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Explicit solutions of the Rand Equation

A. Huber [1,2]

[1] *Address constantly: Prottesweg 2a, A-8062 Kumberg, AUSTRIA*
[2] *Institute of Theoretical Physics – Computational Physics, Technical University Graz, Petersgasse 16, A-8010 Graz, AUSTRIA*
E-mail: soliton.alf@web.de

Abstract

In this paper the meaning of a nonlinear partial differential equation (nPDE) of the third-order is shown to the first time. The equation is known as the 'Rand Equation' and belongs to a class of less studied nPDEs. Both the explicit physical meaning as well as the behaviour is not known until now. Therefore we believe it is indispensable to study this evolution equation in detail. We perform a classical Lie Group analysis to analyze the point symmetries. By using a similarity reduction we are able to deduce more classes of solutions of general character. Special nonlinear transformations are given in a most general form. In addition, we also study Lie's non-classical case relating to potential and generalized symmetries. Both the potential and approximate symmetries are discussed to the first time leading to new results. So we expect a better understanding and concrete physical as well as technical application in future.

Keywords: Nonlinear partial differential equations, evolution equations, symmetries, similarity solutions, Rand Equation.

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AMS-Code: 35L05, 35Q53, 14H05.

1. Introduction - outline of the problem

The scaled nPDE in (1+1) dimension under consideration is given by:

$$3\left(\frac{\partial u}{\partial t}\right)^3 = u^2\,\frac{\partial^3 u}{\partial x^3}\,,\;\; u = u(x,t)\,,\; u \in R^3(-\infty,\infty)\,,\; t > 0\,, \tag{1}$$

where the function $u(x,t)$ is related to a variation of a physical quantity depending upon the positive time t. We seek for classes of solutions for which $u = F(x,t)$, where $F \in R^3$ and $D \subset R^2$ is an open set and further we exclude $D := \left\{(u,(x,t)) \in \widetilde{D} : u(x,t) \neq 0\right\}$. Suitable classes of solutions are $u \in I$ an interval so that $I \subseteq D$ and $u : I \rightarrow R^2$. Note:

We suppress the item 'classes', so that 'classes of solutions' are simply 'solutions'. Firstly, taking a look at eq.(1) concluding the following: Unlike classical evolution equations (e.g. the Korteweg de Vries equation and many others (Whitham, 1974; Drazin and Johnson, 1989; Eilenberger, 1983; Ablowitz and Clakson, 1991; Dodd *et al*., 1988; Huber, 2010)), where the nonlinear part is counterbalanced by a linear part and therefore responsible for stable waves. Here, on the contrary, we have no such balance. That means we cannot expect a priori classical wave motion where the steepening effect is counterbalanced by some linear parts since both the l.h.s. and the r.h.s. of eq.(1) are nonlinear.

In other words, e.g. the effect of beach wave breaking cannot occur. This leads to the assumption that other types of waves might appear (if we assume that the eq.(1) at least admits wave solutions).

This is the main task of the given paper whereby we are interested in questions about the meaning, validity and existence of solutions. Another question of importance is how we can associate a concrete physical and/or technical application with the eq.(1).

1.2. Classical symmetry analysis - algebraic group properties

We take up now the developments given in (Ibragimov, 1994a; Olver, 1986; Bluman and Kumei, 1989, Gaeta, 1994) omitting all technical details.

To use symmetry groups in any application we first deduce the symmetries of eq.(1).

The result is a well-defined system of eight linear homogeneous PDEs (describing the point symmetries) for the infinitesimals $\xi_i = \xi_i(x,u)$ and $\phi_i = \phi_i(x,u)$. These constitute the so-called determining equations for the symmetries of eq.(1) generated by Fréchet's derivative (Ibragimov, 1994a; Huber, 2007a,b; Huber, 2008a,b; Huber, 2009):

$$\frac{\partial \xi_1}{\partial u} = \frac{\partial \xi_2}{\partial u} = \frac{\partial \xi_2}{\partial x} = \frac{\partial^2 \phi}{\partial u^2} = 0\,, \tag{1.1}$$

$$9\frac{\partial \xi_1}{\partial t} - \frac{\partial^3 \xi_1}{\partial x^3} + 3\frac{\partial^3 \phi}{\partial x^2 \partial u} = 0\,, \tag{1.2}$$

$$9\frac{\partial \phi}{\partial t} - \frac{\partial^3 \phi}{\partial x^3} = 0\,, \tag{1.3}$$

$$\frac{\partial^2 \phi}{\partial x \partial u} - \frac{\partial^2 \xi_1}{\partial x^2} = 0\,, \tag{1.4}$$

$$\frac{\partial \xi_2}{\partial t} - 3\frac{\partial \xi_1}{\partial x} = 0\,. \tag{1.5}$$

Solving the above given set of equations (1.1) to (1.5) we derive at the infinitesimals:

$$\begin{aligned} \xi_1 &= k_3 + k_4 x \\ \xi_2 &= k_2 + 3k_4 t \\ \phi &= k_1 u(x,t) + F(x,t)\,. \end{aligned} \tag{1.6}$$

The result shows that the symmetry group of eq.(1) constitutes an infinite four-dimensional point group where the group parameters are denoted by k_i, $\mathrm{i} = 1,2,3$. The infinite part of the group is generated by the function $F(x,t)$ whereby the latter function has to satisfy the linear third-order equation: $9F_t - F_{xxx} = 0$. The arbitrary function $F(x,t)$ does not satisfy any further equation(s).

So, in what follows we have the freedom to set the function $F(x,t)$ equal to zero (or individually otherwise). Eq.(1) admits the four-dimensional Lie algebra L of its classical infinitesimal point symmetries related to the following vector fields:

$$V_1 = \partial_x \ \ , \ V_2 = \partial_t \ \ , \ V_3 = 3t\ \partial_t + x\partial_x \ \ , \ V_4 = u\,\partial_u \ . \tag{1.7}$$

This group of four vector fields contains translations in time and space so that $\{t' \to t + \lambda,\ x' \to x + \lambda\}$ holds for $\{\mathrm{V}_1, \mathrm{V}_2\}$ and the associated differential operators V_3 and V_4 are related to dilatation operations. The symmetry vector fields form a Lie algebra L by:

$$[V_2, V_4] = 3V_2 \ \ , \ [V_3, V_4] = V_3 \ \ , \ [V_4, V_2] = -3V_2 \ \ , \ [V_4, V_3] = -V_3 \ . \tag{1.8}$$

For this four-dimensional Lie algebra the commutator table for the V_i is a (4 x 4)- table whose $(\mathrm{i},\mathrm{j})\mathrm{th}$ entry expresses the Lie Bracket $\lfloor V_i, V_j \rfloor$ given in (1.8). The table is skew-symmetric and the diagonal elements vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the $(\mathrm{i},\mathrm{j})\mathrm{th}$ entry of Table 1 and the related structure constants can be read from Table 1:

$$C_{2,4,2} = -3,\ C_{3,4,3} = -1,\ C_{4,2,2} = 3,\ C_{4,3,3} = 1\ . \tag{1.9}$$

Table 1 The commutator table of the Rand Equation

	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	0	$-3V_2$
V_3	0	0	0	$-V_3$
V_4	0	$3V_2$	V_3	0

Theorem: The Lie algebra of eq.(1) is solvable.

Proof: A Lie algebra L is called solvable if $V^{(n)} = 0$ for some n > 0. It can be shown that L is reducible to $V^{(4)} = 0$ starting by the ideal $\left\{V^{(1)},....,V^{(4)}\right\}$ since the algebra is four-dimensional.

Other useful algebraic group properties are mentioned: Eq.(1) has the Casimir operator by V_1, the group order is four containing 15 subgroups. These subgroups are important below to perform a similarity reduction deducing suitable solutions. The metric ($4 \otimes 4$ Cartanian tensor) satisfies:

$$g_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 12 \end{pmatrix}, \quad \text{with} \quad \det(g) = 0\,, \tag{1.10}$$

and, since the condition $\det(g) = 0$ holds, the given algebra is therefore degenerate and commutative. Note: Alternatively one can write with eq.(1.9) $g_{im} = \sum_{i,k=1}^{n} c_{lk}^{i} c_{mi}^{k}$.

2. Similarity solutions

Let us now discuss the most important three similarity solutions for special subgroups. If we set the group parameters $k_2 = k_3 = 1$ and $k_1 = k_4 = 0$, the following linear ODE of the third-order results:

$$\frac{d^3S}{d\zeta^3} + 9\frac{dS}{d\zeta} = 0\,,\ S: R \times R \to R\,,\ \zeta \in R,\ -\infty \le \zeta \le \infty\,,\ D := \left\{(S,\zeta) \in \widetilde{D} : S(\zeta) \neq 0\right\}. \tag{2}$$

The similarity variable ζ together with the relevant transformation (Case H) reads as $t - x = \zeta$, $S = u$ which is closely related to the case of traveling waves. Following Peanos' theorem we expect that at least solutions exist (locally in this sense) and secondly solutions are unique on the entire real axis. We calculate a superposition of harmonic wave trains by

$$S(\zeta) = C_3 + \frac{1}{3}\left\{C_1 \cos[3\zeta] + C_2 \sin[3\zeta]\right\}, \tag{2.1}$$

where the C_i, $i = 1,2,3$ are arbitrary constants of integration. A compact written form gives

$$S(\zeta) = \frac{4}{3} + \sum_{k=0}^{\infty} \frac{\zeta^{1+k}\left(2\,(-9)^k\,(1+k) - (-1)^k\,3^{1+2k}\,\zeta\right)}{\Gamma(3+2k)}, \tag{2.2}$$

where $\Gamma(.)$ means the gamma function.

At this stage a question is of interest: What physical meaning can we associate with this solution, eq.(2.1) or otherwise, represent this solution a solitary wave? If so, the following condition must hold: $S \to 0$ as $|\zeta| \to \infty$. It is shown that $S \to \infty$ as $|\zeta| \to \infty$, that means no solitary motion is possible. So we have periodic wave trains on the entire real axis which is seen in Figure 1.

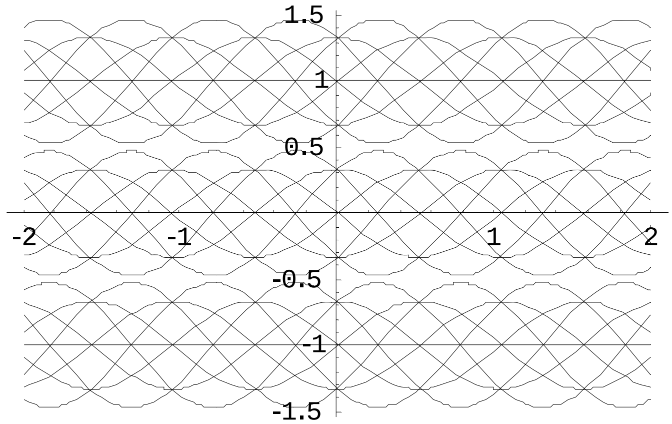

Figure 1 A planar sketch of the periodic wave train solution eq.(2.1) by using different values of the integration constants C_i, $i = 1,2,3$. We chose $-1 < C_i < 1$ for the domain of the constants and the periodic wave trains are stable.

The further behaviour strongly depends on the choice of the constant C_3. If we set $C_3 = 0$ the function vanishes $\forall \zeta \in R \setminus \{-\pi/12, \pi/4\}$, otherwise, if $C_3 \in N^{\pm}$, especially $C_3 = 1$ the function vanishes $\forall \zeta \in \Re \setminus \left\{-\frac{1}{3} arccos\left[\frac{1}{2}\left(-3 \pm \sqrt{7}\right)\right]\right\}$, that is numerically $\approx -(0{,}8 + 0{,}5\,i)$.

If we consider another choice for the group parameters, i.e. $k_3 = k_4 = 1$ and $k_1 = k_2 = 0$ (corresponding to Case C) the transformation is $\frac{t}{(1+x)^3} = \zeta$, $u = S$ and the following linear homogeneous ODE of the third-order with non-constant coefficients result:

$$9\zeta^3 \frac{d^3S}{d\zeta^3} + 36\zeta^2 \frac{d^2S}{d\zeta^2} + (20\zeta + 3)\frac{dS}{d\zeta} = 0\,,\ S: R \times R \to R,\ \zeta \in R,\ -\infty \le \zeta \le \infty. \tag{2.3}$$

This is solved explicitly by introducing the transformation $S' = p(\zeta)$ to get a second-order equation

$$9\zeta^3 p'' + 36\,\zeta^2 p' + (20\zeta + 3)p = 0\,,\ D := \left\{(p,\zeta) \in \widetilde{D} : p(\zeta) \ne 0\right\} \tag{2.4}$$

where the prime means the derivation $d/d\zeta$. This is solved by Bessel functions of broken order:

$$p(\zeta) = \frac{C_1}{\sqrt{\zeta^3}} J_{-1/3}\left[\frac{2}{\sqrt{3\zeta}}\right] + \frac{C_2}{\sqrt{\zeta^3}} J_{1/3}\left[\frac{2}{\sqrt{3\zeta}}\right],\ \zeta \ne 0\,. \tag{2.5}$$

Finally we get the solution function after integrating once to

$$S(\zeta) = C_3 - \frac{2C_1\sqrt[6]{3}}{\sqrt[3]{\zeta}\,\Gamma(5/3)}\, {}_p^qF\left[\left(\frac{1}{3}\right);\left(\frac{2}{3},\frac{4}{3}\right);-\frac{1}{3\zeta}\right] - \frac{9C_2\sqrt[9]{3}\,\Gamma(2/3)}{4\pi\sqrt[3]{\zeta^2}}\, {}_p^qF\left[\left(\frac{2}{3}\right);\left(\frac{4}{3},\frac{5}{3}\right);-\frac{1}{3\zeta}\right]. \tag{2.6}$$

Here $\Gamma(.)$ means the gamma function and ${}_p^qF(a;b_1;b_2;z)$ is the generalized hypergeometric function. In our case we explicitly have the function ${}_1^2F\left[\{a_1....,a_p\};\{b_1.....,b_q\};\zeta\right]$.

For further considerations we are interested in the asymptotic case $\zeta \to \infty$. By using the asymptotic behaviour of the hypergeometric function we have

$${}_1^2F\left[\{a_1....,a_p\};\{b_1.....,b_q\};\zeta\right] \sim \frac{\Gamma(b_1)\Gamma(b_2)}{2\pi\Gamma(a_1)}\zeta^{\frac{1}{2}\left(a_1 - b_1 - b_2 + \frac{1}{2}\right)} e^{2\sqrt{\zeta}} \times \left\{1 + O\left[\frac{1}{\sqrt{\zeta}}\right]\right\}. \tag{2.7}$$

The calculation for the real part leads to the asymptotic representation which is shown in Figure 2:

$$S(\zeta) \sim \frac{\Gamma(2/3)\Gamma(4/3)}{2\pi\,\Gamma(1/3)}\zeta^{-7/6}\cos\left[\frac{1}{\sqrt{3}\,\zeta}\right], \text{ as } \zeta \to \infty. \tag{2.8}$$

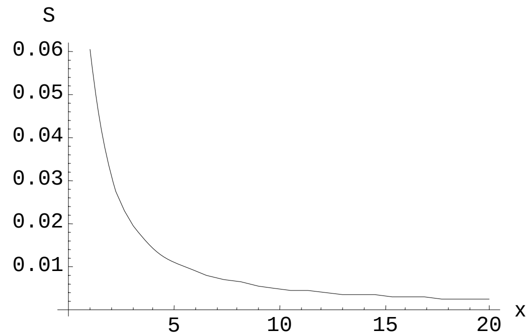

Figure 2 A planar sketch of the asymptotic solution function eq.(2.8) with $C_1 = C_2 = C_3 = 1$. The asymptotic behaviour is clearly seen. The point $\zeta = 0$ is the irregularity of the gamma function and the function decreases rapidly for $\zeta \to \infty$. Here, for relevant applications only the real part is considered.

Note: The solution function eq.(2.5) can be expressed in terms of Airy functions. Imagine the fact that Bessel functions of order 1/3 are expressible so that we have alternatively by putting together

$$p(\zeta) = \frac{3^{5/6}}{\zeta^{4/3}} Ai\left[-\sqrt[3]{\frac{3}{\zeta}}\right], \ \zeta > 0\ . \tag{2.9}$$

Both expressions are similar and do not describe any periodic or traveling wave motions. Finally, the Case N is of interest if we use the following choice for the group parameters:

$k_1 = k_2 = k_3 = 1$ and $k_4 = 0$. The transformation reads as $t - x = \zeta$ and $e^{-x}u = S$. This choice represents the case of traveling waves and we have to solve the following nODE of the third-order:

$$\frac{d^3S}{d\zeta^3} - 3S^2\frac{d^2S}{d\zeta^2} + 12S^2\frac{dS}{d\zeta} - S^3 = 0\,, S: R \times R \to R\,,\ \zeta \in R,\ -\infty \le \zeta \le \infty\ . \tag{2.10}$$

This equation cannot be solved explicitly so we decided to perform a power series representation up to order five:

$$S(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \frac{1}{6}\left(a_0^3 - 12\,a_0^2\,a_1 + 3\right)\zeta^3 + \frac{1}{8}\left(a_0\left(a_0\,a_1 - 8a_1^2 - 8\,a_0\,a_2\right)\right)\zeta^4 + O[\zeta]^5, \tag{2.11}$$

with arbitrary chosen coefficient a_i, $i = 0,1,2$ whereby this polynomial solution is continuously differentiable on the entire real axis. A graphical overview is given in Figure 3.

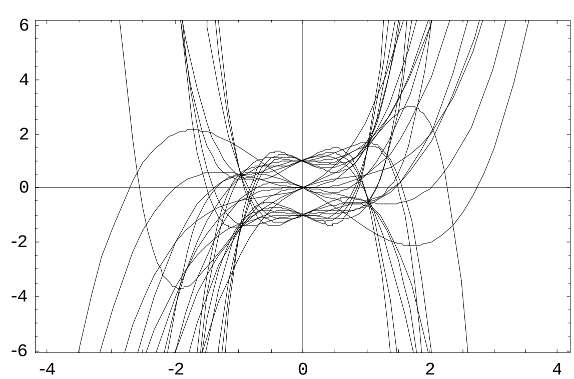

Figure 3 The behaviour of the solution function eq.(2.11) for the similarity function S. This polynomial solution was generated by the choice of the coefficients a_i for the values $-1 < a_1 < 1$ and the same for a_2 and a_3.

If we transform by $t-x=\zeta$ and $e^{-x}u=S$ we show a sequence of solution surfaces depending on the independent variable x and t considering special values for the parameter λ in Figure 4.
This shows that a traveling wave solution does not appear.
If we introduce the notation $P(\zeta)$ for the polynomial part of eq.(2.11), one can write the complete solution $u(x,t)=e^{x}P(\zeta)$ in short and comparing with the animations given, we conclude:

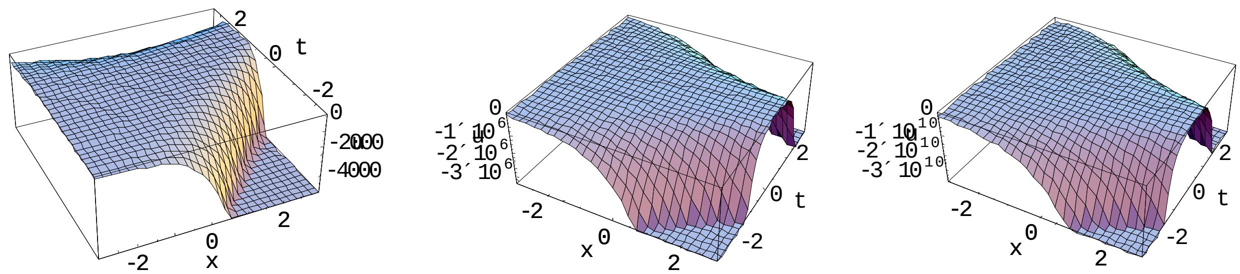

Figure 4 An animation of different solution surfaces of eq.(2.11), left: $\lambda=2$, middle: $\lambda=10$, right: $\lambda=-10$, all $a_i=1$.

The exponential part influences the solution in the sense of a damping effect whereby the exponential part either does not decrease or increase. So, this part covers a domain of saturation. After we have discussed all similarity cases of relevance we finish this chapter to proceed further with the analysis.
For completeness, in Table 2 we show all relevant nonlinear transformations by considering special values of the group parameters.

Table 2 Symmetry calculation and nonlinear transformations for the Rand Equation

Case	Choice of the group parameters	Transformation for ζ	Transformation for S
A	$k_1=k_2=k_3=0$, $k_4=1$	$tx^{-1}=\zeta$	$u=S$
B	$k_1=k_2=k_4=0$, $k_3=1$	$t=\zeta$	$u=S$
C	$k_1=k_2=0$, $k_3=k_4=1$	$t(1+x)^{-3}=\zeta$	$u=S$
D	$k_1=k_3=k_4=0$, $k_2=1$	$x=\zeta$	$u=S$
E	$k_1=k_3=0$, $k_2=k_4=1$	$(1+3t)3x^{-3}=\zeta$	$u=S$
F	$k_1=0$, $k_2=k_3=k_4=1$	$(1+3t)3(1+x)^{-3}=\zeta$	$u=S$
G	$k_2=k_3=k_4=0$, $k_1=1$	non-solvable	non-solvable
H	$k_1=k_4=0$, $k_2=k_3=1$	$t-x=\zeta$	$u=S$
I	$k_2=k_3=0$, $k_1=k_4=1$	$tx^{-3}=\zeta$	$ux^{-1}=S$
J	$k_2=k_4=0$, $k_1=k_3=1$	$t=\zeta$	$ue^{-x}=S$
K	$k_2=0$, $k_1=k_3=k_4=1$	$t(1+x)^{-3}=\zeta$	$u(1+x)^{-1}=S$
L	$k_3=k_4=0$, $k_1=k_2=1$	$x=\zeta$	$u=S$
M	$k_3=0$, $k_1=k_2=k_4=1$	$(1+3t)3x^{-3}=\zeta$	$ux^{-1}=S$
N	$k_4=0$, $k_1=k_2=k_3=1$	$t-x=\zeta$	$ue^{-x}=S$
O	$k_1=k_2=k_3=k_4=1$	$(1+3t)3(1+x)^{-3}=\zeta$	$u(1+x)^{-1}=S$

3. Analysis by the dominant balance method

Again, consider the Rand Equation, eq.(1). If we introduce the similarity 'ansatz' $u(x,t)=f(\xi)$, $\xi=x-\lambda t$, we derive the following nODE of the third-order:

$$f^2 \frac{d^3 f}{d\xi^3} - 3\left(\frac{df}{d\xi}\right)^3 = 0,\ f = f(\xi),\ D := \left\{(f,\zeta) \in \widetilde{D} : f(\zeta) \neq 0,\ \{f', f'', f'''\} \neq 0\right\}. \quad (3)$$

Let the domain $\widetilde{D} = D \times R^1 \subseteq R^3 \times R^1$. We seek proper solutions on the interval I, $I \in D$ and $f : I \to R^2$. Unfortunately, this nODE cannot be solved analytically in a closed form. Therefore, we apply the Dominant Balance Method in order to generate new solutions. Generally, from eq.(3) follows that $f^2 f''' = 3 f'^3$ and $f^2 f''' = 0 \to f^2 = 0 \vee f''' = 0$. This is not possible since we require both the existence of the function and their derivation. By balancing we have to treat the following cases considering a two-term balance:

Case(i): $f^2 f_{xxx} \approx 0$ requiring that $f'' >> \left| 3(f')^3 / f^2 \right|$ otherwise the condition holds: $f^2 f_{xxx} > 0 \to 3(f')^3 < 0$ from eq.(3) for a suitable balance.

Case(ii): $-3(f')^3 \approx 0$ requiring that $\left|(f')^3\right| = 0$ with the condition $(f')^3 > 0 \to f^2 f_{xxx} < 0$.

The ODE for Case(i) is solved explicitly by $f(\xi) \sim c_1 + c_2 \xi + c_3 \xi^3$.

For proper of solutions (e.g. if $\zeta \in R^{\pm}$) it is seen that this solution contradicts the given inequalities. So we conclude that the polynomial of the third-order represents a consistent balance solution and therefore we have

$$u(x,t) \sim c_1 + c_2 (x - \lambda t) + c_3 (x - \lambda t)^3,\ \lambda \neq 0, \quad (3.1)$$

as proper solutions as before with suitable chosen coefficients c_i, but $c_2 \neq 0$, $c_3 \neq 0$.

For practical calculations we perform a series representation of the nODE, eq.(3) with arbitrary constants a_i, $i = 0,1,2$ up to order four:

$$f(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + \frac{a_1^3}{2 a_0^2} \xi^3 + \frac{a_1^2 \left(3 a_0 a_2 - a_1^2\right)}{4 a_0^3} \xi^4 + O[\xi]^5. \quad (3.2)$$

For this series solution we give a graphical overview in Figure 5 by using suitable chosen values for the parameters a_i.

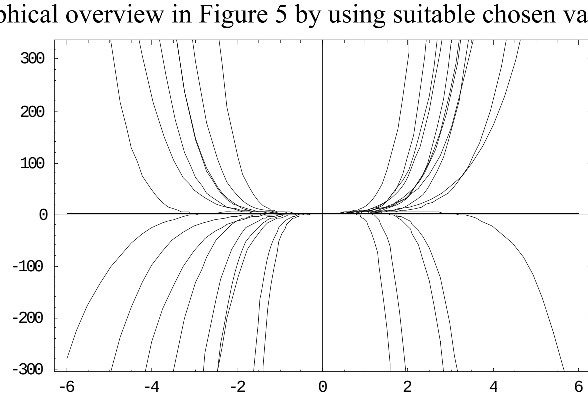

Figure 5 A planar plot of the series solution, eq.(3.2) generated with the choice of the parameters for all a_i so that the domain of the parameters is given by $0 < a_i < 2$, the curves are all symmetrically to the x-axis.

4. The non-classical case I: Potential symmetries

For more technical details we refer to (Olver, 1986; Bluman and Kumei, 1989, Gaeta, 1994, Huber, 2009) respectively. For the Rand Equation, eq.(1) we found the following: The equation admits only one possible potential system, Ψ_1 consisting of two relations. The systems can be formulated for the dependent variable V_1 and can be treated in their derivations w.r.t. the independent variables (x,t) denoted by subscripts:

$$u u_{xx} + \frac{\partial V_1}{\partial t} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0,\quad \frac{\partial V_1}{\partial x} + \frac{9u^2}{2} = 0. \quad (4)$$

Unlike other evolution equations having two or more potential systems, here we are confronted with an unexpected case: Calculating the infinitesimals we see that no new potential symmetry occurs:

$$\xi_1 = k_3 + \frac{k_5 x}{3} \qquad \xi_2 = k_4 + k_5 t$$
$$\varphi_1 = \frac{1}{6}(3k_2 - k_5)u \qquad \varphi_2 = k_1 + k_2 V_1 \tag{4.1}$$

It is of interest to compare with the classical case, eq.(1.6). The dimension of the group increases (we have a five-dimensional group) and also an infinite group generated by the function $u(x,t)$ is seen. Here no new potential symmetry could observe.

5. The non-classical case II: Generalized symmetries

We find it advisable to mention some basic notes. It is obvious from Lie theory that point symmetries are a subset of generalized symmetries (Ibragimov, 1985; 1994c). The determination of the characteristics for the general case follows by a similar algorithm as in the case of point transformations (PT) in the classical case.
Classical symmetries of a (n)PDE (assumed to be in a general form $\Delta = 0$) are PT which guarantee the invariance of the solution space and so, PT are created by infinitesimal transformations.
The determining equations for the characteristics GS_α are consequences of the relation

$$pr\,\vec{\upsilon}_{GS}\,\Delta\big|_{\Delta=0} = 0\,, \tag{5}$$

where $pr\,\vec{\upsilon}_{GS}$ denotes the prolongation of the vector field υ_{GS} and 'GS' means generalize symmetry. The main difference however is the fact that in general the characteristics depend on derivatives of an infinite order. If the order is equal to identity we arrive at the so-called contact transformations. By increasing the order of derivatives $n > 1$ we shall find higher order GS.
In case of the Rand Equation, eq.(1) we found GS depending on the first derivative:

$$GS_1(x,t,u,u_x,u_t) = k_2 u + k_1 u_x\,. \tag{5.1}$$

This symmetry also changes from the symmetries given in (1.6), (4.1). Here we are confronted with a two-dimensional finite group of transformations where the second part $\partial u/\partial x$ is related to dilatation operations. For the case $n = 2$ by assuming second partial derivatives we further found

$$GS_2(x,t,u,u_x,u_t) = k_2 u + k_1 u_x\,. \tag{5.2}$$

as a quite similar result.

6. Approximate symmetries

In this section we follow (Ibragimov, 1985, 1994c; Huber, 2009) respectively and our intension is to present new results without referring too much theory. However, some remarks will be indicated.
Definition**:** Approximate symmetries:
We assume that $x = (x_1 x_2,..)$ are independent coordinates of functions which are analytic in their arguments. Let us further assume that ε is a small parameter on which the functions additionally
depend. We will denote the involved infinitesimal small functions of order ε^{p+1} by $\theta_p(x,\varepsilon)$, where
$p \le 0$ and p is a positive constant. This condition is expressed by $\theta_p(x,\varepsilon) = O[\varepsilon]^p$. In addition an equivalent representation of this condition can be written by

$$\lim_{\varepsilon \to 0} \frac{\Theta_p(x,\varepsilon)}{\varepsilon^p}\,. \tag{6}$$

Let f and g be analytic functions in x. We define an approximation of order p, $f \approx g$ by the relation

$$f(z,\varepsilon) = g(z,\varepsilon) + O[\varepsilon]^p \tag{6.1}$$

for some fixed value of $p \le 0$.
This definition is the basis of all calculations we will carry out in the following.
Let us introduce ε as a small parameter measuring the influence of the nonlinear term of the eq.(1) so that we can write $3\varepsilon\, u_t^2 = u^2 u_{xxx}$. First order approximate symmetries follow by

$$\xi_1 = k_2(1+\varepsilon)$$
$$\xi_2 = k_3(1+\varepsilon) \qquad (6.2)$$
$$\phi = k_4 + k_1\varepsilon \, .$$

Here we have another unexpected situation comparing with the symmetries as above. The order remains equal (four-dimensional) toward the classical case but the dimension is finite. The generating vector fields containing the perturbation parameter reads as

$$V_1 = \varepsilon\partial_t \, , V_2 = (1+\varepsilon)\partial_u \, , V_3 = (1+\varepsilon)\partial_x \, , V_4 = \partial_t \, , \qquad (6.3)$$

and the associated coefficients of these vector fields are given by

$$\{(0,\varepsilon),(0)\}, \quad \{(0,0),(1+\varepsilon)\} \quad \{(1+\varepsilon),(0)\} \quad \{(0,1),(0)\}. \qquad (6.4)$$

In total we have four possible combinations for the vector fields. Possible reductions can be calculated by combining several sub-groups, that is $V_l \otimes V_m \otimes V_n$ with $\{l,m,n\} = 1,2,3,4$.

We now restrict the analysis to the most important case, the case of traveling waves.

This case arises by calculating the combination $V_l \otimes V_3 \otimes V_4$ which gives the traveling-wave transformation for the similarity variable $\zeta = t - x$ and $u = S$ for the similarity function once again.

The relating linear ODE of the third-order is similar as in the classical case, eq.(2), however the difference is the occurrence of a linear part:

$$\varepsilon\frac{d^3S}{d\zeta^3} + 9S = 0, \; S: R\times R \to R, \; \zeta \in R, \; -\infty \le \zeta \le \infty, \; \varepsilon \ne 0, \qquad (6.5)$$

which is solved explicitly by

$$S(\zeta) = C_1\exp\left[-(-3)^{2/3}\zeta\right] + C_2\exp\left[-(3)^{2/3}\zeta\right] + C_3\exp\left[(-1)^{1/3}3^{2/3}\zeta\right], \qquad (6.6)$$

where C_1, C_2 and C_3 are arbitrary constants of integration.

Considering special values of the constants, say, $C_1 = C_2 = C_3 = 1$ the real part of the solution, eq.(6.6) is written as

$$S(\zeta) = \exp\left[-(3)^{2/3}\zeta\right] + 2\exp\left[\tfrac{1}{2}(3)^{2/3}\zeta\right] + \cos\left[\tfrac{3}{2}3^{1/6}\zeta\right]. \qquad (6.7)$$

The function has a finite value at $\zeta = 0$ and it is further proven that $\lim_{\zeta\to 0} S(\zeta) = 3$ holds.

The limiting behaviour is $\lim_{\zeta\to+\infty} S(\zeta) = \infty$ but vanishes for $\zeta \to -\infty$, that is $\lim_{\zeta\to+\infty} S(\zeta) = 0$. Since the second derivative vanishes as $\zeta = 0$, takes positive real-valued as $\zeta = -1$ and negative real-valued as $\zeta = 1$, one can conclude that the solution is not stable at least in the domain $-1 < \xi < 1$.

In addition a compact written form of eq.(6.7) follows by the representation

$$S(\zeta) = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{1+2k}\zeta^{3k}}{(3k)!}. \qquad (6.8)$$

In Figure 6 we give a graphical overview of the behaviour of the real-valued function, eq.(6.7) by considering special values of the integration constants.

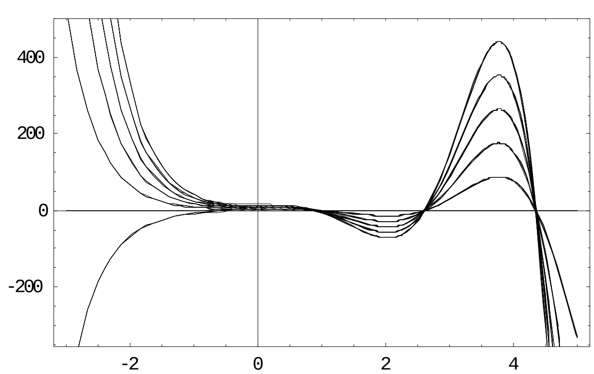

Figure 6 A planar plot of the real-valued solution, eq.(6.7) by using different values of the integration constants C_i ; that is $0 < C_1 < 5$, $-1 < C_2 < 9$, $-3 < C_3 < 5$. The solution is unstable in the domain $-1 < \xi < 1$.

Marked maxima and minima could observe on the real positive domain and both the maxima and minima lay on a vertical straight line. Consider the unusual behaviour in the range near the origin recognizing the instability.

7. Conclusion

The present paper represents a valuable contribution to the understanding of a rarely studied evolution equation, the so called Rand Equation. Unlike many other evolution equations well known in this field, properties and behaviour of the Rand Equation are not available. This paper however is suitable to extend the spectrum of knowledge to get a deeper insight in the solution manifold. As a first step we performed a classical Lie Group analysis to study the point symmetries. Applying the procedure of similarity reduction several cases of interest were shown especially the traveling wave reduction leading to periodic wave trains. Due to complexity of the underlying nODEs asymptotic solutions were given as well as some series representations using in practical calculations were shown. A complete symmetry table containing nonlinear transformations was performed. By using the Dominant Balance Method further similar solutions could derive. Secondly, the non-classical cases were studied. We show how one can derive potential as well as generalized symmetries. Here, interestingly the nPDE does not admit potential symmetries. Otherwise, the nPDE behave in a same kind to other evolution equations relating to generalize symmetries. By increasing the order (remains two-dimensional finite) the symmetry does not change and physically speaking the group transformation correlates with dilatation operations.

As a last fact of interest approximate symmetries were studied in order to show how one can calculate new solutions. The special case of traveling waves leads to unstable solutions at least in a certain domain. We also did not found any kinds of solitons, neither line solitons nor loop or cusp solitons and actually, the highly nonlinear eq.(1) admits therefore no physical description correlating to known processes. Further studies in future done by the author are necessary, especially concerning the following: We shall stress some theoretical basic questions such like the complete integrability. If the eq.(1) can be integrated completely we further can show the existence of a related Bäcklund system. Another question of interest is the affiliation to a known hierarchy also closely related to the integrability. Further it is of interest to know if the eq.(1) possesses the Painlevé property. If so, one can proof that the highly nonlinear eq.(1) can be solved in principle by the Inverse Scattering Method. Otherwise the Painlevé algorithm is suitable to solve the eq.(1) on the singular manifold. At this stage we can expect the limitation of calculating intensions, since, due to the complexity of the eq.(1), future theoretically derivations might fail.

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Biographical notes

Dipl.-Ing. Dr. techn. Huber Alfred is a distinguished lecturer at the Institute of Theoretical Physics – Computational Physics at the Technical University Graz, Austria following his habilitation treatise. He did his diploma thesis titled 'Systematic in the physics of elementary particles focusing the quarkonium states' in the field of elementary particle physics at the former Institute of Nuclear Physics at the Technical University Graz, Austria. He completed his scientific education with the doctoral programme of technical sciences at the Institute of Chemical Technology of Inorganic Compounds at the Technical University Graz, Austria subject to nuclear solid state physics and advanced electrochemistry. Thesis titled 'Synthesis and characterization of doped γ-manganese dioxides'. Also the author has a learnt vocation for a chemical assistant at the Research Centre of Electron Microscopy at the former Technical High School Graz, Austria. He is the author of 27 articles which have appeared in world-wide renowned scientific journals. His research interests are nonlinear partial differential equations (nPDE) of higher order with applications especially in physics and chemistry. The author developed several new algebraic procedures for solving nPDE. Special interests are further given in classical and non-classical symmetry methods, nonlinear transformations and the application of nonlinear methods in describing electrochemical interfaces, nonlinear wave propagation and further nonlinear topics of advanced character.

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