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Decomposıtıon of elastıc constant tensor ınto orthogonal parts

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Abstract

In this paper, we have elaborated on the decomposition methods such as irreducible decomposition, orthonormal tensor basis, harmonic and spectral decomposition for elastic constant tensor. Irreducible decomposition and orthonormal tensor basis methods are developed by using the results of existing theories in the literature. As examples to each decomposition method, we give results for the decomposition of elastic constant tensor in triclinic symmetry as well as materials with isotropic and transversely isotropic symmetry. Numerical examples serve to illustrate and verify each of the four decomposition methods. These examples are used to compare the decomposition methods explicitly. As a result of comparison process, it is stated that the spectral method is a non-linear invariant decomposition method that yields non-linear orthogonal parts contrary to the other three methods which are linear invariant decomposition methods. It is also shown that total scalar (isotropic) part is decomposed into two physically meaningful orthogonal parts by irreducible decomposition, orthonormal tensor basis and spectral methods. While in harmonic decomposition method, decomposition of total scalar part is not orthogonal. We propose that it is possible to make these parts orthogonal to each other.

Keywords: Elastic constant tensor; irreducible decomposition method; orthonormal tensor basis method; harmonic decomposition method; non-linear invariant decomposition method.

1. Introduction

Most of the elastic materials in engineering are anisotropic; metal crystals, fiber-reinforced composites, polycrystalline textured materials, biological tissues, rock structures. In order to understand the physical properties of the anisotropic materials, use of tensors by decomposing them is inevitable. The constitutive relation for linear anisotropic elasticity, defined by using stress and strain tensors, is the generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{1}$$

This formula demonstrates the well known general linear relation between the stress tensor (symmetric second order tensor) whose components are σ_{ij} and the strain tensor (symmetric second order tensor) whose components are ε_{kl}. The coefficients of linearity, namely C_{ijkl} are the components of elastic constant tensor (elasticity tensor) and satisfies three important symmetry restrictions. These are

$$C_{ijkl}=C_{jikl} \quad C_{ijkl}=C_{ijlk} \quad C_{ijkl}=C_{klij}, \tag{2}$$

which follow from the symmetry of the stress tensor, the symmetry of the strain tensor and the elastic strain energy. These restrictions reduce the number of independent elastic constants C_{ijkl} from 81 to 21.

In the literature, orthogonal decomposition methods are mainly distinguished as orthonormal tensor basis, irreducible, harmonic and spectral decomposition methods for elastic constant tensor. There are also other works for general decomposition of any rank tensors, these can be summarized as, Spencer (1970) and Jarić (2003) employed only elementary algebra and makes no group theory and gave a method in which a general tensor of any rank n can be expressed in terms of traceless symmetric tensors of rank n or less. Zou *et al.* (2001) realized orthogonal decomposition for any n rank tensors.

The purpose of this work is to study and elaborate on each orthogonal decomposition method for elastic constant tensor and compare these methods. In the present paper, irreducible decomposition, orthonormal tensor basis, harmonic decomposition, spectral methods and as examples of these methods to triclinic, isotropic and transversely isotropic materials are given in sections 2, 3, 4 and 5 respectively. In section 6, all decomposition methods are compared. Finally, in the last section, the results of comparisons for orthogonal decomposition methods are discussed and conclusions pertinent to this work are stated.

2. Irreducible Decomposition Method

We have encountered many works done related with irreducible decomposition in the literature. For instance; Jerphagnon *et al.* (1978) derived certain results for the irreducible tensors in their natural form. Andrews and Ghoul (1982) followed the technique of Jerphagnon *et al.* and gave the reduction of a fourth rank cartesian tensor into irreducible parts under the three-dimensional rotation group. Walpole (1984) and independently Kunin (1982) realized algebraic decomposition to simplify tensor functions operating on elastic constant tensor using the irreducible tensor algebra. Surrel (1993) used the group of rotations associated with elastic symmetry provides an irreducible representation. There are various related ways of considering elastic constant tensor in terms of rotational group properties of tensors based on complex vectors and tensors. (See also, Mochizuki (1988)) Finally Radwan (2001) carried out the method to elastic compliance tensor.

We follow the works of Jerphagnon *et al.* (1978) and Andrews and Ghoul (1982). Any rank-n cartesian tensor can be written as the direct sum of irreducible tensors in the cartesian representation. The term irreducible indicates sets that cannot be resolved into subsets with seperate linear transformations. The reduction of a (rank-n) cartesian tensor $\mathbf{T}_{(n)}$ generaly results in a sum of irreducible tensors, with some weights (*j*) represented more than once. (where $0 \leq j \leq n$), it is can be accompolished by the formula: $\mathbf{T}_{(n)} = \sum_{j=0}^{n} \sum_{q=1}^{N_n^{(j)}} \mathbf{T}_{(n)}^{(j;q)}$, where *q* is called the seniority index of the irreducible tensor $\mathbf{T}_{(n)}^{(j;q)}$ (irreducible cartesian tensor which is symmetric and traceless) and $N_n^{(j)}$ is the multiplicity of weight j in this reduction, it denotes the number of independent weight- j irreducible tensor parts.

$$N_n^{(j)} = \sum_k (-1)^k \binom{n}{k} \binom{2n-3k-j-2}{n-2}, \tag{3}$$

where $0 \leq k \leq [(n-j)/3]$. Each irreducible tensor has (2j+1) independent components. So that the total number of components in the reduction is $\sum_{j=0}^{n} (2j+1) N_n^{(j)} = 3^n$.

The natural projection of x^j onto the irreducible subspace H_j^j of traceless symmetric tensors of order *j* is denoted by $E^{(j)} = E^{(j)}_{k_1k_2...k_j;l_1l_2...l_j}$. The principal element in the reduction procedure is the mappings $Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j}$ of the minimal rank tensor subspace $H^j_{j,q}$ onto $H^n_{j,q}$, we have chosen the mappings $Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j}$ such that they are orthonormal and g_{pq} will be reduced to identity matrix, δ_{ij}, where g_{pq} is a symmetric matrix which was used and defined in Andrews and Ghoul (1982) through the relation

$$g_{pq} E^{(j)}_{k_1k_2....k_j;l_1l_2....l_j} = Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j} Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j}. \tag{4}$$

In this work, this relation is reduced to

$$\delta_{pq} E^{(j)}_{k_1k_2....k_j;l_1l_2....l_j} = Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j} Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j}. \tag{5}$$

The mappings $\widetilde{Q}^{(0;p)}_{k_1k_2...k_j;i_1i_2...i_n}$ dual to $Q^{(0;p)}_{i_1i_2...i_n;k_1k_2...k_j}$ are defined by the relation

$$\widetilde{Q}^{(0;p)}_{k_1k_2....k_j;i_1i_2...i_n} = Q^{(0;p)}_{i_1i_2....i_n;k_1k_2....k_j}. \tag{6}$$

The dual mappings extract the natural forms $t^{(j;p)}_{s_1s_2...s_j}$ from the tensor $T_{i_1i_2...i_n}$ as

$$t^{(j;p)}_{s_1s_2...s_j} = \widetilde{Q}_{s_1s_2.....s_ji_1i_2...i_n} T_{i_1i_2......i_n}. \tag{7}$$

These tensors can be embedded in the tensor space of order n through the mapping

$$T^{(j;q)}_{i_1i_2...i_n} = Q^{(0;q)}_{i_1i_2...i_n;k_1k_2...k_j} t^{(j;p)}_{k_1k_2...k_j}, \tag{8}$$

or

$$T^{(j,q)}_{i_1i_2....i_n} = Q^{(0;q)}_{i_1i_2....i_n;k_1k_2....k_j} \widetilde{Q}^{(0;p)}_{k_1k_2....k_j;l_1l_2....l_n} T_{l_1l_2....l_n}. \tag{9}$$

These results of Andrews and Ghoul (1982) can be developed such as making decomposed parts orthogonal. Since decomposition of the elastic constant tensor into irreducible parts can be obtainable from Andrews and Ghoul (1982) is different than the following results presented in eqs. (11), (12), (13), (14) and (15). Our irreducible parts are orthonormal to each other but theirs are not. Before giving the results of the reduction of elastic constant tensor, it is illustrated that the orthogonality condition is satisfied during the decomposition procedure. According to orthogonal process, given in Appendix A and taking into account the elastic symmetries such as eq. (2), elastic constant tensor is decomposed as

$$C_{ijkl} = C^{(0;1)}_{ijkl} + C^{(0;2)}_{ijkl} + C^{(2;1)}_{ijkl} + C^{(2;2)}_{ijkl} + C^{(4;1)}_{ijkl}, \tag{10}$$

where

$$C^{(0;1)}_{ijkl} = \frac{1}{9}\delta_{ij}\delta_{kl}C_{ppqq}, \tag{11}$$

$$C^{(0;2)}_{ijkl} = \frac{1}{90}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})(3C_{pqpq} - C_{ppqq}), \tag{12}$$

$$C^{(2;1)}_{ijkl} = \frac{1}{5}(\delta_{ik}C_{jplp} + \delta_{ik}C_{jplp} + \delta_{il}C_{jpkp} + \delta_{il}C_{ipkp}) - \frac{2}{15}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})C_{pqpq}, \tag{13}$$

$$\begin{aligned} C^{(2;2)}_{ijkl} = {} & \frac{1}{7}\delta_{ij}(5C_{klpp} - 4C_{kplp}) + \frac{1}{7}\delta_{kl}(5C_{ijpp} - 4C_{ipjp}) - \frac{2}{35}\delta_{ik}(5C_{jlpp} - 4C_{jplp}) - \frac{2}{35}\delta_{jl}(5C_{ikpp} - 4C_{ipkp}) - \\ & \frac{2}{35}\delta_{il}(5C_{jkpp} - 4C_{jpkp}) - \frac{2}{35}\delta_{jk}(5C_{ilpp} - 4C_{iplp}) + \frac{2}{105}(2\delta_{jk}\delta_{il} + 2\delta_{ik}\delta_{jl} - 5\delta_{ij}\delta_{kl})(5C_{ppqq} - 4C_{pqpq}), \end{aligned} \tag{14}$$

$$\begin{aligned} C^{(4;1)}_{ijkl} = {} & (C_{ijkl} + C_{iklj} + C_{iljk})/3 - [(C_{ijmm} + 2C_{imjm})\delta_{kl} + (C_{klmm} + 2C_{kmlm})\delta_{ij} + \\ & (C_{ikmm} + 2C_{imkm})\delta_{jl} + (C_{jlmm} + 2C_{jmlm})\delta_{ik} + (C_{ilmm} + 2C_{imlm})\delta_{jk} + (C_{jkmm} + C_{jmkm})\delta_{il}]/21 + \\ & (C_{ppmm} + 2C_{pmpm})(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/105, \end{aligned} \tag{15}$$

where $C^{(0;1)}_{ijkl}$, $C^{(0;2)}_{ijkl}$ are scalar parts, $C^{(2;1)}_{ijkl}$, $C^{(2;2)}_{ijkl}$ are deviators and $C^{(4;1)}_{ijkl}$ is the nonor part. As it is given by the group representation theory for the elastic constant tensor that $2D^{(0)} + 2D^{(2)} + D^{(4)}$, where the superscripts denote the weight of the representation, (See, Heine (1960)). This is the decomposition for triclinic materials which are anisotropic materials with no elastic symmetries.

2. 1 For Isotropic Materials

The traditional form of decomposition in isotropic media which is well known in the literature as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{16}$$

where λ and μ are invariant elastic constants and they are also called Lame constants. The traditional form of stress-strain relation for isotropic solids can be defined as

$$\sigma_{ij} = \lambda \varepsilon_{rr} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{17}$$

It is also well known that stress tensor is decomposed into spherical and deviatoric parts and it is given as

$$\sigma_{ij} = \frac{1}{3}\sigma_{rr}\delta_{ij} + (\sigma_{ij} - \frac{1}{3}\sigma_{rr}\delta_{ij}). \tag{18}$$

For irreducible decomposition method, there are only two irreducible parts for isotropic materials which are the scalar parts: $C_{ijkl}^{(0;1)}$ and $C_{ijkl}^{(0;2)}$ mentioned in the previous section. By writing these parts in matrix form, we get

$$C_{mn}^{(0;1)} = K\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{mn}^{(0;2)} = \frac{G}{3}\begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{19}$$

If we add those two parts, isotropic elastic constant tensor is obtained by irreducible decomposition method, isotropic elastic constant tensor can be rewritten instead of eq. (16), as follows

$$C_{ijkl} = K\delta_{ij}\delta_{kl} + 2G(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}), \tag{20}$$

where $K = (C_{11} + 2C_{12})/3$ and $G = \mu = (C_{11} - C_{12})/2,$ where K is the bulk modulus and G is the shear modulus. . In eq. (20) the decomposed parts are orthogonal to each other whereas the decomposed parts in traditional form given in eq. (16) are not orthogonal. From eqs. (1) and (20), the stress-strain relation, for isotropic materials, can be obtained as

$$\sigma_{ij} = K\varepsilon_{rr}\delta_{ij} + 2G(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij}). \tag{21}$$

This is also different from the traditional form and it is a new form in stress-strain relations in the literature. Equation (21) was also obtained in Landau and Lifshitz (1959) by a different method which was based on the expansion of the strain energy density function in powers of ε_{ij}.

2. 2 For Transversely Isotropic Materials

There are five irreducible parts for transversely isotropic materials which are two scalar parts: $C_{ijkl}^{(0;1)}$, $C_{ijkl}^{(0;2)}$, two deviators: $C_{ijkl}^{(2;1)}$, $C_{ijkl}^{(2;2)}$, and a nonor part $C_{ijkl}^{(4;1)}$, mentioned in the previous section. These parts are as follows:

$$C_{ijkl}^{(0;1)} = \frac{\alpha}{9}\delta_{ij}\delta_{kl}, \tag{22}$$

$$C_{ijkl}^{(0;2)} = \frac{\beta}{90}\left(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\right), \tag{23}$$

$$C_{ijkl}^{(2;1)} = \frac{\gamma}{30}\left(2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} - 3\delta_{3i}\delta_{3k}\delta_{jl} - 3\delta_{3j}\delta_{3k}\delta_{il} + \delta_{3i}\delta_{3l}\delta_{jk} + \delta_{3j}\delta_{3l}\delta_{ik}\right), \tag{24}$$

$$C_{ijkl}^{(2;2)} = \frac{\eta}{105}(-4\delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk} + 10\delta_{ij}\delta_{kl} - 15\delta_{3i}\delta_{3j}\delta_{kl} - 15\delta_{3k}\delta_{3l}\delta_{ij} + 6\delta_{3i}\delta_{3k}\delta_{jl} + 6\delta_{3j}\delta_{3l}\delta_{ik} + 6\delta_{3j}\delta_{3k}\delta_{il} + 6\delta_{3i}\delta_{3l}\delta_{jk}), \tag{25}$$

$$C_{ijkl}^{(4;1)} = \frac{\lambda}{35}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} - 5\delta_{3i}\delta_{3j}\delta_{kl} - 5\delta_{3k}\delta_{3l}\delta_{ij} - 5\delta_{3i}\delta_{3k}\delta_{jl} - 5\delta_{3j}\delta_{3k}\delta_{il} - 5\delta_{3i}\delta_{3l}\delta_{jk} - 5\delta_{3j}\delta_{3l}\delta_{ik} + 35\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}), \tag{26}$$

where $\alpha = 2C_{11} + 2C_{12} + 4C_{13} + C_{33}$, $\beta = 7C_{11} - 5C_{12} + 2C_{33} + 12C_{44} - 4C_{13}$,
$\gamma = 3C_{11} - 2C_{33} - 2C_{44} - C_{12}$, $\eta = 7C_{12} - C_{33} - C_{11} - 5C_{13} + 4C_{44}$
and
$\lambda = C_{11} + C_{33} - 2C_{13} - 4C_{44}$.

If we add those five parts, elastic constant tensor for transversely isotropic material is obtained by irreducible decomposition method as the same form in eq. (10).

3. Orthonormal Tensor Basis Method

In the literature, orthonormal tensor basis method had been studied in different names such as integrity basis and form-invariant. First orthonormal tensor basis method was proposed by Gazis *et al.* (1963), developed by Tu (1968). He used the method `integrity basis' and treated the strain energy function as a polynomial in the strain components and lead to determination of integrity basis for invariant functions of the strain components for each one of the 32 crystallographic point groups. Using the integrity basis, orthonormal tensor basis which spans the space of elastic constants was derived.

In form-invariant method, a physical property of tensor is resolved along the triad $v_1, v_{2,} v_3$ denoting the unit vectors along the crystallographic axes. The process of resolution yields the invariants. Forming invariant is an indispensable step to construct orthonormal tensor basis elements needed for decomposition process. Srinivasan (1998) proposed form invariant method which was developed by Ghaith and Akgöz (2005) for second and third rank tensors such as piezoelectric tensors.
In this paper, we have used form-invariant method which is a different one from the integrity basis method (by Tu (1968)). It is shown that two existing decomposition theories have close relationship since they give the same results for decomposition of elastic constant tensor under the title of orthonormal tensor basis method. It is first time that we apply form-invariant method in order to decompose elastic constant tensor for triclinic materials. The form invariant expression for the components of elastic constant tensor, the elastic stiffness coefficients is

$$C_{ijkl} = v_{ai} v_{bj} v_{ck} v_{dl} A_{abcd}, \tag{27}$$

where summation is implied by repeated indices, v_{ai} are the components of the unit vectors v_a ($a = 1,2,3$) along the material direction axes. A_{abcd} is invariant in the sense that when the Cartesian system is rotated to a new orientation $Ox'y'z'$, then eq. (27) takes the following form:

$$C'_{ijkl} = v'_{ai} v'_{bj} v'_{ck} v'_{dl} A_{abcd}, \tag{28}$$

where v_1, v_2, v_3 form a linearly independent basis in three dimensions but they are not necessarily always orthogonal. Their relative orientations in the seven crystal systems are well known and given by Nye (1957). The corresponding reciprocal triads must satisfy the following relation (taken from Srinivasan, Nigam (1969) and Srinivasan (1998)):

$$v_{ai} v_{aj} = \delta_{ij} \tag{29}$$

Form-invariant expression of isotropic symmetry is formed by the following two basis elements (see for instance; Fedorov (1968)):

$$\delta_{ij}\delta_{kl}, \qquad \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{30}$$

The decomposition of C_{ijkl} for tricilinic system with no elastic symmetries is given in terms of its orthonormalized basis elements as (see also; Tu (1968) and Srinivasan (1998))

$$C_{ijkl} = \sum_K (\mathbf{C}, A^K_{ijkl}) A^K_{ijkl}, (K = I \ldots XXI), \tag{31}$$

where $(\mathbf{C}, A^K_{ijkl})$ represents the inner product of C_{ijkl} and K^{th} elements, A^K_{ijkl}, of the basis, where

$$(\mathbf{C}, A^I_{ijkl}) = \frac{1}{3}[(C_{11} + C_{22} + C_{33}) + 2(C_{12} + C_{13} + C_{23})],$$

$$(\mathbf{C}, A^{II}_{ijkl}) = \frac{1}{3\sqrt{5}}[2(C_{11} + C_{22} + C_{33}) + 6(C_{44} + C_{55} + C_{66}) - 2(C_{12} + C_{13} + C_{23})],$$

$$(\mathbf{C}, A^{III}_{ijkl}) = \frac{1}{\sqrt{6}}[3C_{33} - (C_{11} + C_{22} + C_{33})], \quad (\mathbf{C}, A^{IV}_{ijkl}) = \frac{1}{\sqrt{3}}[3C_{13} + 3C_{23} - 2(C_{12} + C_{13} + C_{23})],$$

$$(\mathbf{C}, A^{V}_{ijkl}) = \frac{1}{\sqrt{30}}[-2(C_{11} + C_{22} + C_{33}) + 4(C_{44} + C_{55} + C_{66}) + 2(C_{12} + C_{13} + C_{23})],$$

$$(\mathbf{C}, A_{ijkl}^{VI}) = \sqrt{\frac{2}{3}}[C_{44} + C_{55} - 2C_{66}], \quad (\mathbf{C}, A_{ijkl}^{VII}) = \sqrt{\frac{1}{2}}[C_{11} - C_{22}], \quad (\mathbf{C}, A_{ijkl}^{VIII}) = [C_{13} - C_{23}],$$

$$(\mathbf{C}, A_{ijkl}^{IX}) = \sqrt{2}[C_{44} - C_{55}], \quad (\mathbf{C}, A_{ijkl}^{X}) = 2\sqrt{2}C_{46}, \quad (\mathbf{C}, A_{ijkl}^{XI}) = 2C_{35}, \quad (\mathbf{C}, A_{ijkl}^{XII}) = 2C_{15},$$

$$(\mathbf{C}, A_{ijkl}^{XIII}) = 2C_{25}, \quad (\mathbf{C}, A_{ijkl}^{XIV}) = 2\sqrt{2}C_{45}, \quad (\mathbf{C}, A_{ijkl}^{XV}) = 2C_{16}, \quad (\mathbf{C}, A_{ijkl}^{XVI}) = 2C_{26},$$

$$(\mathbf{C}, A_{ijkl}^{XVII}) = 2C_{36}, \quad (\mathbf{C}, A_{ijkl}^{XVIII}) = 2\sqrt{2}C_{56}, \quad (\mathbf{C}, A_{ijkl}^{XIX}) = 2C_{24}, \quad (\mathbf{C}, A_{ijkl}^{XX}) = 2C_{34},$$

$$(\mathbf{C}, A_{ijkl}^{XXI}) = 2C_{14}.$$

Here, elastic constants are given in Voigt notation.

3. 1 For Isotropic Materials

The elastic constant tensor for isotropic materials is decomposed as same as the form given in eq. (19) and stress-strain relation is also identical with the expression presented in eqs. (18), (21). So decomposition for isotropic materials give the same decomposed parts with irreducible decomposition method.

The decomposition of C_{ijkl} for the isotropic system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = \sum_K (\mathbf{C}, A_{ijkl}^K) A_{ijkl}^K = (\mathbf{C}, A_{ijkl}^I) A_{ijkl}^I + (\mathbf{C}, A_{ijkl}^{II}) A_{ijkl}^{II}, (K = I, II), \tag{32}$$

where $(\mathbf{C}, A_{ijkl}^K)$ denotes the inner product of C_{ijkl} and $A_{ijkl}^I = \frac{1}{3}\alpha_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$, $A_{ijkl}^{II} = \frac{1}{6\sqrt{5}}(3\beta_{ijkl} - 2\alpha_{ijkl})$.

These are orthonormalized basis elements for isotropic system. it is possible to compute inner products for isotropic system, they are

$$(\mathbf{C}, A_{ijkl}^I) = \frac{1}{3}[C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}], \tag{33}$$

$$(\mathbf{C}, A_{ijkl}^{II}) = \frac{1}{6\sqrt{5}}[4C_{11} - 4C_{12} - 4C_{13} + 4C_{22} + 4C_{33} - 4C_{23} + 12C_{44} + 12C_{55} + 12C_{66}]. \tag{34}$$

Same procedure is valid for other symmetry types but number of basis elements are changing depending on the number of independent elastic constants of material symmetry. The orthonormalized basis elements for isotropic system are constructed by performing the several steps given in Appendix B.

3. 2 For Transversely Isotropic Materials

The form invariant expression for transversely isotropic materials

$$C_{ijkl} = \lambda_1\alpha_{ijkl} + \lambda_2\beta_{ijkl} + \lambda_3\gamma_{ijkl} + \lambda_4\delta_{ijkl} + \lambda_5\varepsilon_{ijkl}, \tag{35}$$

where $\alpha_{ijkl} = \delta_{ij}\delta_{kl}$, $\beta_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, $\gamma_{ijkl} = \nu_{3i}\nu_{3j}\nu_{3k}\nu_{3l}$,

$\delta_{ijkl} = \nu_{3i}\nu_{3j}\delta_{kl} + \nu_{3k}\nu_{3l}\delta_{ij}$ and

$\varepsilon_{ijkl} = \nu_{3i}\nu_{3k}\delta_{jl} + \nu_{3j}\nu_{3l}\delta_{ik} + \nu_{3i}\nu_{3l}\delta_{jk} + \nu_{3j}\nu_{3k}\delta_{il.}$

λ_1, λ_2, λ_3, λ_4 and λ_5 are invariant elastic constants for transversely isotropic system. (See, Srinivasan and Nigam (1969))

The decomposition of C_{ijkl} for transversely isotropic system is given in terms of the orthonormalized basis elements as

$$C_{ijkl} = (\mathbf{C}, A_{ijkl}^I) A_{ijkl}^I + (\mathbf{C}, A_{ijkl}^{II}) A_{ijkl}^{II} + (\mathbf{C}, A_{ijkl}^{III}) A_{ijkl}^{III} + (\mathbf{C}, A_{ijkl}^{IV}) A_{ijkl}^{IV} + (\mathbf{C}, A_{ijkl}^{V}) A_{ijkl}^{V}, \tag{36}$$

where $(\mathbf{C}, A_{ijkl}^K)$ denotes the inner product of C_{ijkl} and

$$A_{ijkl}^{III} = \frac{1}{6\sqrt{5}}(15\gamma_{ijkl} - \beta_{ijkl} - \alpha_{ijkl}), \quad A_{ijkl}^{IV} = \frac{1}{12}(9\delta_{ijkl} - 15\gamma_{ijkl} + \beta_{ijkl} - 5\alpha_{ijkl}),$$

$$A_{ijkl}^{V} = \frac{1}{4}(2\varepsilon_{ijkl} - \delta_{ijkl} + 3\gamma_{ijkl} - \beta_{ijkl} + \alpha_{ijkl}),$$

which are orthonormalized basis elements for transversely isotropic system. Since first two orthonormalized basis elements of transversely isotropic system are same with isotropic symmetry, inner products are also same, the other inner products for transversely isotropic system are found as

$$(\mathbf{C}, A^{III}_{ijkl}) = \frac{1}{6\sqrt{5}}[-3C_{11} - 3C_{22} + 12C_{33} - 2C_{12} - 2C_{13} - 2C_{23} - 4C_{44} - 4C_{55} - 4C_{66}], \tag{37}$$

$$(\mathbf{C}, A^{IV}_{ijkl}) = \frac{1}{12}[-3C_{11} - 3C_{22} - 10C_{12} + 8C_{13} + 8C_{23} + 4C_{44} + 4C_{55} + 4C_{66}], \tag{38}$$

$$(\mathbf{C}, A^{V}_{ijkl}) = \frac{1}{4}[-C_{11} - C_{22} + 2C_{12} + 4C_{44} + 4C_{55} - 4C_{66}]. \tag{39}$$

This method is orthogonal and it is proved by constructing orthonormalized basis elements in Appendix B. The decomposed parts are different from the expressions given in eqs. (22)-(26) obtained by irreducible decomposition method.

4. Harmonic Decomposition Method

In the literature, harmonic decomposition had been studied extensively. Firstly, Backus (1970) proposed a representation of elastic constant tensor in terms of harmonic tensors. These are based on an isomorphism between the space of homogeneous harmonic polynomials of degree *q* and the space of totally symmetric tensors of order *q*. Furthermore according to Sirotin (1975) elastic constant tensor was decomposed with respect to general linear group and then orthogonal group O(3). Baerheim (1993) followed Backus (1970) and developed the method.

In harmonic decomposition, the action of $SO(3)$ on a vector space is said to be irreducible when there are no proper invariant subspaces. It is deduced that there is a decomposition of the space of elastic constant tensors ($\mathbf{E}$ la) into a direct sum of orthogonal subspaces on which the action of $SO(3)$ is irreducible. An important theorem of group representation theory can be summarized as: every space on which the group of rotations acts irreducibly is isomorphic through an $SO(3)$ -invariant map with an appropriate space of harmonic polynomials. In view of isomorphism, there is a decomposition of $\mathbf{E}$ la into a direct sum of spaces of harmonic tensors. (See, for instance; Forte and Vianello (1996))

Besides, there is an $SO(3)$ -invariant isomorphism between $\mathbf{E}$ la and the direct sum R $\oplus$ R $\oplus$ Dev $\oplus$ Dev $\oplus$ **Hrm**. We give a brief review for this method as follows: The decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry, we obtain

$$C_{ijkl} = H_{ijkl} + [\delta_{ij}H_{kl} + \delta_{kl}H_{ij} + \delta_{ik}H_{jl} + \delta_{jl}H_{ik} + \delta_{il}H_{jk} + \delta_{jk}H_{il}] + H[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + \delta_{ij}h_{kl} + \delta_{kl}h_{ij} - \frac{1}{2}\delta_{ik}h_{jl} - \frac{1}{2}\delta_{jl}h_{ik} - \frac{1}{2}\delta_{il}h_{jk} - \frac{1}{2}\delta_{jk}h_{il} + h(\delta_{ij}\delta_{kl} - \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{il}\delta_{jk}). \tag{40}$$

Where $H = \frac{1}{45}(C_{ppqq} + 2C_{pqpq}), \quad h = \frac{1}{9}(C_{ppqq} - C_{pqpq}),$

$$H_{ij} = \frac{1}{21}\left[C_{ijkk} - \frac{1}{3}C_{jjkk}\delta_{ij} + 2\left(C_{ikjk} - \frac{1}{3}C_{jkjk}\delta_{ij}\right)\right], \; h_{ij} = \frac{2}{3}(C_{ijpp} - C_{ipjp}) - \frac{2}{9}\delta_{ij}(C_{rrpp} - C_{rprp}).$$

In this method, we use the notation of Baerheim (1993).

The total scalar (isotropic) part of harmonic decomposition (obtained from eq. (40)) is (denoted as S)

$$S = \frac{\delta_{ij}\delta_{kl}}{15}(2C_{ppqq} - C_{pqpq}) + \frac{(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}{30}(3C_{pqpq} - C_{ppqq}). \tag{41}$$

Furthermore the total deviatoric part or second rank traceless tensor is composed of summation of the linear combination of second order tensors(H_{ij} and h_{ij}) given in eq. (40), which is (denoted as D)

$$D = \frac{1}{7}\delta_{kl}(5C_{ijpp} - 4C_{ipjp}) + \frac{1}{7}\delta_{ij}(5C_{klpp} - 4C_{kplp}) + \frac{1}{7}\delta_{ik}(3C_{jplp} - 2C_{jlpp}) + \frac{1}{7}\delta_{il}(3C_{jpkp} - 2C_{jkpp}) + \frac{1}{7}\delta_{jl}(3C_{ipkp} - 2C_{ikpp}) + \frac{1}{7}\delta_{jk}(3C_{iplp} - 2C_{ilpp}) + \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\left(\frac{4}{21}C_{pprr} - \frac{2}{7}C_{rprp}\right) + \delta_{ij}\delta_{kl}(\frac{8}{21}C_{rprp} - \frac{10}{21}C_{pprr}). \tag{42}$$

From eq. (40), harmonic (nonor) part is obtained as the same expression as $C_{ijkl}^{(4;1)}$ given in eq. (15).

Moreover, the results for elastic constant tensor decomposition are given by Onat (1984, 1994) and quoted by Cowin (1989), Forte and Vianello (1996, 2006), He (2004), Ting and He (2006), Zheng (2007), Annin and Ostrosablin (2008) in which elastic constant tensor is decomposed into two scalar, two deviatoric and nonor parts. These decompositions are the same as harmonic decomposition method since scalar, deviatoric and nonor parts are common and they are identical with those obtained from harmonic decomposition method, only difference here is notations used for scalar, traceless symmetric second rank tensors and nonor parts.

According to these studies, decomposition of elastic constant tensor for anisotropic materials possessing triclinic symmetry is expressed as follows:

$$C_{ijkl} = \frac{1}{15}(2C_{ppqq} - C_{pqpq})\delta_{ij}\delta_{kl} + \frac{1}{30}(3C_{pqpq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ij}A_{kl} + \delta_{kl}A_{ij} + \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il} + Z_{ijkl}, \tag{43}$$

From eq. (43), total scalar part is

$$\frac{1}{15}(2C_{ppqq} - C_{pqpq})\delta_{ij}\delta_{kl} + \frac{1}{30}(3C_{pqpq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{44}$$

The total deviatoric part is

$$\delta_{ij}A_{kl} + \delta_{kl}A_{ij} + \delta_{ik}B_{jl} + \delta_{jl}B_{ik} + \delta_{il}B_{jk} + \delta_{jk}B_{il}. \tag{45}$$

The components of deviatoric part are

$$A_{ij} = (15C_{ijkk} - 12C_{ikjk} - 5\delta_{ij}C_{ppkk} + 4\delta_{ij}C_{pkpk})/21,$$

$$B_{ij} = (-6C_{ijkk} + 9C_{ikjk} + 2\delta_{ij}C_{ppkk} - 3\delta_{ij}C_{pkpk})/21.$$

Finally nonor part is the same as $C_{ijkl}^{(4;1)}$ given in eq. (15).

Furthermore the decomposition of elastic constant tensor given in Forte and Vianello (1996, 2006) contains misprints in components of scalar part and total deviatoric part. In eqs. (44) and (45), these parts are corrected.

4. 1 For Isotropic Materials

Like irreducible decomposition method, there are two irreducible parts constituted total isotropic (scalar) parts which are found by using eqs. (43) and (44) and these decomposed parts are

$$\frac{1}{15}(2C_{ppqq} - C_{pqpq})\delta_{ij}\delta_{kl} \tag{46}$$

and

$$\frac{1}{30}(3C_{pqpq} - C_{ppqq})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{47}$$

The sum of eqs. (46) and (47) gives total scalar part which is obtained in eq. (44).

4. 2 For Transversely Isotropic Materials

There are five irreducible parts for transversely isotropic materials which are two scalars and two deviators and a nonor (harmonic) part. The decomposed parts of total scalar part are the same as those given in eqs. (46) and (47) and decomposed parts of total deviatoric part which are $C_{ijkl}^{(2;1)}$ and $C_{ijkl}^{(2;2)}$ found in eqs. (24) and (25) respectively, are obtained by arranging equation (42) so we get total deviatoric part in harmonic decomposition method.

Nonor part is equal to $C_{ijkl}^{(4;1)}$ given in eq. (26), we obtain harmonic part for transversely isotropic materials in harmonic decomposition method.

5. Spectral Decomposition Method

The spectral decomposition of $\mathbf{C}$ (elastic constant tensor) for triclinic materials which is well known in the literature, given by Cowin *et al.* (1991), Sutcliffe (1992) and Rychlewski (2000)

$$\mathbf{C} = \lambda_k(\vec{n}^{(k)} \otimes \vec{n}^{(k)}) \quad \text{summation over k}, \quad (k = 1,....,6.) \tag{48}$$

where λ_k are the eigenvalues of elastic constant tensor and $\overset{\rightarrow(k)}{n}$ are the corresponding eigenvectors of elastic constant tensor. The symbol $\otimes$ is the dyadic (tensor) product.

$\overset{\rightarrow(k)}{n} \otimes \overset{\rightarrow(k)}{n} = \mathbf{N}$ are the eigentensors (idempotent tensors) and they are orthogonal to each other.

This decomposition method is a non-linear orthogonal method since eigenvalues constituting decomposed parts, are expressed in terms of elastic constants.

In spectral method; the orthogonality condition is checked by using orthonormalized basis elements. In order to construct orthonormalized basis, the following properties must be satisfied

$$\mathbf{I} = \overset{\rightarrow(1)}{n} \otimes \overset{\rightarrow(1)}{n} + \dots\dots\dots + \overset{\rightarrow(6)}{n} \otimes \overset{\rightarrow(6)}{n},$$

which yields

$$\mathbf{I} = N_1 + N_2 + N_3 + N_4 + N_5 + N_6$$

(Where $\mathbf{I}$ is the identity matrix.)

$$N_1, \quad N_2, \dots, N_6, \quad N_K \cdot N_K = N_K$$

$$N_K \cdot N_L = 0 \quad (\text{ if } K \neq L)$$

So by using this method, decomposed parts of elastic constant tensor is orthogonal to each other.

5. 1 For Isotropic Materials

For isotropic symmetry, there are two eigenvalues which are $C_{11} + 2C_{12}$ (λ_1) and $(C_{11} - C_{12})/2$ (λ_2). These eigenvalues are found by the formula: $|\mathbf{C} - \lambda_k I| = 0,$ the normalized eigenvectors are obtained by the formula: $(\mathbf{C} - \lambda_k I)\overset{\leftrightarrow(k)}{n} = 0,$ where $\overset{\leftrightarrow(k)}{n}$ are normalized eigenvectors.

The elastic constant tensor for isotropic materials can be written as

$$\mathbf{C} = \lambda_1(\overset{\leftrightarrow(1)}{n} \otimes \overset{\leftrightarrow(1)}{n}) + \lambda_2(\sum_{k=2}^{6} \overset{\leftrightarrow(k)}{n} \otimes \overset{\leftrightarrow(k)}{n}). \tag{49}$$

Recall that $\lambda_2 = \lambda_3 = \quad \lambda_4 = \lambda_5 = \lambda_6,$ and tensors that have the same eigenvalues can be combined and the elastic constant tensor can be written as two parts, we obtain the same parts in eqs. (19) and (31) given in irreducible decomposition and orthonormal tensor basis methods for isotropic materials.

5. 2 For Transversely Isotropic Materials

For transversely isotropic symmmetry like isotropic symmetry, eigenvalues are found by the formula; $|\mathbf{C} - \lambda_k I| = 0,$ corresponding normalized eigenvectors are obtained by the formula:

$(\mathbf{C} - \lambda_k I)\overset{\leftrightarrow(k)}{n} = 0.$ From these formulas, eigenvalues are

$$\lambda_1 = (C_{11} + C_{12} + C_{33} - \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2,$$

$$\lambda_2 = (C_{11} + C_{12} + C_{33} + \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2,$$

$$\lambda_3 = C_{11} - C_{12}, \quad \lambda_4 = \lambda_5 = 2C_{44} \qquad \lambda_6 = C_{11} - C_{12}.$$

The corresponding normalized eigenvectors are

$$\overset{\leftrightarrow(1)}{n} = \frac{1}{\left(\sqrt{2C_{13}^2 + b_1^2}\right)}[C_{13}, C_{13}, b_1, 0, 0, 0]^T, \quad \overset{\leftrightarrow(2)}{n} = \frac{1}{\left(\sqrt{2C_{13}^2 + b_2^2}\right)}[C_{13}, C_{13}, b_2, 0, 0, 0]^T,$$

$$\overset{\leftrightarrow(3)}{n} = \frac{1}{\sqrt{2}}[1, -1, 0, 0, 0, 0]^T, \quad \overset{\leftrightarrow(4)}{n} = [0, 0, 0, 0, 1, 0]^T, \quad \overset{\leftrightarrow(5)}{n} = [0, 0, 0, 1, 0, 0]^T,$$

$$\overset{\leftrightarrow(6)}{n} = [0, 0, 0, 0, 0, 1]^T,$$

where $b_1 = \left(-C_{11} - C_{12} + C_{33} - \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2})/2\right)/2$ and

$$b_2 = \left(-C_{11} - C_{12} + C_{33} + \sqrt{(C_{11} + C_{12} - C_{33})^3 + 8C_{13}^2}\,)/2\right)/2.$$

By using these eigenvalues and eigenvectors, elastic constant tensor for transversely isotropic material can be decomposed by spectral method as

$$\mathbf{C} = \sum_{k=1}^{6} \lambda_k (\overset{\leftrightarrow}{n}^{(k)} \otimes \overset{\leftrightarrow}{n}^{(k)}) \tag{50}$$

Since spectral decomposition is a non-linear method, it gives decomposed parts in terms of eigenvalues which are functions of elastic constants. The results are different from the other three decomposition methods.

6. Numerical Examples

To support the analytic results of four methods, we give illustrative numerical examples for triclinic, transversely isotropic and isotropic symmetries. The main purpose of these examples from various symmetry systems to figure out not only the significant differences but also the critical similarities among all decomposition methods more explicitly.

For triclinic material, let $\mathbf{C}$ be an elastic constant tensor of Low Albite (taken from Brown *et al.* (2006)) which has the matrix form in GPa,

$$C_{ij} = \begin{bmatrix} 69.1 & 34 & 30.8 & 5.1 & -2.4 & -0.9 \\ 34 & 183.5 & 5.5 & -3.9 & -7.7 & -5.8 \\ 30.8 & 5.5 & 179.5 & -8.7 & 7.1 & -9.8 \\ 5.1 & -3.9 & -8.7 & 24.9 & -2.4 & -7.2 \\ -2.4 & -7.7 & 7.1 & -2.4 & 26.8 & 0.5 \\ -0.9 & -5.8 & -9.8 & -7.2 & 0.5 & 33.5 \end{bmatrix}. \tag{51}$$

Applying irreducible decomposition method, we use the formulas given in section 2 and we obtain the scalar, deviatoric and nonor parts respectively. Hence elastic constant tensor for Low Albite can be decomposed as

$$C_{ij} = \begin{bmatrix} 63.6333 & 63.6333 & 63.6333 & 0 & 0 & 0 \\ 63.6333 & 63.6333 & 63.6333 & 0 & 0 & 0 \\ 63.6333 & 63.6333 & 63.6333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 54.88 & -27.44 & -27.44 & 0 & 0 & 0 \\ -27.44 & 54.88 & -27.44 & 0 & 0 & 0 \\ -27.44 & -27.44 & 54.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 41.16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 41.16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 41.16 \end{bmatrix} +$$

$$\begin{bmatrix} -57.1467 & 0 & 0 & 0 & -1 & -3.64 \\ 0 & 32.8533 & 0 & 1.48 & 0 & -3.64 \\ 0 & 0 & 24.2933 & 1.48 & -1 & 0 \\ 0 & 1.48 & 1.48 & 14.2867 & -1.82 & -0.5 \\ -1 & 0 & -1 & -1.82 & -8.2133 & 0.74 \\ -3.64 & -3.64 & 0 & -0.5 & 0.74 & -6.0733 \end{bmatrix} +$$

$$\begin{bmatrix} 0.0419 & -0.4333 & 0.5381 & -1.3571 & 0.7371 & -0.1971 \\ -0.4333 & -0.2152 & -0.1048 & -0.2714 & 3.6857 & -0.1971 \\ 0.5381 & -0.1048 & 0.1733 & -0.2714 & 0.7371 & -0.9857 \\ -1.3571 & -0.2714 & -0.2714 & 0.0419 & 0.3943 & -1.4743 \\ 0.7371 & 3.6857 & 0.7371 & 0.3943 & -0.2152 & 0.5429 \\ -0.1971 & -0.1971 & -0.9857 & -1.4743 & 0.5429 & 0.1733 \end{bmatrix} +$$

$$\begin{bmatrix} 7.6915 & -1.76 & -5.9314 & 6.4571 & -2.1371 & 2.9371 \\ -1.76 & 32.349 & -30.589 & -5.1086 & -11.386 & -1.9629 \\ -5.9314 & -30.589 & 36.52 & -9.9086 & 7.3629 & -8.8143 \\ 6.4571 & -5.1086 & -9.9086 & -30.589 & -0.9743 & -5.2257 \\ -2.1371 & -11.386 & 7.3629 & -0.9743 & -5.9315 & -0.7829 \\ 2.9371 & -1.9629 & -8.8143 & -5.2257 & -0.7829 & -1.76 \end{bmatrix}. \tag{52}$$

Using orthonormal tensor basis method, we apply the formula given in eq. (31). For this reason inner products are calculated as:

$(\mathbf{C}, A^{I}_{ijkl}) = 190.9, \quad (\mathbf{C}, A^{II}_{ijkl}) = 184.0731,$

$(\mathbf{C}, A^{III}_{ijkl}) = 43.4376, \quad (\mathbf{C}, A^{IV}_{ijkl}) = -18.302, \quad (\mathbf{C}, A^{V}_{ijkl}) = -69.8894, \quad (\mathbf{C}, A^{VI}_{ijkl}) = -12.4924,$

$(\mathbf{C}, A^{VII}_{ijkl}) = -80.893, \quad (\mathbf{C}, A^{VIII}_{ijkl}) = 25.3, \quad (\mathbf{C}, A^{IX}_{ijkl}) = -2.687, \quad (\mathbf{C}, A^{X}_{ijkl}) = -20.3647,$

$(\mathbf{C}, A^{XI}_{ijkl}) = 14.2, \quad (\mathbf{C}, A^{XII}_{ijkl}) = -4.8, \quad (\mathbf{C}, A^{XIII}_{ijkl}) = -15.4, \quad (\mathbf{C}, A^{XIV}_{ijkl}) = -6.7882,$

$(\mathbf{C}, A^{XV}_{ijkl}) = -1.8, \quad (\mathbf{C}, A^{XVI}_{ijkl}) = -11.6, \quad (\mathbf{C}, A^{XVII}_{ijkl}) = -19.6, \quad (\mathbf{C}, A^{XVIII}_{ijkl}) = 1.4142,$

$(\mathbf{C}, A^{XIX}_{ijkl}) = -7.8, \quad (\mathbf{C}, A^{XX}_{ijkl}) = -17.4, \quad (\mathbf{C}, A^{XXI}_{ijkl}) = 10.2.$

The elastic constant tensor for triclinic material, C_{ijkl}, can be represented in the following matrix form:

$$C_{pq} = \begin{bmatrix} 63.6333 & 63.6333 & 63.6333 & 0 & 0 & 0 \\ 63.6333 & 63.6333 & 63.6333 & 0 & 0 & 0 \\ 63.6333 & 63.6333 & 63.6333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 54.88 & -27.44 & -27.44 & 0 & 0 & 0 \\ -27.44 & 54.88 & -27.44 & 0 & 0 & 0 \\ -27.44 & -27.44 & 54.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 41.16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 41.16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 41.16 \end{bmatrix} +$$

$$\begin{bmatrix} -17.7333 & 0 & 0 & 0 & 0 & 0 \\ 0 & -17.7333 & 0 & 0 & 0 & 0 \\ 0 & 0 & 35.4667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 10.5667 & -5.2833 & 0 & 0 & 0 \\ 10.5667 & 0 & -5.2833 & 0 & 0 & 0 \\ -5.2833 & -5.2833 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$
\begin{bmatrix} 25.52 & -12.76 & -5.2833 & 0 & 0 & 0 \\ -12.76 & 25.52 & -12.76 & 0 & 0 & 0 \\ -12.76 & -12.76 & 25.52 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12.76 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12.76 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12.76 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.55 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.55 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5.1 \end{bmatrix} +
$$

$$
\begin{bmatrix} -57.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 57.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 12.65 & 0 & 0 & 0 \\ 0 & 0 & -12.65 & 0 & 0 & 0 \\ 12.65 & -12.65 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.95 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.95 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7.2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -2.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.4 & 0 \\ 0 & 0 & 0 & -2.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -0.9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.9 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5.8 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -9.8 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8.7 & 0 & 0 \\ 0 & 0 & -8.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 5.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{53}
$$

By applying harmonic decomposition method, we use the formulas given in section 4 and we obtain the scalar, deviatoric and nonor parts respectively. Hence elastic constant tensor for Low Albite can be decomposed as

$$C_{ij} = \begin{bmatrix} 36.19 & 36.19 & 36.19 & 0 & 0 & 0 \\ 36.19 & 36.19 & 36.19 & 0 & 0 & 0 \\ 36.19 & 36.19 & 36.19 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 82.32 & 0 & 0 & 0 & 0 & 0 \\ 0 & 82.32 & 0 & 0 & 0 & 0 \\ 0 & 0 & 82.32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 41.16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 41.16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 41.16 \end{bmatrix} +$$

$$\begin{bmatrix} -57.1467 & 0 & 0 & 0 & -1 & -3.64 \\ 0 & 32.8533 & 0 & 1.48 & 0 & -3.64 \\ 0 & 0 & 24.2933 & 1.48 & -1 & 0 \\ 0 & 1.48 & 1.48 & 14.2867 & -1.82 & -0.5 \\ -1 & 0 & -1 & -1.82 & -8.2133 & 0.74 \\ -3.64 & -3.64 & 0 & -0.5 & 0.74 & -6.0733 \end{bmatrix} +$$

$$\begin{bmatrix} 0.0419 & -0.4333 & 0.5381 & -1.3571 & 0.7371 & -0.1971 \\ -0.4333 & -0.2152 & -0.1048 & -0.2714 & 3.6857 & -0.1971 \\ 0.5381 & -0.1048 & 0.1733 & -0.2714 & 0.7371 & -0.9857 \\ -1.3571 & -0.2714 & -0.2714 & 0.0419 & 0.3943 & -1.4743 \\ 0.7371 & 3.6857 & 0.7371 & 0.3943 & -0.2152 & 0.5429 \\ -0.1971 & -0.1971 & -0.9857 & -1.4743 & 0.5429 & 0.1733 \end{bmatrix} +$$

$$\begin{bmatrix} 7.6915 & -1.76 & -5.9314 & 6.4571 & -2.1371 & 2.9371 \\ -1.76 & 32.349 & -30.589 & -5.1086 & -11.386 & -1.9629 \\ -5.9314 & -30.589 & 36.52 & -9.9086 & 7.3629 & -8.8143 \\ 6.4571 & -5.1086 & -9.9086 & -30.589 & -0.9743 & -5.2257 \\ -2.1371 & -11.386 & 7.3629 & -0.9743 & -5.9315 & -0.7829 \\ 2.9371 & -1.9629 & -8.8143 & -5.2257 & -0.7829 & -1.76 \end{bmatrix}. \tag{54}$$

By applying the formula given in eq. (48) and using these eigenvalues and eigenvectors, elastic constant tensor for Low Albite can be decomposed by spectral method and we obtain six decomposed parts which are

$$C_{ij} = \begin{bmatrix} 20.9639 & 47.5364 & 39.4175 & -2.1899 & -0.7517 & -3.9091 \\ 47.5364 & 107.7906 & 89.3808 & -4.9656 & -1.7046 & -8.8640 \\ 39.4175 & 89.3808 & 74.1153 & -4.1175 & -1.4135 & -7.3501 \\ -2.1899 & -4.9656 & -4.1175 & 0.2288 & 0.0785 & 0.4083 \\ -0.7517 & -1.7046 & -1.4135 & 0.0785 & 0.0270 & 0.1402 \\ -3.9091 & -8.8640 & -7.3501 & 0.4083 & 0.1402 & 0.7289 \end{bmatrix} +$$

$$\begin{bmatrix} 0.0317 & -1.5176 & 1.8074 & -0.0635 & 0.1638 & -0.0587 \\ -1.5176 & 72.5973 & -86.4599 & 3.0353 & -7.8373 & 2.8088 \\ 1.8074 & -86.4599 & 102.9695 & -3.6149 & 9.3339 & -3.3451 \\ -0.0635 & 3.0353 & -3.6149 & 0.1269 & -0.3277 & 0.1174 \\ 0.1638 & -7.8373 & 9.3339 & -0.3277 & 0.8461 & -0.3032 \\ -0.0587 & 2.8088 & -3.3451 & 0.1174 & -0.3032 & 0.1087 \end{bmatrix} +$$

$$\begin{bmatrix} 46.8766 & -11.8390 & -9.9058 & 12.0941 & -4.1776 & 1.5609 \\ -11.8390 & 2.9900 & 2.5018 & -3.0544 & 1.0551 & -0.3942 \\ -9.9058 & 2.5018 & 2.0933 & -2.5557 & 0.8828 & -0.3298 \\ 12.0941 & -3.0544 & -2.5557 & 3.1202 & -1.0778 & 0.4027 \\ -4.1776 & 1.0551 & 0.8828 & -1.0778 & 0.3723 & -0.1391 \\ 1.5609 & -0.3942 & -0.3298 & 0.4027 & -0.1391 & 0.052 \end{bmatrix} +$$

$$\begin{bmatrix} 0.4252 & 0.0124 & 0.0036 & -1.9074 & 0.4902 & 3.4427 \\ 0.0124 & 0.0004 & 0.0001 & -0.0555 & 0.0143 & 0.1001 \\ 0.0036 & 0.0001 & 0.0000 & -0.0161 & 0.0041 & 0.0291 \\ -1.9074 & -0.0555 & -0.0161 & 8.5565 & -2.1991 & -15.4437 \\ 0.4902 & 0.0143 & 0.0041 & -2.1991 & 0.5652 & 3.9691 \\ 3.4427 & 0.1001 & 0.0291 & -15.4437 & 3.9691 & 27.8742 \end{bmatrix} +$$

$$\begin{bmatrix} 0.2215 & 0.0553 & -0.1861 & -0.0971 & 2.3357 & -0.4139 \\ 0.0553 & 0.0138 & -0.0464 & -0.0242 & 0.5827 & -0.1032 \\ -0.1861 & -0.0464 & 0.1564 & 0.0816 & -1.9624 & 0.3477 \\ -0.0971 & -0.0242 & 0.0816 & 0.0426 & -1.0241 & 0.1815 \\ 2.3357 & 0.5827 & -1.9624 & -1.0241 & 24.6255 & -4.3636 \\ -0.4139 & -0.1032 & 0.3477 & 0.1815 & -4.3636 & 0.7732 \end{bmatrix} +$$

$$\begin{bmatrix} 0.5859 & -0.2492 & -0.3233 & -2.7413 & -0.4595 & -1.5244 \\ -0.2492 & 0.1060 & 0.1375 & 1.1662 & 0.1955 & 0.6485 \\ -0.3233 & 0.1375 & 0.1784 & 1.5128 & 0.2536 & 0.8412 \\ -2.7413 & 1.1662 & 1.5128 & 12.8263 & 2.1499 & 7.1323 \\ -0.4595 & 0.1955 & 0.2536 & 2.1499 & 0.3604 & 1.1955 \\ -1.5244 & 0.6485 & 0.8412 & 7.1323 & 1.1955 & 3.9661 \end{bmatrix}. \tag{55}$$

For Low Albite, eqs. (52), (53), (54) prove that total scalar parts (sum of the first two decomposed parts) are common in irreducible, orthonormal tensor basis and harmonic decomposition methods whereas eq. (55) illustrates that total scalar part is different from other decomposition methods.

For transversely isotropic material, let $\mathbf{C}$ be an elastic constant tensor of Polystyrene (see, Wright *et al.,* 1971) which has the matrix form in GPa,

$$C_{ij}=\begin{bmatrix}5.20 & 2.75 & 2.75 & 0 & 0 & 0\\ 2.75 & 5.20 & 2.75 & 0 & 0 & 0\\ 2.75 & 2.75 & 5.70 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.30 & 0 & 0\\ 0 & 0 & 0 & 0 & 1.30 & 0\\ 0 & 0 & 0 & 0 & 0 & 1.225\end{bmatrix} \tag{56}$$

Applying irreducible decomposition method, we use the formulas given in section (2.2) and we obtain the scalar, deviatoric and nonor parts respectively. Hence elastic constant tensor for Polystyrene can be decomposed as

$$C_{ij}=\begin{bmatrix}3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0\\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0\\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}1.7178 & -0.8589 & -0.8589 & 0 & 0 & 0\\ -0.8589 & 1.7178 & -0.8589 & 0 & 0 & 0\\ -0.8589 & -0.8589 & 1.7178 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.2883 & 0 & 0\\ 0 & 0 & 0 & 0 & 1.2883 & 0\\ 0 & 0 & 0 & 0 & 0 & 1.2883\end{bmatrix}+$$

$$\begin{bmatrix}-0.1533 & 0 & 0 & 0 & 0 & 0\\ 0 & -0.1533 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.3067 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.0383 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.0383 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.0767\end{bmatrix}+\begin{bmatrix}-0.0038 & -0.019 & 0.0095 & 0 & 0 & 0\\ -0.019 & -0.0038 & 0.0095 & 0 & 0 & 0\\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.0038 & 0 & 0\\ 0 & 0 & 0 & 0 & -0.0038 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.0076\end{bmatrix}+$$

$$\begin{bmatrix}0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0\\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0\\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.0229 & 0 & 0\\ 0 & 0 & 0 & 0 & -0.0229 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.0057\end{bmatrix}. \tag{57}$$

By orthonormal tensor basis method, we apply the formula given in eq. (36). For this reason, inner products must be calculated as

$$(\mathbf{C},A^{I})=10.8667,\quad (\mathbf{C},A^{II})=5.7616,\quad (\mathbf{C},A^{III})=0.4025,\quad (\mathbf{C},A^{IV})=0.05,\quad (\mathbf{C},A^{V})=0.15.$$

The elastic constant tensor for Polystyrene can be represented in the form

$$C_{ij}=\begin{bmatrix}3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0\\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0\\ 3.6222 & 3.6222 & 3.6222 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}1.7178 & -0.8589 & -0.8589 & 0 & 0 & 0\\ -0.8589 & 1.7178 & -0.8589 & 0 & 0 & 0\\ -0.8589 & -0.8589 & 1.7178 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.2883 & 0 & 0\\ 0 & 0 & 0 & 0 & 1.2883 & 0\\ 0 & 0 & 0 & 0 & 0 & 1.2883\end{bmatrix}+$$

$$\begin{bmatrix}-0.09 & -0.03 & -0.03 & 0 & 0 & 0\\ -0.03 & -0.09 & -0.03 & 0 & 0 & 0\\ -0.03 & -0.03 & 0.36 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.03 & 0 & 0\\ 0 & 0 & 0 & 0 & -0.03 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.03\end{bmatrix}+\begin{bmatrix}-0.0125 & -0.0208 & 0.0167 & 0 & 0 & 0\\ -0.0208 & -0.0125 & 0.0167 & 0 & 0 & 0\\ 0.0167 & 0.0167 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.0042 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.0042 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.0042\end{bmatrix}$$

$$+\begin{bmatrix}-0.0375 & 0.0375 & 0 & 0 & 0 & 0\\ 0.0375 & -0.0375 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.0375 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.0375 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.0375\end{bmatrix}. \tag{58}$$

Elastic constant tensor of Polystyrene can be represented by harmonic decomposition method as

$$C_{ij}=\begin{bmatrix}2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0\\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0\\ 2.7633 & 2.7633 & 2.7633 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}2.5767 & 0 & 0 & 0 & 0 & 0\\ 0 & 2.5767 & 0 & 0 & 0 & 0\\ 0 & 0 & 2.5767 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.2883 & 0 & 0\\ 0 & 0 & 0 & 0 & 1.2883 & 0\\ 0 & 0 & 0 & 0 & 0 & 1.2883\end{bmatrix}+$$

$$\begin{bmatrix}-0.1533 & 0 & 0 & 0 & 0 & 0\\ 0 & -0.1533 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.3067 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.0383 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.0383 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.0767\end{bmatrix}+\begin{bmatrix}-0.0038 & -0.019 & 0.0095 & 0 & 0 & 0\\ -0.019 & -0.0038 & 0.0095 & 0 & 0 & 0\\ 0.0095 & 0.0095 & 0.0076 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.0038 & 0 & 0\\ 0 & 0 & 0 & 0 & -0.0038 & 0\\ 0 & 0 & 0 & 0 & 0 & 0.0076\end{bmatrix}+$$

$$\begin{bmatrix} 0.0171 & 0.0057 & -0.0229 & 0 & 0 & 0 \\ 0.0057 & 0.0171 & -0.0229 & 0 & 0 & 0 \\ -0.0229 & -0.0229 & 0.0457 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0229 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0057 \end{bmatrix}. \tag{59}$$

By using the formula given in eq. (50), elastic constant tensor for Polystyrene can be decomposed by spectral method and we get following six decomposed parts which are different from those obtained by other three methods:

$$C_{ij} = \begin{bmatrix} 3.474 & 3.474 & 3.693 & 0 & 0 & 0 \\ 3.474 & 3.474 & 3.693 & 0 & 0 & 0 \\ 3.693 & 3.693 & 3.926 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.501 & 0.501 & -0.943 & 0 & 0 & 0 \\ 0.501 & 0.501 & -0.943 & 0 & 0 & 0 \\ -0.943 & -0.943 & 1.774 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 1.225 & -1.225 & 0 & 0 & 0 & 0 \\ -1.225 & 1.225 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.225 \end{bmatrix}. \tag{60}$$

From eqs. (57), (58), (59) and it is obvious that decomposed parts of total scalar (isotropic) parts are identical in both orthonormal tensor basis and irreducible decomposition methods besides total scalar parts are the same in all methods except spectral decomposition illustrated in eq. (60).

For isotropic material , RPV Steel (see, Cheong *et al.,* 1999) is selected as an example,

$$C_{ij} = \begin{bmatrix} 277.001 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 277.001 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 277.001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{61}$$

Using irreducible decomposition method, we apply the formula given in section (2.1). So elastic constant tensor for RPV Steel can be decomposed as

$$C_{ij} = \begin{bmatrix} 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 105.524 & -52.762 & -52.762 & 0 & 0 & 0 \\ -52.762 & 105.524 & -52.762 & 0 & 0 & 0 \\ -52.762 & -52.762 & 105.524 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{62}$$

By orthonormal tensor basis method, we apply the formula given in eq. (32). For this reason inner products must be calculated as; $(\mathbf{C}, A^{I}) = 514.431, \quad (\mathbf{C}, A^{II}) = 353.94.$

The symmetric fourth rank tensor for RPV Steel can be represented in the form

$$C_{ij} = \begin{bmatrix} 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 105.524 & -52.762 & -52.762 & 0 & 0 & 0 \\ -52.762 & 105.524 & -52.762 & 0 & 0 & 0 \\ -52.762 & -52.762 & 105.524 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{63}$$

Like irreducible decomposition method, there is only scalar part for RPV Steel in harmonic decomposition method and the elastic constant tensor for it, represented as follows:

$$C_{ij} = \begin{bmatrix} 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 118.715 & 118.715 & 118.715 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 158.286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 158.286 & 0 & 0 & 0 & 0 \\ 0 & 0 & 158.286 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{64}$$

By applying the formula given in eq. (49) and using these eigenvalues and eigenvectors, elastic constant tensor for RPV Steel can be decomposed by spectral method. We obtain the same decomposed parts as irreducible decomposition and orthonormal tensor basis methods.

$$C_{ij} = \begin{bmatrix} 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 171.477 & 171.477 & 171.477 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 105.524 & -52.762 & -52.762 & 0 & 0 & 0 \\ -52.762 & 105.524 & -52.762 & 0 & 0 & 0 \\ -52.762 & -52.762 & 105.524 & 0 & 0 & 0 \\ 0 & 0 & 0 & 79.143 & 0 & 0 \\ 0 & 0 & 0 & 0 & 79.143 & 0 \\ 0 & 0 & 0 & 0 & 0 & 79.143 \end{bmatrix}. \tag{65}$$

From eqs. (62), (63), (64) and (65), it is seen that harmonic decomposition method yields different decomposed parts for isotropic materials while these parts are identical in the other three methods. The summation expressed in eqs. (62), (63), (64) and (65) states that total of these decomposed parts for isotropic materials are the same.

7. Comparison of the Decomposition Methods

For comparison purposes, we find out critical relationships between irreducible and harmonic decomposition methods. Not only these relations but also the comparison of the four decomposition methods are summarized in this section.

In section 4, it is seen that there are many works done harmonic decomposition method in the literature. Like irreducible decomposition, two scalars, two deviators and the nonor part are obtained in harmonic decomposition method. Hence total scalar, total deviatoric parts and nonor part are identical in two methods. This is the first relationship between these methods.

In irreducible method, components of total scalar parts; $C_{mn}^{(0;1)}$ and $C_{mn}^{(0;2)}$ are orthogonal to each other. Contrary to it, decomposed parts of total scalar part are not orthogonal to each other in harmonic decomposition method due to the expression for decomposition of elastic constant tensor given in eqs. (40) and (43). Since $\delta_{ij}\delta_{kl}$ is not orthogonal to $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ (denoted as $2I_{ijkl}$). If we replace $\delta_{ij}\delta_{kl}$ and $2I_{ijkl}$ by the hydrostatic and deviatoric operators

$$I^h_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}, \qquad I^d_{ijkl} = I_{ijkl} - I^h_{ijkl}, \tag{66}$$

respectively, then we obtain back the matrices in eq. (19) in which decomposed parts of total scalar parts are orthogonal to each other. So the components of total scalar part in harmonic decomposition method takes the form of $C^{(0;1)}_{mn}$ and $C^{(0;2)}_{mn}$. This case is a significant innovation for both decomposition methods for elastic constant tensor. It is the second relationship between both decomposition methods.

Besides, in irreducible decomposition method, by using equations (18), (21), we obtain two physically meaningful parts which are

$$J_1 = 3KI_1, \tag{67}$$

and

$$(\sigma_{ij} - \frac{1}{3}\sigma_{rr}\delta_{ij}) = 2G(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{rr}\delta_{ij}), \tag{68}$$

where $J_1 = \sigma_{ii}$ and $I_1 = \varepsilon_{ii}$ are the first fundamental invariants of stress and strain tensors, respectively.

Equation (67) represents volume-change without distortion under hydrostatic stress and eq. (68) represents shape-change without volume-change under deviatoric stress.

So irreducible decomposition method gives two isotropic (scalar) parts which have significant physical meanings. On the other hand, scalar parts obtained from harmonic decomposition method do not have any physical meanings like irreducible decomposition method. By using eq. (66), we can make scalar parts orthogonal to each other for harmonic decomposition method and then these parts also have same physical meanings as those in irreducible decomposition method. So orthogonal decomposition of elastic constant tensor is important.

As a result, components of scalar and deviatoric parts in irreducible method are not equal to those in harmonic decomposition method, so it proves that there is not a unique decomposition for both deviatoric and scalar parts, in other words total deviatoric and scalar parts can be decomposed into infinitely many independent components. This case also indicates that total scalar, deviatoric and nonor parts of elastic constant tensor obtained from irreducible decomposition methods are the same as those of harmonic decomposition method. (See, Rychlewski (2000))

Furthermore, in order to designate the similarities between the all decomposition methods as well as differences, we compare them. The following results are found out:

1. Irreducible decomposition, orthonormal tensor basis, harmonic and spectral decomposition methods are suitable for the orthogonal decomposition of the elastic constant tensor.
2. Orthonormal tensor basis method gives twenty one decomposed parts and spectral decomposition yields only six terms at most for triclinic system whereas irreducible and harmonic decomposition methods decompose elastic constant tensor into five parts at most for triclinic materials and these five parts are composed of two scalars, two deviators and one nonor part.
3. Decomposed parts of isotropic material are identical in orthonormal tensor basis, irreducible decomposition and spectral methods, contrary to these methods, components of isotropic material are different in harmonic decomposition method.
4. Total scalar (isotropic) parts from irreducible and harmonic decomposition methods are identical with the isotropic parts of lower symmetry types such as transversely isotropic symmetry obtained from orthonormal tensor basis method.

Following Ryclewski (2000), we can call spectral method `**non-linear invariant decomposition**' and the other three decomposition methods `**linear invariant decomposition**'.

8. Discussion

Decomposition methods as irreducible decomposition, orthonormal tensor basis, harmonic decomposition and spectral have many applications in different subjects of physics and engineering (atomic and molecular physics and the physics of condensed matter). The decomposition methods of elastic constant tensor are applied to different fields of science and engineering. For instance, Geophysicists have used it in geophysical applications. (See, for instance; Chevrot and Browaeys (2004)) Furhermore for very valuable materials like diamond or quartz used in mining, it is difficult to measure its elastic constants because of its small samples. Applying orthonormal tensor basis method, we are able to specify the elastic constants of these types of materials (this case is proved by Tu (1968)). As an application for harmonic decomposition, it is possible to decide which type of symmetry a material has when the elastic constants are measured relative to an arbitrary coordinate system. A second rank symmetric tensor associated to the elastic constant tensor can be used to verify if the coordinate axes are the symmetry axes of the material and determine a symmetry coordinate system (examples for this case are given in Baerheim (1993)). So comprehending the decomposition methods is considerable to understand the idea behind these decomposition methods as well as the physical properties of anisotropic materials.

9. Conclusions

In conclusion, it is seen that orthonormal tensor basis, irreducible and harmonic decomposition methods give orthogonal and linear decomposed parts, while spectral method is a non-linear and orthogonal decomposition method since decomposed parts are expressed in terms of functions of elastic constants. In harmonic decomposition method, we are able to construct the decomposed parts, obtained from irreducible decomposition method, by making decomposed parts of total scalar part orthogonal. One of the important contribution of this work is that the sum of the total scalar and deviatoric parts are the same in both irreducible and harmonic decomposition methods. Finally, we hope this paper prepares interested readers to appreciate a deep understanding of application the orthonormal tensor basis and irreducible decomposition methods to elastic constant tensor and general review of harmonic decomposition and spectral methods for elastic constant tensor and for comparison purposes as well as the non-linear property of spectral method.

Appendix A

For irreducible decomposition method; the orthogonality condition is checked by using orthonormalized basis elements, the procedure is shown as

For fourth rank tensor $n=4$ and $j=0$, q will take the values $1,2$ and 3 . For $q=1$,

$$Q_{ijkl}^{(0,1)} = a\delta_{ij}\delta_{kl}. \tag{A1}$$

Applying the normalization condition, eq. (A1) takes the following form:

$$Q_{ijkl}^{(0,1)}Q_{ijkl}^{(0,1)} = a^2\delta_{ij}\delta_{kl}\delta_{ij}\delta_{kl} = 1$$

$a=\frac{1}{3}$, So the first idempotent is

$$Q_{ijkl}^{(0,1)} = \frac{1}{3}\delta_{ij}\delta_{kl}, \tag{A2}$$

For $q=2$,

$$Q_{ijkl}^{(0,2)} = (a\delta_{ik}\delta_{jl} + b\delta_{il}\delta_{jk}), \tag{A3}$$

According to orthogonality condition (we take the inner product of both eqs. (A2) and (A3))

$$Q_{ijkl}^{(0,1)}Q_{ijkl}^{(0,2)} = 0\ ,\ Q_{ijkl}^{(0,2)}Q_{ijkl}^{(0,2)} = 1$$

so for $q=2$, $a=\frac{1}{2\sqrt{3}}$, $b=-\frac{1}{2\sqrt{3}}$. The second idempotent is

$$Q_{ijkl}^{(0,2)} = \frac{1}{2\sqrt{3}}\delta_{ik}\delta_{jl} - \frac{1}{2\sqrt{3}}\delta_{il}\delta_{jk} \tag{A4}$$

By using the same procedure, the last idempotent is found as

$$Q_{ijkl}^{(0,3)} = \frac{1}{6\sqrt{5}}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}) \tag{A5}$$

As an example to how we obtain the irreducible parts from idempotent is demonstrated, $C_{ijkl}^{(0;1)}$ is given

$C_{ijkl}^{(0;1)} = Q_{ijkl}^{(0;1)}\widetilde{Q}_{ijkl}^{(0;1)}C'_{ijkl}$. This gives

$$C_{ijkl}^{(0;1)} = \frac{1}{9}\delta_{ij}\delta_{kl}C_{ppqq}. \tag{A6}$$

This is the first decomposed part of scalar part. Second decomposed part $C_{ijkl}^{(0;2)}$ is found as

$C_{ijkl}^{(0;2)} = Q_{ijkl}^{(0;2)}\widetilde{Q}_{ijkl}^{(0;2)}C'_{ijkl}$. This gives

$$C_{ijkl}^{(0;2)} = \frac{1}{90}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})(3C_{pqpq} - C_{ppqq}). \tag{A7}$$

Using same procedure, the other parts can be found.

Appendix B

For orthonormal tensor basis method; the orthogonality condition is checked by using orthonormalized basis elements, the procedure is shown as

Any isotropic tensor C_{ijkl} can be written as follows

$$C_{ijkl} = aA^{I}_{ijkl} + bA^{II}_{ijkl}. \tag{B1}$$

Taking inner products of both sides with A^{I}, eq. (B1) takes the following form:

$$(\mathbf{C}, A^{I}) = a(A^{I}, A^{I}) + b(A^{II}, A^{I}). \tag{B2}$$

In order to satisfy the orthogonality conditions

$(A^{I}, A^{I}) = 1$ and $(A^{II}, A^{I}) = 0.$ So we get; $a = (\mathbf{C}, A^{I})$

In order to find $b,$ again we can take inner products of both sides with A^{II}, eq. (B1) takes the following form:

$$(\mathbf{C}, A^{II}) = a(A^{I}, A^{II}) + b(A^{II}, A^{II}). \tag{B3}$$

By using same procedure, $b = (\mathbf{C}, A^{II}).$ Then we get the following:

$$C_{ijkl} = (\mathbf{C}, A^{I})A^{I}_{ijkl} + (\mathbf{C}, A^{II})A^{II}_{ijkl}, \tag{B4}$$

where $A^{I}_{ijkl} = d\alpha_{ijkl}$,

$(A^{I}_{ijkl}, A^{I}_{ijkl}) = d^{2}\alpha_{ijkl}\alpha_{ijkl}$

Then $d = \frac{1}{3}.$ So the first orthonormalized basis element A^{I}_{ijkl} can be found as

$$A^{I}_{ijkl} = \frac{1}{3}\alpha_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$$

A^{II}_{ijkl} can be obtained by applying the same procedure step by step. Let $A^{II}_{ijkl} = d\alpha_{ijkl} + e\beta_{ijkl}$

Now taking inner product with A^{II}_{ijkl} and

$(A^{II}_{ijkl}, A^{II}_{ijkl}) = (d\alpha_{ijkl} + e\beta_{ijkl})(d\alpha_{ijkl} + e\beta_{ijkl}),$ which gives

$$1 = 9d^{2} + 12de + 24e^{2}$$

$(A^{II}_{ijkl}, A^{I}_{ijkl}) = \frac{1}{3}\alpha_{ijkl}(d\alpha_{ijkl} + e\beta_{ijkl}),$ this causes

$$0 = \frac{d}{3}\alpha_{ijkl}\alpha_{ijkl} + \frac{e}{3}\alpha_{ijkl}\beta_{ijkl},$$

where $\alpha_{ijkl}\alpha_{ijkl} = (\delta_{ij}\delta_{kl})(\delta_{ij}\delta_{kl}) = 9,\ \alpha_{ijkl}\beta_{ijkl} = (\delta_{ij}\delta_{kl})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = 6$

Then $d = -\frac{2}{6\sqrt{5}}$, $e = \frac{1}{2\sqrt{5}}$. So, the second orthonormalized basis element for isotropic system A^{II}_{ijkl} can be found as

$$A^{II}_{ijkl} = \frac{1}{6\sqrt{5}}(3\beta_{ijkl} - 2\alpha_{ijkl}).$$

For transversely isotropic system, the other orthonormalized basis elements for transversely isotropic system are constructed by performing the following steps:

A transversely isotropic tensor can be written as follows:

$A^{III}_{ijkl} = d_{1}\alpha_{ijkl} + d_{2}\beta_{ijkl} + d_{3}\gamma_{ijkl}$. By taking inner products of both sides with A^{III}_{ijkl}, the above equation takes the form:

$(A^{III}_{ijkl}, A^{III}_{ijkl}) = [d_{1}\alpha_{ijkl} + d_{2}\beta_{ijkl} + d_{3}\gamma_{ijkl}][d_{1}\alpha_{ijkl} + d_{2}\beta_{ijkl} + d_{3}\gamma_{ijkl}]$. In order to satisfy the orthogonality conditions,

$(A^{III}, A^{III}) = 1$ and $(A^{III}, A^{I}) = 0.$ So we get

$$(A^{III}, A^{III}) = 1 = d_1^2\alpha_{ijkl}\alpha_{ijkl} + d_1 d_2\alpha_{ijkl}\beta_{ijkl} + d_1 d_3\alpha_{ijkl}\gamma_{ijkl} + d_2 d_1\beta_{ijkl}\alpha_{ijkl} + d_2^2\beta_{ijkl}\beta_{ijkl} + d_2 d_3\beta_{ijkl}\gamma_{ijkl} + d_3 d_1\gamma_{ijkl}\alpha_{ijkl} + d_3 d_2\gamma_{ijkl}\beta_{ijkl} + d_3^2\gamma_{ijkl}$$

By rearranging the above equation, we obtain

$$1 = 9d_1^2 + 12d_1 d_2 + 4d_2 d_3 + 2d_1 d_3 + 24d_2^2 + d_3^2 \tag{B5}$$

$$(A^{III}, A^{I}) = 0 = \frac{1}{3}\alpha_{ijkl}[d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl}] \Rightarrow \quad 0 = d_1\alpha_{ijkl}\alpha_{ijkl} + d_2\alpha_{ijkl}\beta_{ijkl} + d_3\alpha_{ijkl}\gamma_{ijkl},$$

$$0 = 9d_1 + 6d_2 + d_3 \tag{B6}$$

$(A^{III}, A^{II}) = 0 = \dfrac{1}{6\sqrt{5}}[3\beta_{ijkl} - 2\alpha_{ijkl}][d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl}]$. By rearranging the above equation, we get

$$d_2 = -\frac{1}{15}d_3 \tag{B7}$$

By substituting eq. (B7) into eq. (B6) we get

$$d_1 = -\frac{1}{15}d_3 \tag{B8}$$

By putting eqs. (B7) and (B8) into eq. (B5), the constants are found as

$$d_1 = d_2 = -\frac{1}{6\sqrt{5}}, \quad d_3 = \frac{15}{6\sqrt{5}} \tag{B9}$$

The third basis element of transversely isotropic system is

$$A_{ijkl}^{III} = \frac{1}{6\sqrt{5}}(15\gamma_{ijkl} - \beta_{ijkl} - \alpha_{ijkl}) \tag{B10}$$

For the other element, similar steps are performed then we obtain

$$A_{ijkl}^{IV} = d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl}$$

By taking inner products of both sides with A_{ijkl}^{IV}, the above equation takes the form

$$(A_{ijkl}^{IV}, A_{ijkl}^{IV}) = [d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl}][d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl}]$$

Orthogonality conditions are

$$(A_{ijkl}^{IV}, A_{ijkl}^{IV}) = 1 = d_1^2\alpha_{ijkl}\alpha_{ijkl} + d_1 d_2\alpha_{ijkl}\beta_{ijkl} + d_1 d_3\alpha_{ijkl}\gamma_{ijkl} + d_1 d_4\gamma_{ijkl}\alpha_{ijkl} + d_2 d_1\beta_{ijkl}\alpha_{ijkl} + d_2^2\beta_{ijkl}\beta_{ijkl} + d_3 d_2\gamma_{ijkl}\beta_{ijkl} + d_4 d_2\gamma_{ijkl}\beta_{ijkl} + d_1 d_3\alpha_{ijkl}\beta_{ijkl} + d_1 d_3\alpha_{ijkl}\gamma_{ijkl} + d_2 d_3\gamma_{ijkl}\beta_{ijkl} + d_3^2\gamma_{ijkl}\gamma_{ijkl} + d_3 d_4\gamma_{ijkl}\delta_{ijkl} + d_{41}\delta_{ijkl}\alpha_{ijkl} + d_4 d_2\delta_{ijkl}\beta_{ijkl} + d_3 d_4\gamma_{ijkl}\delta_{ijkl} + d_4^2\gamma_{ijkl}\gamma_{ijkl}$$

$$\Rightarrow 1 = 9d_1^2 + 6d_1 d_2 + d_1 d_3 + 6d_1 d_4 + 6d_1 d_2 + 24d_2^2 + 2d_2 d_3 + 4d_2 d_4 + d_1 d_3 + 2d_2 d_3 + 2d_3 d_4 + d_3^2 + 6d_1 d_4 + 4d_2 d_4 + 2d_4 d_3 + 8d_4^2$$

By rearranging the above equation, we obtain

$$1 = 9d_1^2 + 12d_1 d_2 + 2d_1 d_3 + 12d_1 d_4 + 24d_2^2 + 4d_2 d_3 + 8d_2 d_4 + 4d_4 d_3 + 8d_4^2 \tag{B11}$$

Another equation is obtained from the orthogonality condition which is

$$(A_{ijkl}^{IV}, A_{ijkl}^{III}) = 0 = \frac{1}{6\sqrt{5}}[15\gamma_{ijkl} - \beta_{ijkl} - \alpha_{ijkl}][d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl}]$$

By rearranging the above equation, we get

$$d_3 = -\frac{5}{3}d_4 \tag{B12}$$

In order to find the constant, steps should be continued as follows

$$(A^{IV}_{ijkl}, A^{II}_{ijkl}) = 0 = \frac{1}{6\sqrt{5}}[3\beta_{ijkl} - 2\alpha_{ijkl}][d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl}]$$

From the above equation, we obtain

$$d_2 = \frac{1}{9}d_4 \tag{B13}$$

$$(A^{IV}_{ijkl}, A^{I}_{ijkl}) = 0 = \frac{1}{3}\alpha_{ijkl}[d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl}]$$

From above equation, we get

$$d_1 = -\frac{5}{9}d_4 \tag{B14}$$

By substituting eqs. (B14), (B13), (B12) into eq. (B11), we can find the constants as

$d_1 = -\frac{5}{12}, \quad d_2 = \frac{1}{12}, d_3 = -\frac{15}{12}, \quad d_4 = \frac{9}{12}$. So the fourth basis element of transversely isotropic system is

$$A^{IV}_{ijkl} = \frac{1}{12}(9\delta_{ijkl} - 15\gamma_{ijkl} + \beta_{ijkl} - 5\alpha_{ijkl}) \tag{B15}$$

For the other element, similar steps are performed then we get

$$A^{V}_{ijkl} = d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}$$

$$(A^{V}_{ijkl}, A^{V}_{ijkl}) = [d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}][d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}] = 1$$

$$\Rightarrow 9d_1^2 + 12d_1d_2 + 2d_1d_3 + 12d_1d_4 + 24d_2^2 + 4d_2d_3 + 8d_2d_4 + 16d_2d_5 + d_3^2 + 4d_3d_4 + 8d_4^2 + 8d_5^2 = 1 \tag{B16}$$

$$(A^{V}_{ijkl}, A^{III}_{ijkl}) = \frac{1}{6\sqrt{5}}[15\gamma_{ijkl} - \beta_{ijkl} - \alpha_{ijkl}][d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}] = 0$$

$$0 = 12d_3 + 20d_4 - 8d_5 \tag{B17}$$

$$(A^{V}_{ijkl}, A^{IV}_{ijkl}) = \frac{1}{12}(9\delta_{ijkl} - 15\gamma_{ijkl} + \beta_{ijkl} - 5\alpha_{ijkl})[d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}] = 0$$

From the above equation two constants are found as

$$d_4 = -\frac{1}{2}d_5, \quad d_3 = \frac{3}{2}d_5 \tag{B18}$$

$$(A^{V}_{ijkl}, A^{II}_{ijkl}) = \frac{1}{6\sqrt{5}}(3\beta_{ijkl} - 2\alpha_{ijkl})[d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}] = 0$$

$$\Rightarrow d_2 = -\frac{1}{2}d_5 \tag{B19}$$

$$(A^{V}_{ijkl}, A^{I}_{ijkl}) = \frac{1}{3}\alpha_{ijkl}[d_1\alpha_{ijkl} + d_2\beta_{ijkl} + d_3\gamma_{ijkl} + d_4\delta_{ijkl} + d_5\varepsilon_{ijkl}] = 0$$

$$\Rightarrow d_1 = \frac{1}{2}d_5 \tag{B20}$$

By putting eqs. (B20) , (B19) and (B18) into eq. (B16), we can find the constants as

$$\Rightarrow \quad d_1 = \frac{1}{4}, \quad d_2 = d_4 = -\frac{1}{4}, \quad d_3 = \frac{3}{4}, \quad d_5 = \frac{1}{2},$$

Then we can find the last orthonormalized basis element for transversely isotropic system as

$$A^{V}_{ijkl} = \frac{1}{4}(2\varepsilon_{ijkl} - \delta_{ijkl} + 3\gamma_{ijkl} - \beta_{ijkl} + \alpha_{ijkl}) \tag{B21}$$

References

Andrews D.L. and Ghoul W.A., 1982. Irreducible fourth-rank cartesian tensors. *PhysicalReview A*, Vol. 25, pp. 2647-2657.

Annin B.D. and Ostrosablin N.I., 2008. Anisotropy of elastic properties of materials. *Journal of Applied Mechanics and Technical Physics*, Vol. 49, pp. 998-1014.

Backus G., 1970. A Geometrical picture of anisotropic elastic tensors. *Reviews of Geophysics*, Vol. 8, pp. 633-671.

Baerheim R., 1993. Harmonic decomposition of the anisotropic elasticity tensor. *Q.JI. Mech.Appl. Math*., Vol. 46, pp. 391-418.

Brown J.M., Abramson E.H. and Angel R.J., 2006. Triclinic elastic constants for low albite. *Phys.Chem. Minerals,* Vol. 33, pp. 256-265.

Cheong,Yong-Moo et al., 1999. Dynamic elastic constants of Weld HAZ of SA 508 CL.3 steel using resonant ultrasound spectroscopy. Korea Atomic Energy Research Institute, Korea.

Chevrot S., Browaeys J.T., 2004. Decomposition of the elastic tensor and geophysical applications. *Geophys. J. Int.*, Vol. 159, pp. 667-678.

Cowin S., 1989. Properties of the anisotropic elasticity tensor. *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 42, pp. 249-267.

Cowin S.C., Mehrabadi M.M. and Sadegh A.M., 1991. Kelvin formulation of the anisotropic Hooke's law, in modern theory of anisotropic elasticity and applications. pp.340-356.

Fedorov, F.I.,1968. *Theory of Elastic Waves in Crystals*, Plenum, New York.

Forte S. and Vianello M., 1996. Symmetry classes for elasticity tensors. *Journal of Elasticity*, Vol. 43, pp. 81-108.

Forte S. and Vianello M., 2006. Restricted invariants on the space of elasticity tensors. *Mathematics and Mechanics of Solids*, Vol. 11, pp. 48-82.

Gazis D.C., Tadjbakhsh and Toupin R. A., 1963. The elastic tensor of given symmetry nearest to an anisotropic elastic tensor. *Acta Crsyt*., Vol. 16, pp. 917-922.

Ghaith M.S. and Akgö z C.Y., 2005, A new representation for the properties of anisotropic elastic fiber reinforced composite materials. *Rev. Adv. Mater. Sci.*, Vol. 10, pp. 138-142.

He Q.-C., 2004. Characterization of the anisotropic materials capable of exhibiting an isotropic Young or shear or area Modulus. *Int.J Engineering Sci.*, Vol. 42, pp. 2107-2118.

Heine V., 1960. Group Theory in Quantum Mechanics, Pergamon Press, Oxford.

Jarić J.P., 2003. On the decomposition of symmetric tensors into traceless symmetric tensors. *Int.J. Engineering Sci.*, Vol. 41, pp. 2123-2141.

Jerphagnon J., Chemla D. and Bonneville R., 1978. The description of the physical properties of condensed matter using irreducible tensors. *Advances in Physics*, Vol. 27, pp. 609-650.

Kunin, I.A., 1982. *Elastic Media with Microstructure*, Springer, Berlin Heidelberg New York.

Landau L.D. and Lifshitz E.M., 1959. *Theory of Elasticity*, Pergamon Press.

Mochizuki, E., 1988. Spherical harmonic decomposition of an elastic tensor. *Geophysical Journal*, Vol. 93, pp. 521-526.

Nye J.F.,1957. *Physical Properties of Crystals; their representation by tensors and matrices*. Oxford: At the Clarendon Press.

Onat E.T., 1984. Effective properties of elastic materials that contain penny shaped voids. *Int. J. Enginnering Sci.*, Vol. 22, pp. 1013-1021.

Onat E.T., Boehler J.P. and Kirillov Jr., 1994. On the polynomial invariants of the elasticity tensor. *Journal of Elasticity*, Vol. 34, pp. 97-110.

Radwan Fae'q A.A., 2001. Irreducible parts of elastic compliance tensor and anisotropy. *Pakistan Journal of Applied Sciences*, Vol. 1, No. 3, pp. 270-274.

Rychlewski J., 2000. A qualitative approach to Hooke's tensors. Part I., *Arch. Mech*., Vol. 52, pp. 737-759.

Sirotin Yu.I., 1975. Decomposition of material tensors into irreducible parts. *Sov. Phys.Crystallogr*., Vol. 19, pp. 565-568.

Spencer A.J.M., 1970. A note on the decomposition of tensors into traceless symmetric tensors. *Int.J. Engineering Sci.*, Vol. 8, pp. 475-481.

Srinivasan T.P. and Nigam S.D., 1969. Invariant elastic constants for crystals. *Journal of Mathematics and Mechanics,* Vol. 19, 411-420.

Srinivasan T.P., 1998. Decomposition of tensors representing physical properties of crystals. *J. Phys.: Condens. Matter*, Vol. 10, pp. 3489-3496.

Surrel Y., 1993. A new description of the tensors of elasticity based upon irreducible representations. *Eur. J. Mech., A/Solids*, Vol. 12, pp. 219-235.

Sutcliffe S., 1992. Spectral decomposition of the elasticity tensor. *Transactions of the ASME*, Vol. 59, pp. 762-773.

Ting T.C.T. and He Q.-C., 2006. Decomposition of elasticity tensors and tensors that are structurally invariant in three dimensions. *Q.JI Mech. Appl. Math.* Vol. 59, pp. 323-341.

Tu Yih-O, 1968. The decomposition of an anisotropic elastic tensor. *Acta Cryst.*, Vol. A24, pp. 273-282.

Walpole L.J., 1984. Fourth-rank tensors of the thirty-two crystal classes: multiplication tables. *Proc. R. Soc. Lond.*, Vol. A391, pp. 149-179.

Wright, A., Faraday, C.S.N., White, E.F.T. and Treloar L. R. G. 1971. Elastic constants of oriented glassy polymers. *Journal of Physics D- Applied Physics*, Vol. 4, pp. 2002.
Zheng X., 2007. Identification of symmetry planes in weakly anisotropic elastic media. *Geophys.J.Int.*, Vol. 170, pp. 468-478.
Zou W.-N, Zheng Q.-S., Du D.-X and Rychlewski J., 2001. Orthogonal irreducible decompositions of tensors of high orders. *Mathematics and Mechanics of Solids*, Vol. 6, pp. 249-267.

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