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# Solute transport with periodic input point source in one-dimensional heterogeneous porous media

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**Abstract**

In the present study, we presented analytical solutions for solute transport in a semi-infinite heterogeneous adsorbing porous media with time-varying boundary condition. Initially, solute concentration in the domain is function of the space variable. Continuous periodic point source is injected in the domain through left boundary, i.e. $x = 0$. Due to heterogeneity of the medium, dispersion parameter is considered proportional to $(\zeta + 1)$th power of linear function of space variable. The groundwater flow velocity is considered proportional to multiple of temporal function and $\zeta$th power of linear function of space. First-order decay and zero-order production are also considered space as well as time dependent while retardation factor is a space dependent function. Laplace Transformation Technique is employed to get the solution of the proposed problem. Certain new transformations are introduced to convert the variable coefficient into constant coefficient. Comparison with analytical and pdepe MATLAB solution of the transport equation are illustrated graphically and found to be in excellent agreement.

*Keywords:* Advection; Dispersion; Aquifer; Porous Medium ; Retardation; Laplace Transformation.

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## 1. Introduction

The contaminants in aquifer mainly transport with ground water flow that may affect subsurface water table. Solute transport in the subsurface and the groundwater are also affected by a number of physical, chemical and biological properties of the media. The natural hydrological conditions may also affect the behavior of some pollutants because there is potential interaction between the water and the porous medium through which it passes. If groundwater gets polluted once, it is very difficult to clean it. Contaminant transport through porous medium is described by second order partial differential equation of parabolic type which is generally known as advection-dispersion equation (Bear, 1972). In the published literatures, numerous analytical solutions have been published for estimation of the solute transport quantitatively in subsurface, lakes, reservoirs, drains, and canals through mathematical models. Analytical solutions for conservative, non conservative solute in confined/unconfined aquifers have been developed for various types of boundary conditions. Crank (1956) developed the analytical solution of advection-dispersion in one-dimension for a point source pollutant in porous medium. Huyakorn *et al.* (1987) developed an analytical model for predicting contaminant transport perpendicular to the direction of groundwater flow. Park and Zhan (2001) developed analytical solutions of contaminant transport from one, two, and three-dimensional finite sources using Green's function method.

Workman *et al.* (1997) developed an analytical solution for water-table fluctuations in a finite thickness aquifer. Mazumder and Das (1992) and Jiang and Grotberg (1993) studied that the effect of wall absorption on axial dispersion in oscillatory tube flow. One-dimensional solute transport through porous media with or without accounting for zero-order production and first-order decay is developed by van Genuchten *et al.* (2013a&b). Kumar *et al.* (2010) , Yadav *et al.* (2010), Jaiswal *et al.* (2011), Yadav and Jaiswal (2011) obtained analytical solutions for one and two-dimensional advection-diffusion equation with variable coefficients in a longitudinal semi-infinite domain, for temporally and specially dependent dispersion problems. Singh et *al.* (2015) derived analytical solution of advection diffusion equation with variable porosity. Generally groundwater contamination takes place due to infiltration of contaminant through the vadoze zone and reaching to the water-table in the direction of groundwater flow. Chen *et al.* (2008) obtained analytical solution with hyperbolic asymptotic distance-dependent dispersivity in porous media. Sanskrityayn *et al.* (2016) obtained analytical solution of advection dispersion equation with spatially and temporally dependent dispersion using Green's function. Singh and Chatterjee (2016) presented a solution of three dimensional advection dispersion equation with non-point source of in semi-infinite aquifer with specified concentration along an arbitrary plane using Laplace transform technique.

The objective of present study is to develop a mathematical model for conservative solute transport in one-dimensional heterogeneous porous domain of adsorbing nature. The solution of the present study is derived by using Laplace Transformation Technique. Zero concentration gradient is assumed at the exit boundary, i.e. infinity. The adsorption coefficient is taken as a function of space variable. The effect of heterogeneity and physical parameters on water flow and solute transport are well illustrated. Dispersion coefficient is considered proportional to $(\zeta+1)$th power of linear function of space variable. The ground water flow velocity is considered proportional to multiple of temporal function and $\zeta$ th power of linear function of space. First order decay and zero order production are also taken into account. Two form of temporal velocity are considered as especial cases. Solutions have been obtained using a set of hypothetical input data taken from the previous published works. Ground water velocity ranges from 2 m/year to 2 m/day, intermediate values are taken in present study (Todd, 1980).

## 2. Mathematical Description of the Problem

The pollutant reaches mainly in two ways, inside the subsurface, the first advection which is caused by flow of groundwater and the second dispersion caused by mechanical mixing and molecular diffusion.The mathematical relationship of advection–dispersion equation in one-dimension may be given by a second order partial differential equation of parabolic type (Bear, 1972).

$$R(x,t)\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x}-u(x,t)c\right)-\mu(x,t)c+\gamma(x,t) \qquad (1)$$

The Eq. (1) is derived from Darcy's law and the law of conservation of mass (Freeze and Cherry, 1979) and in this equation $c[ML^{-3}]$ is the solute concentration of the pollutant transporting along the flow field through the medium at a position $x[L]$ and time $t[T]$ . $D[L^2T^{-1}]$ and $u[LT^{-1}]$ are the longitudinal dispersion and the seepage velocity along $x$ axis respectively and $\mu[T^{-1}]$, $\gamma[ML^{-3}T^{-1}]$ represent the first order decay and zero order production rate coefficients for solute which represents internal/external production of the solute respectively. First term on the left hand side of the equation (1) represents change in concentration with time in liquid phase and $R$ is retardation factor which is a dimensionless quantity. First term on the right-hand side of the Eq. (1) describes the influence of the dispersion on the concentration distribution in longitudinal direction while second term is the change of the concentration due to convective transport in longitudinal direction.

The coefficient of dispersion is considered directly proportional temporally dependent groundwater velocity (Yim and Mohsen, 1992), i.e.

$$D(x,t)\propto u(x,t) \qquad (2)$$

We have assumed groundwater velocity and dispersion as $u(x,t)=u_0 f(mt)(1+ax)^{\zeta}$ and $D(x,t)=D_0 f(mt)(1+ax)^{\zeta+1}$ where $m[T^{-1}]$ is the unsteady parameter having the dimension inverse of time while $a[L^{-1}]$ is a parameter to regulate the spatially dependency of dispersion coefficient and $\zeta$ is arbitrary real number. Thus $f(mt)$ is an expression of non-dimensional variable. Since it is assumed that dispersion is directly proportional to the temporally dependent seepage velocity i.e., $D\propto u$ or $D=\eta u$, where $\eta$ is constant which depends upon the pore geometry of the porous medium. First order decay $\mu(x,t)$ and zero order production $\gamma(x,t)$ which are temporally proportional to dispersion coefficient, may be defined as $\gamma(x,t)=\gamma_0 f(mt)(1+ax)^{\zeta-1}$, $\mu(x,t)=\mu_0 f(mt)(1+ax)^{\zeta-1}$ respectively and retardation is $R(x,t)=R_0(1+ax)^{\zeta-1}$ where $u_0[LT^{-1}], D_0[L^2T^{-1}]$ , $\mu_0[T^{-1}]$ , $\gamma_0[ML^{-3}T^{-1}]$ and $R_0$ are constants. Therefore, Eq.(1) may be re-written as

$$\begin{aligned}R_0(1+ax)^{\zeta-1}\frac{\partial c}{\partial t}&=\frac{\partial}{\partial x}\left(D_0 f(mt)(1+ax)^{\zeta+1}\frac{\partial c}{\partial x}-u_0 f(mt)(1+ax)^{\zeta}c\right)-\\&\mu_0 f(mt)(1+ax)^{\zeta-1}c+\gamma_0 f(mt)(1+ax)^{\zeta-1}\end{aligned} \qquad (3)$$

Initially, the domain has some concentration which is function of the space variable (Singh *et al*. 2015). It means domain is not solute free. Periodic input condition is assumed at the origin and flux type condition is considered at the other end i.e., at $x = \infty$, of the domain. It appears that pollution spreads in the direction of flow. The mathematically the initial boundary conditions may be written as follows:

$$c(x,t) = c_i + \frac{\gamma}{u}x;\ t = 0,\ x > 0 \tag{4}$$

$$c(x,t) = c_0\left(1+\cos(mt)\right);\ t > 0,\ x = 0 \tag{5}$$

$$\frac{\partial c(x,t)}{\partial x} = 0;\ \text{as}\quad x \to \infty\quad,\ t > 0 \tag{6}$$

where $c_0[ML^{-3}]$ , $c_i[ML^{-3}]$ are reference and resident concentration respectively. The practical significance of the equation (5) is the periodic concentration at source of the boundary i.e., $x = 0$. The field observations indicate the source concentration may not be negative, therefore $c_0(1+\cos(mt))$ is taken.

Let us introduce a new time variable $T$ as: (Crank, 1975),

$$T = \int_0^t f(mt)dt \tag{7}$$

It is evident that the dimension of $T$ will be that of old time variable $t$ hence it is referred as a new time variable. It is also ascertained that $T = 0$ at $t = 0$. So nature of initial condition does not change in the new time domain. In terms of new time variable the advection-dispersion equation (3) reduces into

$$R_0(1+ax)^{\zeta-1}\frac{\partial c}{\partial T} = \frac{\partial}{\partial x}\left(D_0(1+ax)^{\zeta+1}\frac{\partial c}{\partial x} - u_0(1+ax)^{\zeta}c\right) - \mu_0(1+ax)^{\zeta-1}c + \gamma_0(1+ax)^{\zeta-1} \tag{8}$$

The analytical solution is obtained for $f(mt) = |\cos(mt)|$ and $f(mt) = \dfrac{mt}{\sqrt{(mt)^2 + k^2}}$ separately. Where $k$ is any dimensionless real number.

*2.1Case 1. when $f(mt) = |\cos(mt)|$ , (periodic function)*

using $f(mt) = |\cos(mt)|$ , in Eq. (7) we have

$$T = \int_0^t |\cos(mt)|dt \tag{9}$$

Or $$mT = |\sin(mt)| \tag{10}$$

Initial and boundary conditions (4-6) in terms of new time variable $T$ may be written as:

$$c(x,T) = c_i + \frac{\gamma}{u}x;\ 0 \le x < \infty,\quad T = 0 \tag{11}$$

$$c(x,T) = \frac{c_0}{2}\left\{4 - m^2T^2\right\};\ x = 0,\quad T > 0 \tag{12}$$

$$\frac{\partial c}{\partial x} = 0; \text{as} \quad x \to \infty, \ T \geq 0 \tag{13}$$

Since $mT << 1$, so neglecting higher order terms (third and on wards) in binomial expansion of $\left(1 - m^2T^2\right)^{1/2}$ .

In order to reduce the advection dispersion equation (8) into the constant coefficient a new space dependent transformation is introduced as: (Kumar *et al.* 2012)

$$X = \frac{1}{a}\log(1 + ax) \text{ Or } x = \frac{1}{a}\{\exp(aX) - 1\} \tag{14}$$

Dimension of new space variable $X$ is same as of $x$ and $X = 0$ at $x = 0$ .

The Eq. (8) and Eqs. (11-13) are reduced into following form with transformation (14)

$$R_0 \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - (u_0 - \zeta a D_0)\frac{\partial c}{\partial X} - (\mu_0 + \zeta a u)c + \gamma_0$$

Or

$$R_0 \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - w_0 \frac{\partial c}{\partial X} - \mu_1 c + \gamma_0 \tag{15}$$

$$c(X,T) = c_i + \frac{\gamma_0}{u_0 a}\{1 - \exp(-aX)\}; 0 \leq X < \infty, T = 0 \tag{16}$$

$$c(X,T) = \frac{c_0}{2}\{4 - m^2T^2\}; X = 0, \quad T > 0 \tag{17}$$

$$\frac{\partial c}{\partial X} = 0; \text{as} \quad X \to \infty, \ T \geq 0 \tag{18}$$

Where $\mu_1 = \mu_0 + \zeta a u_0$ and $w_0 = u_0 - \zeta a D_0$

Now introducing a transformation in order to reduce the convective term from advection–dispersion Eq. (15)

$$c(X,T) = k(X,T)\exp\left\{\frac{w_0}{2D_0}X - \frac{1}{R_0}\left(\frac{w_0^{\ 2}}{4D_0} + \mu_1\right)T\right\} + \frac{\gamma_0}{\mu_1} \tag{19}$$

With transformation Eq. (19), Eqs. (15–18) reduce into

$$R_0 \frac{\partial k}{\partial T} = D_0 \frac{\partial^2 k}{\partial X^2} \tag{20}$$

$$k(X,T) = \left[\left(c_i + \frac{\gamma_0}{u_0 a} - \frac{\gamma_0}{\mu_1}\right) - \frac{\gamma_0}{u_0 a}\exp(-aX)\right]\exp(-\beta X); \ 0 \leq X < \infty, T = 0 \tag{21}$$

$$k(X,T) = \left\{\frac{c_0}{2}\left(4 - m^2T^2\right) - \frac{\gamma_0}{\mu_1}\right\}\exp\left(\eta^2 T\right); \ X = 0, T > 0 \tag{22}$$

$$\frac{\partial k}{\partial X}+\frac{w_0}{2D_0}k=0\text{; as } X\to\infty, T\geq 0 \tag{23}$$

Where, $\beta=\dfrac{w_0}{2D_0}$ and $\eta=\sqrt{\dfrac{1}{R_0}\left(\dfrac{{w_0}^2}{4D_0}+\mu_1\right)}$

Applying Laplace Transformation on Eqs. (20– 23), we get

$$\frac{d^2\bar{k}}{dX^2}-\frac{pR_0}{D_0}\bar{k}=-\frac{R_0}{D_0}\left[\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\exp(-\beta X)-\frac{\gamma_0}{u_0a}\exp\{-(a+\beta)X\}\right] \tag{24}$$

$$\bar{k}(X,p)=\left(2c_0-\frac{\gamma_0}{\mu_1}\right)\frac{1}{(p-\eta^2)}-\frac{m^2c_0}{(p-\eta^2)^3}\text{; } X=0 \tag{25}$$

$$\frac{d\bar{k}}{dX}+\frac{w_0}{2D_0}\bar{k}=0\text{;}\quad \text{as}\quad X\to\infty \tag{26}$$

Where $\bar{k}(X,p)=\int_0^\infty k(X,T)e^{-pT}dT$ , in which $p$ is the Laplace transformation parameter.

General solution Eq. (24) may be written as:

$$\bar{k}(X,p)=C_1e^{\sqrt{\frac{pR}{D_0}}X}+C_2e^{-\sqrt{\frac{pR}{D_0}}X}+\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\frac{\exp(-\beta X)}{\left(p-\frac{D_0}{R}\beta^2\right)}-\frac{\gamma_0}{u_0a}\frac{\exp\{-(a+\beta)X\}}{\left\{p-\frac{D_0}{R}(a+\beta)^2\right\}} \tag{27}$$

Using conditions Eq. (25, 26) in Eq. (27), the solution of differential Eq. (24) may be obtained as:

$$\bar{k}(X,p)=\left(2c_0-\frac{\gamma_0}{\mu_1}\right)\frac{\exp\left(-\sqrt{\frac{pR_0}{D_0}}X\right)}{(p-\eta^2)}-m^2c_0\frac{\exp\left(-\sqrt{\frac{pR_0}{D_0}}X\right)}{(p-\eta^2)^3}-\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\frac{\exp\left(-\sqrt{\frac{pR_0}{D_0}}X\right)}{\left(p-\frac{D_0}{R_0}\beta^2\right)}+$$

$$\frac{\gamma_0}{u_0a}\frac{\exp\left(-\sqrt{\frac{pR_0}{D_0}}X\right)}{\left\{p-\frac{D_0}{R_0}(a+\beta)^2\right\}}+\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\frac{\exp(-\beta X)}{\left(p-\frac{D_0}{R_0}\beta^2\right)}-\frac{\gamma_0}{u_0a}\frac{\exp\{-(a+\beta)X\}}{\left\{p-\frac{D_0}{R_0}(a+\beta)^2\right\}} \tag{28}$$

Taking inverse Laplace transform of Eq. (28), and using the transformation (19), we get the desired analytical solution as:

$$c(x,t)=\left[\frac{1}{2}\left(2c_0-\frac{\gamma_0}{\mu_1}\right)\left\{\exp\left(\eta^2T-\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\eta\sqrt{T}\right)+\exp\left(\eta^2T+\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\eta\sqrt{T}\right)\right\}-\right.$$

$$\frac{m^2 c_0}{4\eta}\left\{\left(\eta T^2-\frac{XT\sqrt{R_0}}{\sqrt{D_0}}+\frac{X\sqrt{R_0}}{4\eta^2\sqrt{D_0}}+\frac{X^2R_0}{4D_0\eta}\right)\exp\left(\eta^2X-\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\eta\sqrt{T}\right)+\right.$$
$$\left(\eta T^2+\frac{XT\sqrt{R_0}}{\sqrt{D_0}}-\frac{X\sqrt{R_0}}{4\eta^2\sqrt{D_0}}+\frac{X^2R_0}{4D_0\eta}\right)\exp\left(\eta^2T+\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\eta\sqrt{T}\right)-\frac{X\sqrt{R_0T}}{\eta D_0\sqrt{\pi}}\times$$
$$\left.\exp\left(-\frac{X^2R_0}{4D_0T}\right)\right\}-\frac{1}{2}\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\left\{\exp\left(\rho^2T-\frac{\rho\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\rho\sqrt{T}\right)+\right.$$
$$\left.\exp\left(\rho^2T+\frac{\rho\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\rho\sqrt{T}\right)\right\}+\frac{\gamma_0}{u_0a}\left\{\exp\left(\omega^2T-\frac{\omega\sqrt{R_0}X}{\sqrt{D_0}}\right)\times\right.$$
$$\left.erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\omega\sqrt{T}\right)+\exp\left(\omega^2T+\frac{\omega\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\omega\sqrt{T}\right)\right\}+$$
$$\left.\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\exp\left\{\rho^2T-\beta X\right\}-\frac{\gamma_0}{u_0a}\exp\left\{\omega^2T-(a+\beta)X\right\}\right]\times$$
$$\exp\left\{\frac{w_0}{2D_0}X-\frac{1}{R_0}\left(\frac{w_0^{\;2}}{4D_0}+\mu_1\right)T\right\}+\frac{\gamma_0}{\mu_1} \tag{29}$$

Where $\rho=\sqrt{\frac{D_0}{R_0}\beta^2}$ and $\omega=\sqrt{\frac{D_0}{R_0}(a+\beta)^2}$

*2.2 Case 2. when* $f(mt)=\frac{mt}{\sqrt{(mt)^2+k^2}}$ *,(Algebraic sigmoid function)* as Singh *et al.*(2015),

So, with help of Eq.(7) , the new time variable $T$ for the present case may be written as:

$$T=\int_0^t\frac{mt}{\sqrt{(mt)^2+k^2}}dt$$

Or, $$T=\frac{1}{m}\left(\sqrt{(mt)^2+k^2}-k\right) \tag{30}$$

Initial and boundary conditions reduced to

$$c(x,T)=c_i+\frac{\gamma}{u}x\,;0\le x<\infty\ ,\ T=0 \tag{31}$$

$$c(x,T)=\frac{c_0}{2}\left\{4-m^2\left(1-\frac{k^2}{3}\right)T^2-2kmT\right\};x=0,T>0 \tag{32}$$

$$\frac{\partial c(x,T)}{\partial x}=0\ ;\text{ as } x\to\infty\ ,\ T\ge 0 \tag{33}$$

since $mT<<1$ so neglecting third and higher order terms of $mT$ in expansion of $\cos\left\{\sqrt{(mT)^2+2kmT}\right\}$.

Now, adopting similar process as in case 1 i.e., Eqs. (14-29), the solution of the present case may be obtained as:

$$c(x,t)=\Bigg[\frac{c_0}{2}\left(2-\frac{\gamma_0}{\mu_1}\right)\left\{\exp\left(\eta^2T-\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\eta\sqrt{T}\right)+\exp\left(\eta^2T+\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\eta\sqrt{T}\right)\right\}-$$
$$\frac{m^2c_0}{4\eta}\left(1-\frac{k^2}{3}\right)\left\{\left(\eta T^2-\frac{XT\sqrt{R_0}}{\sqrt{D_0}}+\frac{X\sqrt{R_0}}{4\eta^2\sqrt{D_0}}+\frac{X^2R_0}{4D_0\eta}\right)\exp\left(\eta^2T-\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)\times\right.$$
$$erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\eta\sqrt{T}\right)+\left(\eta T^2+\frac{XT\sqrt{R_0}}{\sqrt{D_0}}-\frac{X\sqrt{R_0}}{4\eta^2\sqrt{D_0}}+\frac{X^2R_0}{4D_0\eta}\right)\exp\left(\eta^2T+\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)\times$$
$$\left.erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\eta\sqrt{T}\right)-\frac{X\sqrt{R_0T}}{\eta D_0\sqrt{\pi}}\exp\left(-\frac{X^2R_0}{4D_0T}\right)\right\}-\frac{kmc_0}{4\eta}\left\{\left(2\eta T-\frac{X\sqrt{R_0}}{\sqrt{D_0}}\right)\times\right.$$
$$\exp\left(\eta^2T-\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\eta\sqrt{T}\right)+\left(2\eta T+\frac{X\sqrt{R_0}}{\sqrt{D_0}}\right)\exp\left(\eta^2T+\frac{\eta\sqrt{R_0}X}{\sqrt{D_0}}\right)\times$$
$$\left.erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\eta\sqrt{T}\right)\right\}-\frac{1}{2}\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\left\{\exp\left(\rho^2T-\frac{\rho\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\rho\sqrt{T}\right)+\right.$$
$$\left.\exp\left(\rho^2T+\frac{\rho\sqrt{R_0}X}{\sqrt{D_0}}\right)erfc\left(\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}+\rho\sqrt{T}\right)\right\}+\frac{\gamma_0}{u_0a}\left\{\exp\left\{\omega^2T-\frac{\omega\sqrt{R_0}X}{\sqrt{D_0}}\right\}\times\right.$$
$$\left.erfc\left\{\frac{X\sqrt{R_0}}{2\sqrt{D_0T}}-\omega\sqrt{T}\right\}+\exp\left\{\omega^2T+\frac{\omega\sqrt{R_0}X}{\sqrt{D_0}}\right\}erfc\left\{\frac{x\sqrt{R_0}}{2\sqrt{D_0T}}+\omega\sqrt{T}\right\}\right\}+$$
$$\left.\left(c_i+\frac{\gamma_0}{u_0a}-\frac{\gamma_0}{\mu_1}\right)\exp\left(\rho^2T-\beta X\right)-\frac{\gamma_0}{u_0a}\exp\{\omega^2T-(a+\beta)X\}\right]\times$$
$$\exp\left\{\frac{w_0}{2D_0}X-\frac{1}{R_0}\left(\frac{w_0{}^2}{4D_0}+\mu_1\right)T\right\}+\frac{\gamma_0}{\mu_1} \tag{34}$$

## 4. Result and Discussions

Parameters governing the solute transport through porous domain vary significantly upon the nature of the pollutant of any geological formation. Thus, to illustrate the significant factors accounting for this formulation, a hypothetical case of porous domain and parameters are taken. The numerical values of majority of the parameters considered either from the published literature or determined using existing empirical relationships. The analytical solutions obtained as in Equation (29) & (34) are demonstrated graphically. The concentration values are evaluated in a finite longitudinal space domain, $0 \le x(km) \le 4$. Two forms of $\mathrm{f}(\mathrm{mt})$ are discussed separately.

*4.1 Case1: when pore water velocity is of the form* $\mathrm{f}(\mathrm{mt})=|\cos(\mathrm{mt})|$

Figs.(1-4) are drawn for solution evaluated by Eq.(29) with common parameters $c_0 = 1.0$, $c_i = 0.01$, $u_0 = 0.01(km\ year^{-1})$, $R = 1.2$, $a = 0.085(km^{-1})$, $\mu_0 = 0.05(year^{-1})$ and $\gamma_0 = 0.0007$. The groundwater velocity ranges from $2\,m/day$ to $2\,m/year$ depending upon the pore geometry of porous domain Todd (1980).

Fig.1(a). shows dimensionless concentration profiles in the domain at various times $t(year) = 2,6,10$. It reveals that as the time increases the concentration values continuously increases inside the domain but the level of concentration recorded nearly at same level at the end of the domain. It also reveals that input concentration, $c/c_0$ at the origin, $x(km) = 0$ is nearly $2.0$ at each time and attenuates with position at each considered time. Figure 1(b) obtained from pdepe of Matlab shows same pattern as of Fig.1(a) that validates the accuracy of derived solution.

The Fig.2(a). Illustrates the effects of various dispersion coefficients on concentration profiles at time $t(year)=2$. The input concentration, $c/c_0$ at the origin i.e., at $x(km)=0$, is same at each dispersion coefficient and attenuates with position but concentration level is lower for lower dispersion coefficient while higher for higher. Fig.2(b) ascertains same the concentration pattern.

Fig.3. depicts the concentration profiles, $c/c_0$ verses time interval $0 \le t(year) \le 35$ for different values of $x(km)=0.50, 0.75, 1.00$. It is observed that the contaminant concentration at each mentioned position initially almost same but increases as the time increases and gets stable after some time.

Fig.4(a). depicts the surface distribution of concentration for various position and time with a set of input data. This figure describes the distribution pattern of solute concentration in the medium with a better visualization. The concentration pattern seems to have governed well primarily by boundary condition and then relatively other parameters. As far as transport processes in the subsurface are concerned, the water movement plays a major impact on the spreading of the solutes. Fig.4(b) surface plot obtained from pdepe shows good accuracy of concentration pattern.

*4.2 Case 2: when pore water velocity is of the form* $f(mt)=\dfrac{mt}{\sqrt{(mt)^2+k^2}}$

Fig.(5-8) are drawn for solution evaluated by Eq.(34) with common parameters $c_0=1.0$, $c_i=0.01$, $k=0.1$, $u_0=0.01\left(km\ year^{-1}\right)$, $R=1.2$, $\mu_0=0.05\left(year^{-1}\right)$, $m=0.001\left(year^{-1}\right)$, $a=0.085\left(km^{-1}\right)$, and $\gamma_0=0.0007$.

Fig.5(a). shows dimensionless concentration profiles computed for different times $t(year)=6,10,14$. It reveals that the concentration level at particular position is lower for lower time while higher for higher time. The input concentration, $c/c_0$ at the origin, $x(km)=0$, is nearly $2.0$ at each time and attenuates with position and time. It is also observed that concentration started decreasing with respect to space and increasing with time as in periodic form of pore-water velocity but the concentration values away from the source position for periodic form of velocity at each position are higher than those for the algebraic sigmoid form of velocity. It is observed that the contaminant concentration decreases near the source and emerges at a point near $x(km)=1.0$ and reaching towards the minimum harmless concentration. Figure 5(b) obtained from pdepe of Matlab follows the same pattern as of Figure 5(a) and solution is authenticated from Fig. 5(b).

Fig. 6. Illustrates dimensionless concentration distribution at various positions $x(km)=0.50, 0.75, 1.00$ in a time domain $0 \le t(year) \le 35$. It is observed that the concentration is higher for lower $x$ and lower for higher $x$.

Fig.7(a) demonstrate the concentration profiles for various dispersion coefficients $D_0\left(km^2 year^{-1}\right)=0.1, 0.2, 0.3$. It reveals that concentration value is lower for lower dispersion coefficient near the source boundary and higher for higher. The contaminant concentration values in algebraic sigmoid form of velocity are observed lower than the periodic form of velocity and attained harmless level near to source boundary. The Fig. (7b) shows same pattern as same in Fig. (7a).

Fig.8(a). explores the surface plot for distribution of concentration for various position and time with a set of input data. This figure describes the distribution pattern of solute concentration in the medium with a better visualization. Advection is considered to be the main process driving the movement of solutes from one position to another. Figure 8(b) obtained from pdepe of Matlab authenticates the concentration pattern from Eq. (34).

Fig.(9) demonstrates the comparison of concentration profiles at different positions between periodic function $f(mt)=|\cos(mt)|$ and algebraic sigmoid function $f(mt)=\dfrac{mt}{\sqrt{(mt)^2+k^2}}$ for dispersion $D_0=0.1\left(km^2 year^{-1}\right)$ and unsteady parameter $m\left(year^{-1}\right)=0.001$. It is recorded that at a fixed point in the domain contaminant concentration level remains higher for $f(mt)=|\cos(mt)|$ than that of $f(mt)=\dfrac{mt}{\sqrt{(mt)^2+k^2}}$.

Fig.(10) shows the comparison of concentration pattern in time domain $0 \le t(year) \le 35$ between $f(mt) = |\cos(mt)|$ and $f(mt) = \frac{mt}{\sqrt{(mt)^2 + k^2}}$ for dispersion $D_0 = 0.1(km^2 year^{-1})$ and unsteady parameter $m(year^{-1}) = 0.001$. It reveals that attenuation process is faster for $f(mt) = \frac{mt}{\sqrt{(mt)^2 + k^2}}$ than $f(mt) = |\cos(mt)|$

Table 1 demonstrates the effect of value $\zeta$ in first case $f(mt) = |\cos(mt)|$. As the value of $\zeta$ increases, the concentration at fixed position attenuates slightly fast. It may also be ascertained that effect of the value of $\zeta$ is low for the proposed case.

Table 2 studies changes in the concentration pattern for varying value of $\zeta$ in second case $f(mt) = \frac{mt}{\sqrt{(mt)^2 + k^2}}$. As the value of $\zeta$ increases, the concentration level at fixed position decreases up to distance $x(km) = 0.5$ and increases at the distances $x(km) = 1, 1.5, 2.0, 2.5, 3.0, 3, 3.5 \, and \, 4.0$.

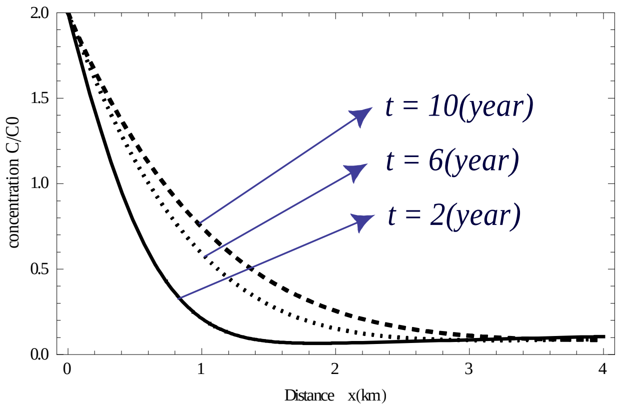


**Figure 1(a).** Concentration distributions versus distance at various time from solution Eq.(29) for fixed $m(year^{-1}) = 0.001$ and $D_0 = 0.1(km^2 year^{-1})$

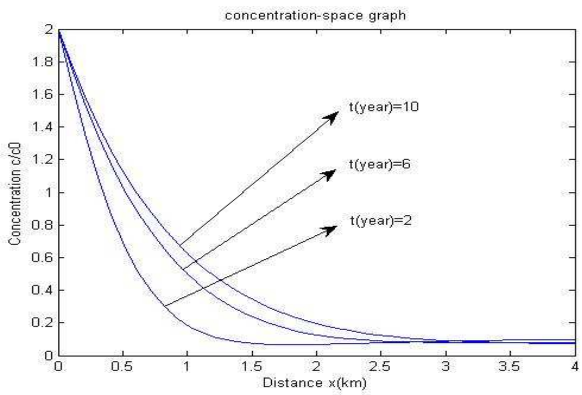


**Figure 1(b).** Concentration distributions versus distance at various time from pdepe of Matlab for fixed $m(year^{-1}) = 0.001$ and $D_0 = 0.1(km^2 year^{-1})$

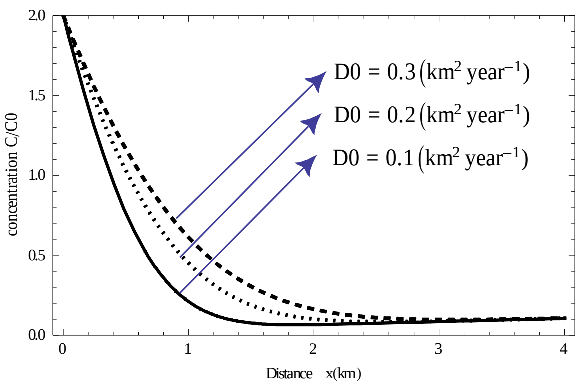


**Figure 2(a).** Concentration distributions for various dispersion coefficient $D_0$ from solution Eq.(29) and fixed $t(year) = 2$ and $m = 0.001(year^{-1})$.

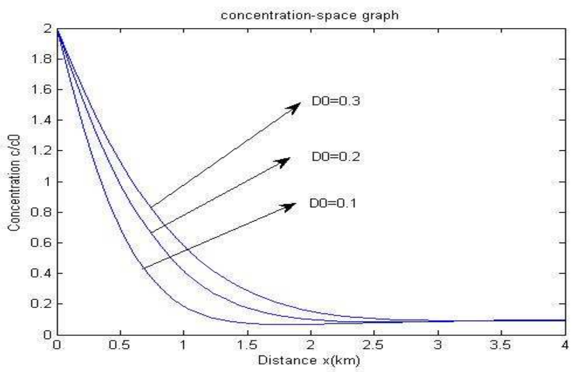


**Figure 2(b).** Concentration distributions for various dispersion coefficient $D_0$ from pdepe of Matlab and fixed $t(year) = 2$ and $m = 0.001(year^{-1})$

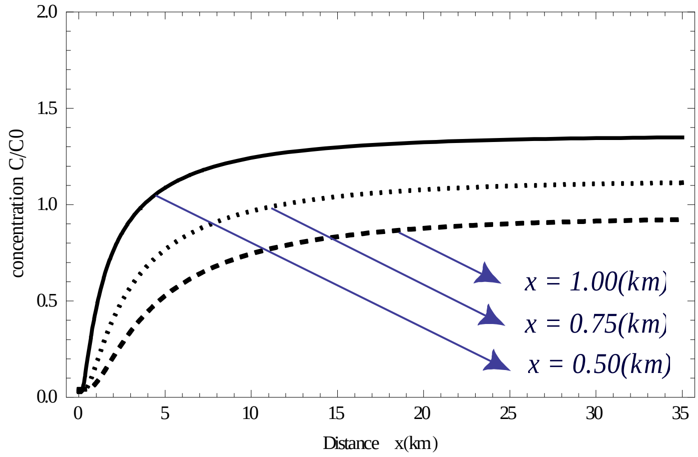


**Figure 3.** Concentration profiles at various positions in domain time $0 \le t(year) \le 35$ from Eq.(29) for fixed $D_0 = 0.1\left(km^2 year^{-1}\right)$ and $m = 0.001\left(year^{-1}\right)$ .

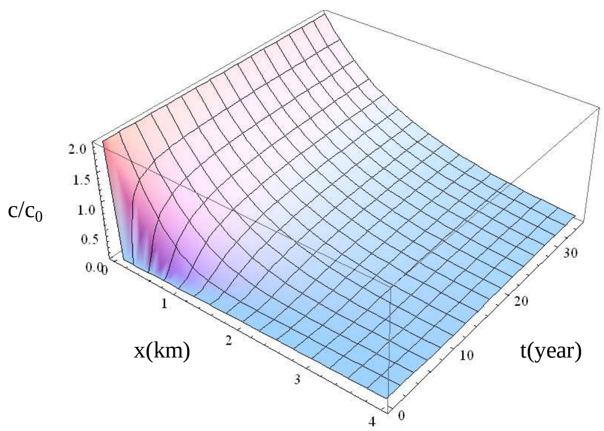


**Figure 4(a).** Surface plot for distribution of concentration from Eq.(29) for fixed $m\left(year^{-1}\right) = 0.001$ and $D_0 = 0.1\left(km^2 year^{-1}\right)$

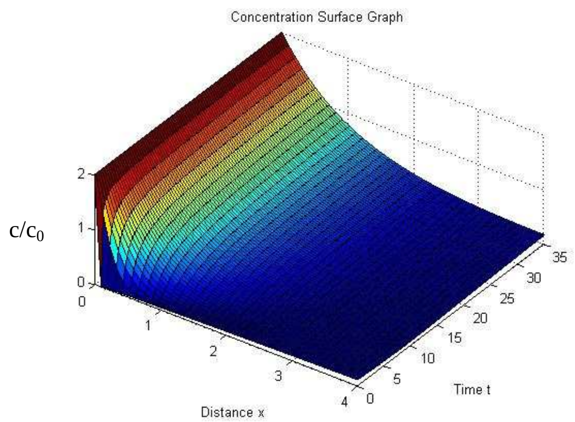


**Figure 4(b).** Surface plot for distribution of concentration from pdepe of Matlab for fixed $m\left(year^{-1}\right) = 0.001$ and $D_0 = 0.1\left(km^2 year^{-1}\right)$

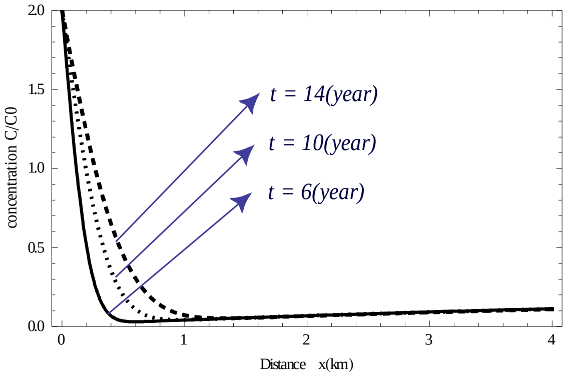


**Figure 5(a).** Concentration distributions versus distance from Eq.(34) for fixed $m(year^{-1}) = 0.001$ and $D_0 = 0.1(km^2 year^{-1})$.

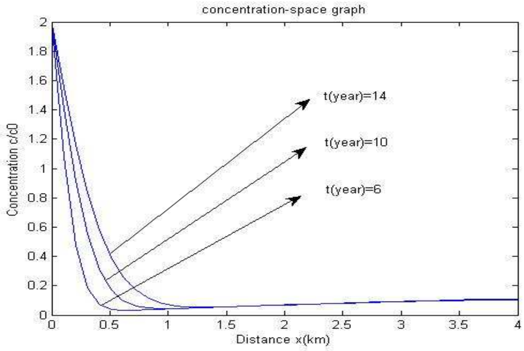


**Figure 5(b).** Concentration distributions versus distance from pdepe Matlab for fixed $m(year^{-1}) = 0.001$ and $D_0 = 0.1(km^2 year^{-1})$

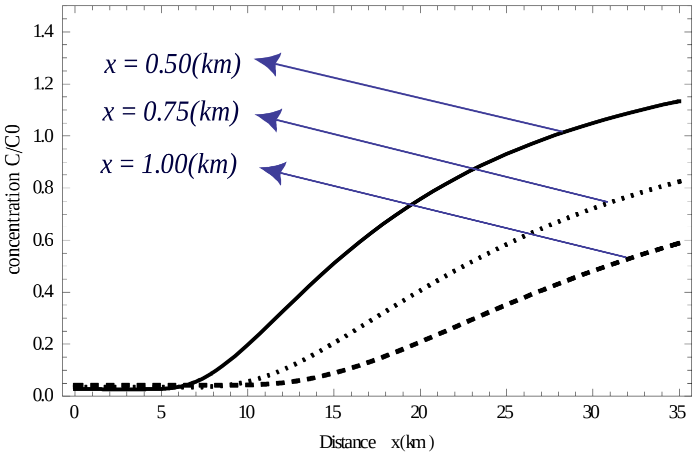


**Figure 6.** Dimensionless concentration distribution at various position in domain time $0 \le t(year) \le 35$ from Eq.(34) for fixed $D_0 = 0.1(km^2 year^{-1})$ and $m = 0.001(year^{-1})$.

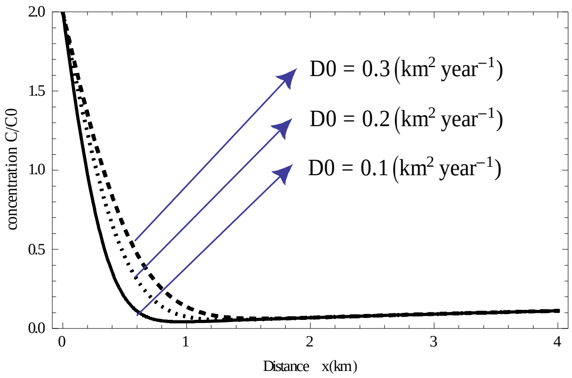


**Figure 7(a).** Concentration distributions for various dispersion coefficient $D_0$ from solution Eq.(34) and fixed $t(year) = 10$ and $m = 0.001(year^{-1})$.

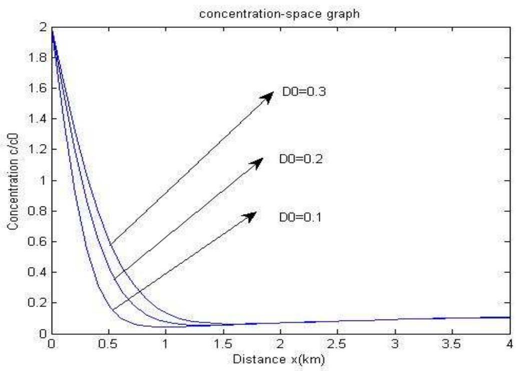


**Figure 7(b).** Concentration distributions for various dispersion coefficient $D_0$ from pdepe of Matlab and fixed $t(year) = 10$ and $m = 0.001(year^{-1})$

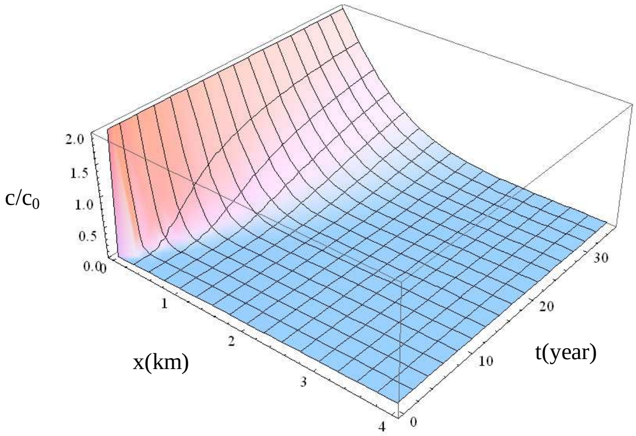


**Fig.8(a).** Surface plot for distribution of concentration from Eq.(34) for fixed $m\left(year^{-1}\right)=0.001$ and $D_0=0.1\left(km^2 year^{-1}\right)$

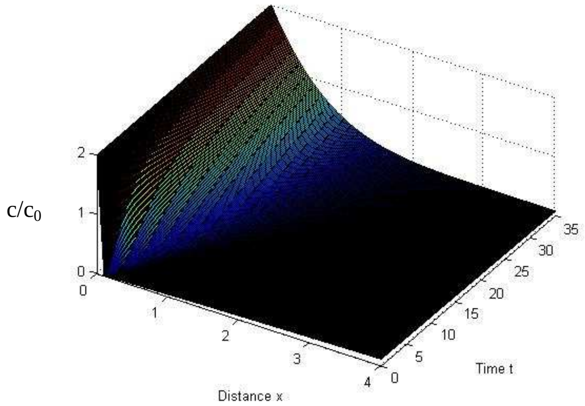


**Fig.8(b).** Surface plot for distribution of concentration from pdepe of Matlab for fixed $m\left(year^{-1}\right)=0.001$ and $D_0=0.1\left(km^2 year^{-1}\right)$

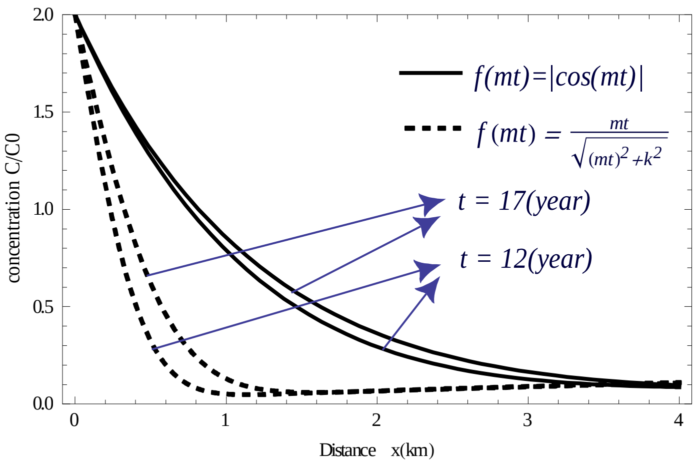


**Figure 9.** Comparison of dimensionless concentration pattern between periodic velocity and velocity including algebraic sigmoid function in a domain $0 \le x(km) \le 4$

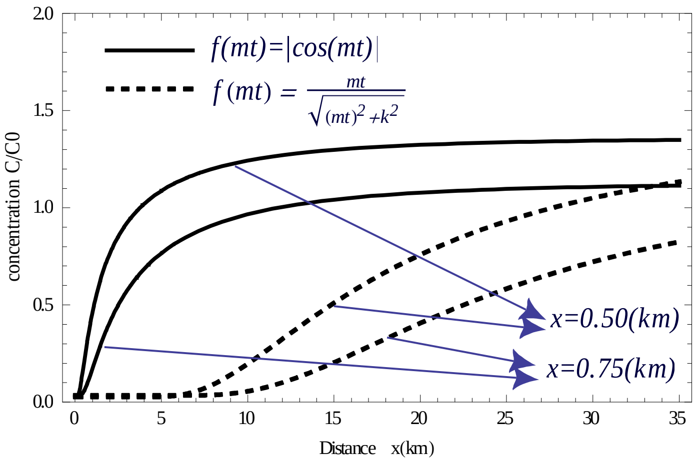


**Figure 10. C**omparison of dimensionless concentration distributions between periodic velocity and velocity including algebraic sigmoid function of time in a domain time $0 \le t(year) \le 35$ at various positions.

**Table 1.** Concentration $c/c_0$ for $f(mt) = |\cos(mt)|$

| $x(km)$ \ $\zeta$ | $\zeta = 2$ | $\zeta = 3.5$ | $\zeta = 5$ |
|---|---|---|---|
| 0 | 1.99995 | 1.99995 | 1.99995 |
| 0.5 | 1.28715 | 1.24239 | 1.19749 |
| 1.0 | 0.79647 | 0.74533 | 0.69594 |
| 1.5 | 0.47688 | 0.43540 | 0.39675 |
| 2.0 | 0.28308 | 0.25488 | 0.22949 |
| 2.5 | 0.17488 | 0.15797 | 0.14321 |
| 3.0 | 0.12008 | 0.11090 | 0.10313 |
| 3.5 | 0.09581 | 0.09122 | 0.08949 |
| 4.0 | 0.08759 | 0.08542 | 0.08366 |

**Table 2.** Concentration $c/c_0$ for $f(mt) = mt\Big/\sqrt{(mt)^2 + k^2}$

| $x(km)$ \ $\zeta$ | $\zeta = 2$ | $\zeta = 3.5$ | $\zeta = 5$ |
|---|---|---|---|
| 0 | 1.99995 | 1.99995 | 1.99995 |
| 0.5 | 0.20287 | 0.19745 | 0.19216 |
| 1.0 | 0.04319 | 0.04321 | 0.04323 |
| 1.5 | 0.05546 | 0.05556 | 0.05567 |
| 2.0 | 0.06844 | 0.06852 | 0.06860 |
| 2.5 | 0.08051 | 0.08057 | 0.08064 |
| 3.0 | 0.09176 | 0.09181 | 0.09186 |
| 3.5 | 0.10228 | 0.10231 | 0.10234 |
| 4.0 | 0.11213 | 0.11214 | 0.11216 |

## 5. Conclusion

This study derives an analytical solution for one-dimensional advective-dispersive transport in semi-infinite heterogeneous porous domain subjected to time-dependent inlet boundary condition assuming horizontal periodic flow direction, Dispersions is considered proportional to multiple of temporally seepage flow and $(\zeta + 1)$th power of the special variable. Two forms of temporally dependent velocities, such as periodic varying and algebraic sigmoid function of time, are considered. The effect of these two flow velocities, on solute transport behavior from a periodic point source injection in domain are explained graphically. The analytical solution is obtained using the Laplace Transformation Technique. Few new transformations are used to transform advection differential equation into ordinary differential equation.The developed analytical solution are compared with solution obtained through Matlab. The present result may helpful to understanding contaminant transport in one-dimensional porous domain with arbitrary time-dependent input function. The derived result may help the determining position and time to reach the minimum/maximum or harmless concentration. The proposed model has not been authenticated against any experimental data for the conditions considered in this study due to unavailable of suitable facilities.

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