{y_n}^{(5)}}{5!\,{x^5}_{n+1}}\begin{vmatrix} \frac{h^3{y_n}^{(3)}}{3!\,{x^3}_{n+1}} & \frac{h^2{y_n}^{(2)}}{2!\,{x^2}_{n+1}} \\ \frac{h^2{y_n}^{(2)}}{2!\,{x^2}_{n+1}} & \frac{h{y_n}^{(1)}}{1!\,x_{n+1}} \end{vmatrix}$$

Leading to;

$$= \frac{h^4{y_n}^{(4)}}{24{x^4}_{n+1}}\left(\frac{h^5{y_n}^{(1)}{y_n}^{(4)} - 2h^5{y_n}^{(2)}{y_n}^{(3)}}{24{x^5}_{n+1}}\right) - \frac{h^3{y_n}^{(3)}}{6{x^3}_{n+1}}\left(\frac{3h^6{y_n}^{(2)}{y_n}^{(4)} - 4h^6\left({y_n}^{(3)}\right)^2}{144{x^6}_{n+1}}\right) - \frac{h^5{y_n}^{(5)}}{120{x^5}_{n+1}}\left(\frac{2h^4{y_n}^{(1)}{y_n}^{(3)} - 3h^4\left({y_n}^{(2)}\right)^2}{12{x^4}_{n+1}}\right)$$

This gives;

$$|A_3| = \frac{h^9}{8640{x^9}_{n+1}}\left[\begin{matrix}15{y_n}^{(1)}\left({y_n}^{(4)}\right)^2 - 60{y_n}^{(2)}{y_n}^{(3)}{y_n}^{(4)} + 40\left({y_n}^{(3)}\right)^3 + 18\left({y_n}^{(2)}\right)^2{y_n}^{(5)} \\ -12{y_n}^{(1)}{y_n}^{(3)}{y_n}^{(5)}\end{matrix}\right] \quad (3.13)$$

By crammer's rule: $q_i = \frac{|A_i|}{|A|}$ $\quad i = 1(1)3$, hence we employ this relation in results (3.10), (3.11), (3.12) and (3.13) to obtain;

$$q_1 = \frac{h}{5x_{n+1}}\left[\frac{6\left({y_n}^{(1)}\right)^2{y_n}^{(5)} - 3y_ny_n{}^{(2)}{y_n}^{(5)} - 20{y_n}^{(1)}\left({y_n}^{(3)}\right)^2 + 5y_ny_n{}^{(3)}{y_n}^{(4)} + 30\left({y_n}^{(2)}\right)^2{y_n}^{(3)} - 15{y_n}^{(1)}{y_n}^{(2)}{y_n}^{(4)}}{3y_ny_n{}^{(2)}{y_n}^{(4)} - 6\left({y_n}^{(1)}\right)^2{y_n}^{(4)} - 4y_n\left({y_n}^{(3)}\right)^2 + 24{y_n}^{(1)}{y_n}^{(2)}{y_n}^{(3)} - 18\left({y_n}^{(2)}\right)^3}\right] \quad (3.14)$$

$$q_2 = \frac{h^2}{20{x^2}_{n+1}}\left[\frac{20{y_n}^{(1)}{y_n}^{(3)}{y_n}^{(4)} - 5y_n\left({y_n}^{(4)}\right)^2 - 12{y_n}^{(1)}{y_n}^{(2)}{y_n}^{(5)} + 4y_ny_n{}^{(3)}{y_n}^{(5)} + 30\left({y_n}^{(2)}\right)^2{y_n}^{(4)} - 40{y_n}^{(2)}\left({y_n}^{(3)}\right)^2}{3y_ny_n{}^{(2)}{y_n}^{(4)} - 6\left({y_n}^{(1)}\right)^2{y_n}^{(4)} - 4y_n\left({y_n}^{(3)}\right)^2 + 24{y_n}^{(1)}{y_n}^{(2)}{y_n}^{(3)} - 18\left({y_n}^{(2)}\right)^3}\right] \quad (3.15)$$

$$q_3 = \frac{h^3}{60x^3{}_{n+1}}\left[\frac{15y_n{}^{(1)}\left(y_n{}^{(4)}\right)^2 - 60y_n{}^{(2)}y_n{}^{(3)}y_n{}^{(4)} + 40\left(y_n{}^{(3)}\right)^3 + 18\left(y_n{}^{(2)}\right)^2 y_n{}^{(5)} - 12y_n{}^{(1)}y_n{}^{(3)}y_n{}^{(5)}}{3y_n y_n{}^{(2)}y_n{}^{(4)} - 6\left(y_n{}^{(1)}\right)^2 y_n{}^{(4)} - 4y_n\left(y_n{}^{(3)}\right)^2 + 24y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(3)} - 18\left(y_n{}^{(2)}\right)^3}\right] \tag{3.16}$$

Suppose:

$$a = 3y_n y_n{}^{(2)}y_n{}^{(4)} - 6\left(y_n{}^{(1)}\right)^2 y_n{}^{(4)} - 4y_n\left(y_n{}^{(3)}\right)^2 + 24y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(3)} - 18\left(y_n{}^{(2)}\right)^3 \tag{3.17}$$

be the common denominator for $q_i = \frac{|A_i|}{|A|}, \forall\ i = 1,2,3.$ If at any point $a = 0 \Leftrightarrow |A| = 0$ and therefore the test for ill-conditioning at any stage is when $a = 0$.

The corresponding numerators are:

$$b = 6\left(y_n{}^{(1)}\right)^2 y_n{}^{(5)} - 3y_n y_n{}^{(2)}y_n{}^{(5)} - 20y_n{}^{(1)}\left(y_n{}^{(3)}\right)^2 + 5y_n y_n{}^{(3)}y_n{}^{(4)} + 30\left(y_n{}^{(2)}\right)^2 y_n{}^{(3)} - 15y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(4)} \tag{3.18}$$

$$c = 20y_n{}^{(1)}y_n{}^{(3)}y_n{}^{(4)} - 5y_n\left(y_n{}^{(4)}\right)^2 - 12y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(5)} + 4y_n y_n{}^{(3)}y_n{}^{(5)} + 30\left(y_n{}^{(2)}\right)^2 y_n{}^{(4)} - 40y_n{}^{(2)}\left(y_n{}^{(3)}\right)^2 \tag{3.19}$$

$$d = 15y_n{}^{(1)}\left(y_n{}^{(4)}\right)^2 - 60y_n{}^{(2)}y_n{}^{(3)}y_n{}^{(4)} + 40\left(y_n{}^{(3)}\right)^3 + 18\left(y_n{}^{(2)}\right)^2 y_n{}^{(5)} - 12y_n{}^{(1)}y_n{}^{(3)}y_n{}^{(5)} \tag{3.20}$$

Hence;

$$q_1 = \frac{bh}{5ax_{n+1}} \quad q_2 = \frac{ch^2}{20ax^2{}_{n+1}} q_3 = \frac{dh^3}{60ax^3{}_{n+1}} \quad \text{Provided } a \neq 0 \tag{3.21}$$

Computing for p_1 and p_2, we have;

$$p_j = \sum_{i=1}^{j} \frac{y_n{}^{(j+1-i)}h^{(j+1-i)}}{(j+1-i)!\,x_{n+1}{}^{(j+1-i)}} q_{i-1} + y_n q_j\,, \quad \text{where } j = 1,2\,,\ k = 3 \text{ and } q_0 = 1$$

That is:

$$p_1 = \frac{y_n{}^{(1)}h}{x_{n+1}} + y_n q_1 \tag{3.22}$$

$$p_2 = \frac{y_n{}^{(2)}h^2}{2x^2{}_{n+1}} + \frac{y_n{}^{(1)}h}{x_{n+1}} q_1 + y_n q_2 \tag{3.23}$$

Using equation (3.21), then equation (3.22) and (3.23) becomes;

$$p_1 = \frac{y_n{}^{(1)}h}{x_{n+1}} + y_n\left[\frac{bh}{5ax_{n+1}}\right]$$

This simplifies to:

$$p_1 = \frac{h}{5ax_{n+1}}\left[5ay_n{}^{(1)} + by_n\right] \qquad \text{Provided } a \neq 0 \tag{3.24}$$

Similarly;

$$p_2 = \frac{y_n{}^{(2)}h^2}{2x^2{}_{n+1}} + \frac{y_n{}^{(1)}h}{x_{n+1}}\left[\frac{bh}{5ax_{n+1}}\right] + y_n\left[\frac{ch^2}{20ax^2{}_{n+1}}\right]$$

Leading to;

$$p_2 = \frac{h^2}{20ax_{n+1}}\left[10ay_n{}^{(2)} + 4by_n{}^{(1)} + cy_n\right] \qquad \text{Provided } a \neq 0 \tag{3.25}$$

By equation (3.7), we have:

$$y_{n+1} = \frac{p_0 + p_1x_{n+1} + p_2x^2{}_{n+1}}{1 + q_1x_{n+1} + q_2x^2{}_{n+1} + q_3x^3{}_{n+1}} \tag{3.26}$$

Substituting equation (3.21), (3.24), (3.25) and $p_0 = y_n$ in equation (3.26), we have;

$$y_{n+1} = \frac{y_n + \frac{h}{5a}\left[5ay_n{}^{(1)} + by_n\right] + \frac{h^2}{20a}\left[10ay_n{}^{(2)} + 4by_n{}^{(1)} + cy_n\right]}{1 + \frac{bh}{5a} + \frac{ch^2}{20a} + \frac{dh^3}{60a}}$$

Simplifying;

$$y_{n+1} = \frac{60ay_n + 12h\left(5ay_n{}^{(1)} + by_n\right) + 3h^2\left(10ay_n{}^{(2)} + 4by_n{}^{(1)} + cy_n\right)}{60a + 12bh + 30ch^2 + dh^3}$$

Further simplification leads us to our composite integrator formula $(k = 3)$ given hereunder as;

$$y_{n+1} = \frac{60ay_n + 12he + 3h^2f}{60a + 12bh + 3ch^2 + dh^3} \tag{3.27}$$

Where: $a = 3y_ny_n{}^{(2)}y_n{}^{(4)} - 6\left(y_n{}^{(1)}\right)^2y_n{}^{(4)} - 4y_n\left(y_n{}^{(3)}\right)^2 + 24y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(3)} - 18\left(y_n{}^{(2)}\right)^3$

$$b = 6\left(y_n{}^{(1)}\right)^2y_n{}^{(5)} - 3y_ny_n{}^{(2)}y_n{}^{(5)} - 20y_n{}^{(1)}\left(y_n{}^{(3)}\right)^2 + 5y_ny_n{}^{(3)}y_n{}^{(4)} + 30\left(y_n{}^{(2)}\right)^2y_n{}^{(3)} - 15y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(4)}$$

$$c = 20y_n{}^{(1)}y_n{}^{(3)}y_n{}^{(4)} - 5y_n\left(y_n{}^{(4)}\right)^2 - 12y_n{}^{(1)}y_n{}^{(2)}y_n{}^{(5)} + 4y_ny_n{}^{(3)}y_n{}^{(5)} + 30\left(y_n{}^{(2)}\right)^2y_n{}^{(4)} - 40y_n{}^{(2)}\left(y_n{}^{(3)}\right)^2$$

$$d = 15y_n{}^{(1)}\left(y_n{}^{(4)}\right)^2 - 60y_n{}^{(2)}y_n{}^{(3)}y_n{}^{(4)} + 40\left(y_n{}^{(3)}\right)^3 + 18\left(y_n{}^{(2)}\right)^2y_n{}^{(5)} - 12y_n{}^{(1)}y_n{}^{(3)}y_n{}^{(5)}$$

$$e = 5ay_n{}^{(1)} + by_n \tag{3.28}$$

$$f = 10ay_n{}^{(2)} + 4by_n{}^{(1)} + cy_n\,. \tag{3.29}$$

4.0 CONVERGENCE AND CONSISTENCY ANALYSIS OF OUR SCHEME

Convergence is a vital property any given numerical formula must attain. Hence, researcher like Lambert (1995) asserts that convergence is a minimal property to expect of a numerical method and that convergence must take place for all initial value problems. Lambert (1995) went on to state that one-step method is said to be convergent if, for all initial value problem satisfying the lipschitz condition then;

$$\lim_{h\to 0}\max_{0\le n\le N}\|y(x_n)-y_n\|=0$$

However, convergence of one-step method implies the consistency of the method, though the converse is not true Lambert (1995). Therefore to show our method is convergent and consistent, we need only to show its convergent which then implies its consistent.

Theorem 3.1: The one-step composite integrator:

$$y_{n+1}=\frac{60ay_n+12eh+3fh^2}{60a+12bh+3ch^2+dh^3} \qquad (3.30)$$

where the functions a, b, c, d, e, and f are specified by (3.17), (3.18), (3.19), (3.20), (3.28) and (3.29) respectively is consistent and convergent.

Proof

We wish to show that: $\lim_{h\to 0}(\frac{y_{n+1}-y_n}{h})=y_n{}^{(1)}=f(x_n,y_n)$
Therefore, from the integrator (3.30), we have:

$$y_{n+1}-y_n=\frac{60ay_n+12eh+3fh^2}{60a+12bh+3ch^2+dh^3}-y_n$$

$$y_{n+1}-y_n=\frac{60ay_n+12h(5ay_n{}^{(1)}+by_n)+3h^2(10ay_n{}^{(2)}+4by_n{}^{(1)}+cy_n)}{60a+12bh+3ch^2+dh^3}-y_n$$

$$=\frac{60ay_n+12h\big(5ay_n{}^{(1)}+by_n\big)+3h^2\big(10ay_n{}^{(2)}+4by_n{}^{(1)}+cy_n\big)-y_n(60a+12bh+3ch^2+dh^3)}{60a+12bh+3ch^2+dh^3}$$

Simplifying;

$$y_{n+1}-y_n=\frac{\mathrm{h}(60\mathrm{a}y_n{}^{(1)}+30ay_n{}^{(2)}h+12by_n{}^{(1)}h+30cy_nh-30cy_nh-dy_nh^2)}{60a+12bh+3ch^2+dh^3}$$

$$\frac{y_{n+1}-y_n}{h}=\frac{60\mathrm{a}y_n{}^{(1)}+30ay_n{}^{(2)}h+12by_n{}^{(1)}h-dy_nh^2}{60a+12bh+3ch^2+dh^3}$$

$$\lim_{h\to 0}\left[\frac{y_{n+1}-y_n}{h}\right]=\frac{60ay_n{}^{(1)}+0+0-0}{60a+0+0+0}=\frac{60ay_n{}^{(1)}}{60a}$$

$$\lim_{h\to 0}\left[\frac{y_{n+1}-y_n}{h}\right] = {y_n}^{(1)} = f(x_n, y_n)$$

Hence our integrator is consistent.

5.0 DEMONSTRATION

Here, we implemented of our new integrator in solving an initial value problem in ordinary differential equations given below: (Problem from Aashikpelokhai, 1991)

$$\begin{bmatrix} {y_1}^{(1)} \\ {y_2}^{(1)} \end{bmatrix} = \begin{bmatrix} -2000 & 1000 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0 \le x \le 5$$

Theoretical solution:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4.9975(-4) & -5.0025(-4) \\ 2.4994(-7) & -1.0002(-3) \end{bmatrix}\begin{bmatrix} e^{-x} \\ e^{-2000x} \end{bmatrix} + \begin{bmatrix} 1(-3) \\ 1(-3) \end{bmatrix}$$

Table A: Solution for the First Component at H = 0.01. Each Solution is multiplied by 10^4

X	THEORETICAL SOLUTION	AASIKPELOKHAI (1991) K = 3	COMPOSITE INTEGRATOR FORMULA, K = 3	N_f
0.5	6.1038	6.1046	2.06256	50
1.0	6.9655	6.9961	2.85022	100
1.5	7.6365	7.6370	3.50510	150
2.0	8.1592	8.1596	4.04258	200
2.5	8.5663	8.5663	4.47905	250
3.0	8.8834	8.8836	4.83045	300
3.5	9.1303	9.1305	5.11141	350
4.0	9.3226	9.3326	5.33480	400
4.5	9.4724	9.4726	5.51163	450
5.0	9.5891	9.5891	5.65112	500

Table B: Solution for the Second Component at H = 0.01

X	THEORETICAL SOLUTION Error × 10^4	AASHIKPELOKHAI (1991), k = 3	COMPOSITE INTEGRATOR FORMULA, K = 3	N_f
0.5	2.2099	2.2100(-4)	1.36683(-7)	50
1.0	3.9327	3.9327(-4)	9.89322(-7)	100
1.5	5.2745	5.2745(-4)	1.65828(-6)	150
2.0	6.3195	6.3196(-4)	2.18031(-6)	200
2.5	7.1335	7.1335(-4)	2.58724(-6)	250
3.0	7.7674	7.7674(-4)	2.90431(-6)	300
3.5	8.2612	8.2612(-4)	3.15133(-6)	350
4.0	8.6457	8.6457(-4)	3.34375(-6)	400
4.5	8.9452	8.9452(-4)	3.49363(-6)	450
5.0	9.1785	9.1785(-4)	3.61038(-6)	500

Index: $a(-b) = a \times 10^{-b}$

Discussion of Results Generated by our method

Whenever a numerical method is used to solve a differential equation, the idea is to produce accurate solution that will override the theoretical solution with minimum error. For instance, if we denote the exact solution as $y(x_i)$ at some pointx_i, then the numerical solution at that point x_i is denoted as y_i. Therefore we are interested in the error given as:

$$\ell_i = |y(x_i) - y_i| \qquad \forall\ 0 \leq i \leq 20 \qquad (5.1)$$

As a matter of fact, we do not expect to be able to know the error (ℓ_i) exactly at given intervals because we do not have the exact solution in general. Hence, it will be of interest to derive a formula that can approximate the solution of IVPs in ODEs whose error accumulates within a specific interval.

A close look at the result on table 'A' above generated by our method (composite integrator formula at k = 3) reveals that the method maintain a reasonable steady low error level throughout the steps even with the increase in step length. However, for the table 'B' above, we observed that the difference between the numerical solution and the theoretical solution was minimal, and the error level varies at different steps due to the inaccuracy inherent in the formula and the arithmetic operations adopted during the computer implementation.

In totality, it is seen that the composite integrator formula was able to produce results that are as accurate as those of other existing methods as our integrator compares favourably with Aashikpelokhai (1991).

Conclusion

In this work, we have been able to derive a composite integrator from Aashikpelokhai (1991) at $k = 3$, analyzed its consistency nature and implemented our integrator on a selected initial value problem in ODEs. Our method compared favorably with the existing methods. However, for derivation of composite integrator at $k > 3$, Cramer's rule may not be appropriate, hence researchers in this line could explore other methods.

Recommendation

Having derived and implemented a new composite integrator formula from Aashikpelokhai (1991) at $k = 3$, capable of solving problems in initial value problems arising from first order in ODEs, we therefore recommend the method for use by ODE solvers and for researchers currently working in this area.

References

Aashikpelokhai, U.S.U (1991): A Class of Non Linear one – step Rational Integrators, PhD Thesis, University of Benin, Benin City, Edo State, Nigeria.

Aashikpelokhai, U.S.U, Agbeboh G.U. and Abhulimen C.E. (2011): A First Course in Differential Equations. Pon Publishers Ltd., Ekpoma, Nigeria.

Aashikpelokhai, U.S.U and Momodu, B (2008): A quadratic based integration scheme for the solution of singulo – stiff differential equation. *IJPS: Vol. 3(4): 97 – 103.*

Abhulimen, C.E. (2014): Exponentially fitted third derivative three step method for numerical integration for stiff IVP s. *Applied mathematics and computation: 243, 446 – 453.*

Abhulimen, C.E. and Otunta, F.O (2007): A Family of Two – Step exponentially fitted multi derivable methods for the numerical integration of stiff IVPs in ODE. *Int. J. Num analysis: Vol. 2, 1 – 21*

Ademiluyi, R.A and Kayode, S.J (2001): Maximum order second derivative hybrid Muti-step methods for integration of initial value problems in Ordinary Differential Equations. *J. Nig. ASS. Maths Phys: Vol. 5: 251 – 262.*

Agbeboh, G.U (2006): Comparison of some one – step integrators for solving singular initial value problem. Ph.D Thesis, AAU, Ekpoma, Nigeria.

Braun M. (1993): "Differential Equations and their Applications". An introduction to Applied Mathematics, 4th Edition.Published by Springer – Verlag, New York Inc.,175 5th Avenue, New York, NY 10010, USA.

Elakhe, A.O (2011): Singulo -Oscillatroy Stiff Rational Integrator. Ph.D Thesis, Ambrose Alli University, Ekpoma, Nigeria. Pp.173.

Fatunla S.O (1978): An implicit Two-Point Numerical Integration Formula for Linear and Non-Linear Stiff Systems of Ordinary Differential Equations, *Mathematics of Computation: 32(141):1-11*

Fatunla, S.O. (1980): Numerical integrators for stiff and highly Oscillatory Differential Equations. *Mathematics of Computation*: 34: 373 – 390.

Fatunla, S.O. (1982): Non – Linear Multi – Step Methods for IVPs. *Compt. Maths Aplic. 18, 231 – 239.*

Lambert, J. D (1973): Computational methods in ODE. John Wiley and Sons New York.

Lambert, J.D and Shaw, B (1965): On the numerical Solution of $y^1 = f(x,y)$ by a class of formulae Based on Rational Approximation. *Mathematics of computation: 19, 456 – 462*

Momodu, I.B.A (2006): Solution of Singulo – Stiff Ordinary Differential Equations. Ph.D Thesis AAU, Ekpoma, Nigeria.

Otunta, F.O and Ikhile, M.N.O (1997): Efficient Rational One – Step NumericalIntegrators for initial value problems in ODE, *Int. J Computer Maths. 72, pp. 49 – 61*

Ukpebor, L.A (2016): Region of Absolute stability of an order 19 Rational Integrator Ph.D Thesis, AAU, Ekpoma, Nigeria.

AUTHOR'S CONTRIBUTIONS

Aashikpelokhai and Akerejola contributed immensely at different stages of this research to ensure its successful completion.